Bogolubov N.N. (JR)

On some asymptotical solution for some problems in statistical mechanics.

Let us briefly discuss the fundamentals of the method of approximating Hamiltonians as applied to the analysis of the models with four-fermion interaction. Among the extensive variety of these models, we choose. first of all, the models of general form for which one can find an exact solution. The models describing the systems of interacting fermions with four-fermion pairwise interaction represent an important example of this type of models. Asymptotically exact solutions of such models were analyzed in [1] by N.N. Bogolyubov, D.N. Zubarev, and Yu.A. Tserkovnikov. In this work, they proposed an approximate method based on the use of "approximating (trial) Hamiltonians," and presented many arguments supporting the assumption that the solution obtained by this method is asymptotically exact in the ordinary thermodynamic limit as $V \to \infty$. Specifically, in [1], a model was considered with the Hamiltonian

$$\begin{gathered} H = H_0 + H_{\text{int}}, \qquad H_0 = \sum_{(p,s)} (E(p) - \mu) a^{\dagger}_{ps} a_{ps}, \\ H_{\text{int}} = - \sum_{(p,p')} \frac{1}{V} J(p,p') a^{\dagger}_{-p,-1/2} a^{\dagger}_{p,1/2} a_{p',1/2} a_{-p',-1/2}, \end{gathered} \tag{1.1}$$

where $a^{\dagger}_{p,\pm 1/2}$ and $a_{p,\pm 1/2}$ are the Fermi operators and V is the volume of the system. It is assumed that the kernel $J(p,p')$ is a real bounded function vanishing outside a certain domain of variation of its arguments. The summation in H_{int} over quasimomenta p and p' is performed within the energy layer $E_F - \omega < E(p) < E_F + \omega$.

As is known, for this type of Hamiltonians, one can obtain an approximate expression for the free energy, which becomes asymptotically exact in the limit as $V \to \infty$. This idea can be realized by introducing the so-called trial Hamiltonian $H_0(C)$, which represents a quadratic form with respect to the Fermi operators and contains an arbitrary complex parameters C. This Hamiltonian can be easily diagonalized, and the corresponding free energy can be calculated explicitly.

In [1], the authors suggested that the approximate free energy $F_0(C)$ is equal to the exact free energy F in the limit as $V \to \infty$. Originally, this result was obtained by the methods of perturbation theory. The assumption that the free energies are identical was supported by the fact that each term of the series in perturbation theory constructed for calculating the corrections to the given approximate solution is asymptotically small in the limit as $V \to \infty$. However, the question of the convergence of the series in perturbation theory was not analyzed in detail. In [2], the same problem was analyzed without using the methods of perturbation theory. The following BCS-type model was investigated:

$$H = \sum_{(f)} T_f a_f^\dagger a_f - \frac{1}{2V} \sum_{(f,f')} J(f,f') a_f^\dagger a_{-f}^\dagger a_{-f'} a_{f'} + \nu \mathcal{U}, \tag{1.2}$$

$$\mathcal{U} = \frac{1}{2} \sum_{(f)} W(f)(a_{-f} a_f + a_f^\dagger a_{-f}^\dagger), \qquad \nu \geq 0,$$

where $f = (p, \sigma)$, $-f = (-p, \sigma)$, σ is the spin quantum number taking the values $1/2$ or $-1/2$. and p is the quasimomentum taking ordinary quasi-discrete values $p_{(d)} = \frac{2\pi n_{(d)}}{L}$, where, for a fixed L ($L^3 = V$), the number $n_{(d)}$ runs over the sequence of integers. $T_f = \frac{p^2}{2m} - \mu$, where μ is the chemical potential, and a_f and $a_f^\dagger$ are operators satisfying ordinary anticommutation relations of the Fermi–Dirac statistics. The functions $J(f, f')$ and $W(f)$ are real functions possessing the properties

$$J(f, f') = J(f', f) = -J(-f, f'), \qquad W(-f) = -W(f).$$

For example,

$$J(f, f') = \frac{1}{2} J(p, p')\{\delta(\sigma - \sigma') - \delta(\sigma + \sigma')\}, \qquad J(p,p') = J(p',p) = J(-p,p'),$$

where $\delta(\sigma - \sigma')$ is the Kronecker delta. The auxiliary term $\nu\mathcal{U}$ in (1.2) is introduced to select the physically meaningful solutions. In [2], a chain of equations for the Green functions was investigated. It was shown that the Green function for an exactly solvable model with the Hamiltonian H_0 satisfies a similar chain of equations obtained for the exact Hamiltonian H with an error of the order of $1/V$. However. from a rigorous mathematical point of view, these arguments could not be regarded as exhaustively convincing. Nevertheless, the works [1, 2] made a significant contribution to the analysis of asymptotically exact solutions. At the same time, we should note that the rigorous proof of asymptotic exactness of the results obtained in [1, 2] has faced substantial mathematical difficulties. The problem of the existence of the asymptotically exact solution as a purely mathematical problem was first investigated by N.N. Bogolyubov for a particular case of zero temperature. He investigated a model described by the Hamiltonian (1.2) under the assumption that the kernel $J(f, f')$ can be factorized, i.e.,

$$J(f, f') = \lambda(f) \cdot \lambda(f').$$

In addition, it was assumed that the functions $\lambda(f)$ and $T(f)$ satisfy the following conditions:

$$\begin{gathered}
\lambda(-f) = -\lambda(f), \qquad T(-f) = T(f), \\
\left.\begin{array}{l} |\lambda(f)| \le \text{const} \\ T(f) \to \infty \end{array}\right\} \quad \text{if} \quad |f| \to \infty, \\
\frac{1}{V}\sum_{(f)} \frac{\lambda^2(f)}{\sqrt{\lambda^2(f)x + T^2(f)}} > 1
\end{gathered} \tag{1.3}$$

for sufficiently small positive x. It is this model that was studied in detail in the zero-temperature case. It was shown that the model is exactly solvable in the limit as $V \to \infty$ in the sense that the asymptotic values for the energy of ground state, for the Green functions, and for the correlation functions characterizing the dynamic behavior of the system can be exactly calculated in this limit. The calculation of the quantities listed above in the case of arbitrary temperature $\theta \neq 0$ is also of considerable interest. However, the direct application of the approach of [3] proved to be impossible in this case.

Thus, initially, this approach was applicable only to the study of the properties of the ground state of a system. Some of the subsequent works, in particular [4, 5], were devoted to the complicated problem of extending the method to the case of model Hamiltonians of the type (1.2) at zero temperature.

Now, consider the model Fermi systems of the form

$$H = \sum_{(f)} T_f a_f^\dagger a_f - \frac{1}{2V} \sum_{(f,f')} \lambda(f) \cdot \lambda(f') a_f^\dagger a_{-f}^\dagger a_{-f'} a_{f'}. \tag{1.4}$$

We use the conventional notations $f = (p, s)$ and $-f = (-p, -s)$ for the set of four quantum numbers, the momentum p and the spin projection σ, which determine the state of a free fermion:

$$V = L^3, \quad p_x = \frac{2\pi n_x}{L}, \quad p_y = \frac{2\pi n_y}{L}, \quad p_z = \frac{2\pi n_z}{L}, \qquad n_x, n_y, n_z \text{ are integers,}$$

and $T_f = \frac{p^2}{2m} - \mu$, where μ is a chemical potential. For the standard Bardeen model, we assume that

$$\lambda(f) = \begin{cases} J\varepsilon(s) = \text{const}, & \left|\dfrac{p^2}{2m} - \mu\right| \le 0, \\[2ex] 0, & \left|\dfrac{p^2}{2m} - \mu\right| > 0, \end{cases} \qquad \varepsilon(s) = \pm 1. \tag{1.5}$$

In this work, we do not literally rely on these stringent constraints imposed on the functions T_f and $\lambda(f)$. For our purposes, it is sufficient to impose the following weaker conditions:

$$\begin{gathered}
\text{The functions } \lambda(f) \text{ and } T_f \text{ are real and } \lambda(-f) = -\lambda(f), \\
\frac{1}{2V}\sum_{(f)} |\lambda_f| \le k_1 = \text{const}, \qquad \frac{1}{V}\sum_{(f)} |T_f \cdot \lambda_f| \le k_2 = \text{const}, \\
\frac{1}{V}\sum_{(f)} \lambda_f^2 \le k_3 = \text{const} \qquad \text{if} \quad V \to \infty.
\end{gathered} \tag{1.6}$$

These conditions are certainly fulfilled in the case (1.5). Note also that, in this case, the free energy per unit volume of a relevant system consisting of noninteracting fermions is finite.

Performing identity transformations, we can rewrite (1.4) as

$$H = H^0 + H_1,$$

where the "approximating Hamiltonian" H^0 is given by

$$H^0 = \sum_{(f)} T_f a_f^\dagger a_f - \left\{ \sum_{(f)} (C a_{-f} a_f + C^* a_f^\dagger a_{-f}) \right\} + 2VC^*C,$$

$$H_1 = 2V \left(\frac{1}{2V} \sum_{(f)} \lambda_f a_f^\dagger a_{-f}^\dagger - C \right) \left(\frac{1}{2V} \sum_{(f)} \lambda_f a_{-f} a_f - C^* \right),$$

and C and C^* are c-numbers. Since H^0 is quadratic with respect to Fermi operators, it can be diagonalized by the u–v-transformation

$$a_f = u(f)\alpha_f - v(f)\alpha_f^\dagger;$$

thus, the free energy per unit volume, defined by

$$f_{H_0}(C) = -\frac{\theta}{V} \ln \operatorname{Sp} e^{-\frac{H^0}{\theta}},$$

can be readily calculated. The complex parameter C entering the trial Hamiltonian H^0 is determined from the condition for the absolute minimum of the free energy per unit volume $f_{H^0}(C)$:

$$f_{H^0}(C) = \min,$$

which yields the equation

$$\frac{\partial f_{H^0}}{\partial C} = 0, \qquad C = \langle J \rangle_{H^0} = \frac{\operatorname{Sp} J e^{-\frac{H^0}{\theta}}}{\operatorname{Sp} e^{-\frac{H^0}{\theta}}}, \tag{1.7}$$

where

$$J = \frac{1}{2V} \sum_{(f)} \lambda(f) \cdot a_f^\dagger a_{-f}^\dagger.$$

We develop a method that allows us to prove that the difference $f_{H^0} - f_H$ of the free energies calculated on the basis of approximating and model Hamiltonians is asymptotically small. For this purpose, it is convenient to consider an auxiliary model system with the Hamiltonian containing the sources whose intensity is characterized by the parameter ν:

$$\Gamma = T - 2VJ \cdot J^\dagger - (\nu J + \nu^* J^\dagger)V. \tag{1.8}$$

When $\nu = 0$, Hamiltonian (1.8) coincides with H, where

$$T = \sum_{(f)} T_f a_f^\dagger a_f.$$

The appropriate trial (approximating) Hamiltonian is given by

$$\Gamma^0 = T - 2V(CJ^\dagger + C^*J) - V(\nu J + \nu^* J^\dagger) + 2V|C|^2. \tag{1.9}$$

Hence, it is obvious that

$$\Gamma = \Gamma^0 + \mathcal{U},$$

where

$$\mathcal{U} = -2V(J - C)(J^\dagger - C^*). \tag{1.10}$$

Now, let us calculate the above difference between the free energies per unit volume. To this end, we note that $\Gamma = \Gamma^0 + \mathcal{U}$ and introduce an intermediate auxiliary Hamiltonian

$$\Gamma^t = \Gamma^0 + t\mathcal{U},$$

which coincides with the trial Hamiltonian Γ^0 if $t = 0$ and with the original Hamiltonian Γ if $t = 1$. We assume that the parameter C in the intermediate Hamiltonian is fixed and independent of t. Consider the statistical sum and the free energy for the intermediate Hamiltonian:

$$f_t(C) = -\frac{\theta}{V}\ln Q_t, \qquad Q_t = \operatorname{Sp} e^{-\frac{\Gamma^t}{\theta}}, \qquad Q_t = e^{-\frac{V\cdot f_t}{\theta}}. \tag{1.11}$$

Note that $f_{t=1}(C) = f_\Gamma$ and, hence, is independent of C. Differentiating (1.11) twice with respect to t, we obtain

$$\frac{\partial^2 Q_t}{\partial t^2} = -\frac{V}{\theta}\frac{\partial^2 f_t}{\partial t^2}Q_t + \frac{V^2}{\theta^2}\left(\frac{\partial f_t}{\partial t}\right)^2 Q_t.$$

On the other hand, taking into account that

$$\frac{\partial^2 Q_t}{\partial t^2} = \frac{1}{\theta^2}\int_0^1 \operatorname{Sp}\left\{\mathcal{U}e^{-\frac{\Gamma^t}{\theta}\tau}\mathcal{U}e^{-\frac{\Gamma^t}{\theta}(1-\tau)}\right\} d\tau,$$

we obtain

$$-\frac{V}{\theta}\frac{\partial^2 f_t}{\partial t^2} + \frac{V^2}{\theta^2}\left(\frac{\partial f_t}{\partial t}\right)^2 = \frac{1}{\theta^2 Q_t}\int_0^1 \operatorname{Sp}\left\{\mathcal{U}e^{-\frac{\Gamma^t}{\theta}\tau}\mathcal{U}e^{-\frac{\Gamma^t}{\theta}(1-\tau)}\right\} d\tau.$$

Taking into account that

$$\frac{\partial f_t}{\partial t} = \frac{1}{V}\frac{\operatorname{Sp}(\mathcal{U}e^{-\frac{\Gamma^t}{\theta}})}{\operatorname{Sp} e^{-\frac{\Gamma^t}{\theta}}} = \frac{1}{V}\langle\mathcal{U}\rangle,$$

we also obtain

$$-\frac{\partial^2 f_t}{\partial t^2} = \frac{1}{\theta V}\left\{\frac{1}{Q_t}\int_0^1 \operatorname{Sp}\left\{\mathcal{U}e^{-\frac{\Gamma^t}{\theta}\tau}\mathcal{U}e^{-\frac{\Gamma^t}{\theta}(1-\tau)}\right\} d\tau - \langle\mathcal{U}\rangle^2\right\}$$

$$= \frac{1}{\theta V Q_t}\int_0^1 \operatorname{Sp}\left\{\mathcal{B}e^{-\frac{\Gamma^t}{\theta}\tau}\mathcal{B}e^{-\frac{\Gamma^t}{\theta}(1-\tau)}\right\} d\tau,$$

where $\mathcal{B} = \mathcal{U} - \langle\mathcal{U}\rangle$. Passing to the matrix representation in which the Hamiltonian is diagonal, we obtain

$$-\frac{\partial^2 f_t}{\partial t^2} = \frac{1}{\theta V Q_t} \int_0^1 \sum_{(n,m)} \mathcal{B}_{nm}\mathcal{B}_{mn} e^{-\frac{(E_m^t - E_n^t)}{\theta}\tau} e^{-\frac{E_n^t}{\theta}} d\tau$$

$$= \frac{1}{\theta V Q_t} \int_0^1 \sum_{(n,m)} |\mathcal{B}_{nm}|^2 e^{-\frac{(E_m^t - E_n^t)}{\theta}\tau} e^{-\frac{E_n^t}{\theta}} d\tau \geq 0,$$

$$-\frac{\partial^2 f_t}{\partial t^2} \geq 0.$$

This, in particular, implies that

$$\frac{\partial f_t}{\partial t} \equiv \frac{1}{V}\langle\mathcal{U}\rangle_t$$

decreases as the parameter t increases. Thus, we have

$$f_{\Gamma^0}(C) - f_\Gamma = -\int_0^1 \frac{\partial f_t}{\partial t} dt = -\int_0^1 \frac{\langle\mathcal{U}\rangle}{V} dt \geq 0.$$

Since this relation holds for arbitrary C, we have

$$\min_{(C)} f_{\Gamma^0}(C) \geq f_\Gamma, \qquad f_{\Gamma^0} \geq f_\Gamma.$$

Integrating both sides of this inequality, we obtain

$$\langle\mathcal{U}\rangle_t \geq \langle\mathcal{U}\rangle_\Gamma, \qquad 0 \leq t \leq 1.$$

Substituting (1.10) instead of $\mathcal{U}$, we can see that the following inequality holds for any C:

$$f_{\Gamma^0}(C) - f_\Gamma \leq 2\langle (J - C)(J^\dagger - C^*)\rangle_\Gamma.$$

In particular, we set $C = \langle J\rangle_\Gamma$ and note that

$$f_{\Gamma^0} = \min f_{\Gamma^0}(C) \leq f_{\Gamma^0}(\langle J\rangle_\Gamma).$$

Thus,

$$f_{\Gamma^0} - f_\Gamma \leq f_{\Gamma^0}(\langle J\rangle_\Gamma) - f_\Gamma \leq \langle (J - \langle J\rangle_\Gamma)(J^\dagger - \langle J^\dagger\rangle_\Gamma)\rangle_\Gamma,$$

and, finally,

$$0 \leq f_{\Gamma^0} - f_\Gamma \leq 2\langle (J - \langle J\rangle)(J^\dagger - \langle J^\dagger\rangle)\rangle. \tag{1.12}$$

Let us return to our main problem. Our goal is to prove that the difference $f_{\Gamma^0} - f_\Gamma$ is asymptotically small in the limit as $V \to \infty$. It follows from (1.12) that, to prove this, we must first prove that the thermodynamic average on the right-hand side of (1.12) is asymptotically small. Let us outline a general method for estimating this average. First, we note that

$$|\Gamma J - J\Gamma| \leq K = \text{const},$$

where $K = |\nu|k_3 + k_2 + 2k_1k_3$. Taking into account that the energy Γ of the system is proportional to V, we can naturally assume that the operators Γ and $J, J^\dagger$ asymptotically commute in the limit as $V \to \infty$. Thus, if we neglected the noncommutativity of the operator Γ with the operators $J, J^\dagger$ for any finite V, then, differentiating the free energy with respect to ν and ν^*, we would obtain

$$-\theta\frac{\partial^2 f}{\partial\nu\partial\nu^*} = V\frac{\mathrm{Sp}(J\cdot J^\dagger e^{-\frac{\Gamma}{\theta}})}{\mathrm{Sp}\, e^{-\frac{\Gamma}{\theta}}} - V\frac{(\mathrm{Sp}\, Je^{-\frac{\Gamma}{\theta}})(\mathrm{Sp}\, J^\dagger e^{-\frac{\Gamma}{\theta}})}{(\mathrm{Sp}\, e^{-\frac{\Gamma}{\theta}})^2} = V\langle (J-\langle J\rangle)(J^\dagger - \langle J^\dagger\rangle)\rangle,$$

or, which is equivalent,

$$-\frac{\theta}{V}\frac{\partial^2 f}{\partial\nu\partial\nu^*} = \langle (J-\langle J\rangle)(J^\dagger - \langle J^\dagger\rangle)\rangle. \tag{1.13}$$

Our problem will be solved if we prove that the second-order derivatives $\frac{\partial^2 f}{\partial\nu\partial\nu^*}$ are bounded. However, we cannot prove this assertion. All we can do is to proceed from the obvious boundedness of the first derivatives:

$$\frac{\partial f}{\partial \nu} = \langle J\rangle, \qquad \left|\frac{\partial f}{\partial\nu}\right| \le |J| \le \frac{1}{2V}\sum_{(f)}|\lambda_f| = k_1 = \mathrm{const}.$$

As we noted above, the operators J and $J^\dagger$ actually do not commute, and, hence, equation (1.13) must be corrected.

The asymptotic smallness of the difference between the two aforementioned free energies per unit volume can be proved in two stages. First, we should construct an estimate for the average (1.13) expressed in terms of the second-order derivative $\frac{\partial^2 f}{\partial\nu\partial\nu^*}$ of free energy with regard to the noncommutativity of the operators Γ and J, J^*. Then, proceeding strictly from this estimate and neglecting the hypothesis that the second-order derivatives of the free energy are bounded, we propose a method by which we prove the asymptotic smallness of the difference $f_{\Gamma^0} - f_\Gamma$ in the limit as $V \to \infty$.

Differentiating the appropriate expression for the free energy, we have

$$-\frac{1}{\theta}\frac{\partial^2 f}{\partial\nu\partial\nu^*} = \frac{V}{\theta^2}\int_0^1 \frac{\mathrm{Sp}(De^{-\frac{\tau}{\theta}\Gamma}D^\dagger e^{-\frac{(1-\tau)}{\theta}\Gamma})\,d\tau}{\mathrm{Sp}\, e^{-\frac{\Gamma}{\theta}}}, \tag{1.14}$$

where $D = J - \langle J\rangle$. Passing to the matrix representation in which the Hamiltonian Γ is diagonal, we obtain

$$\begin{aligned}-\frac{1}{\theta}\frac{\partial^2 f}{\partial\nu\partial\nu^*} &= \frac{V}{\theta}\int_0^1 D_{nm}e^{-\frac{\tau}{\theta}E_m}D^\dagger_{mn}e^{-\frac{(1-\tau)}{\theta}E_n}d\tau\cdot Q^{-1}\\ &= \frac{V}{\theta^2}\sum_{(n,m)}|D_{nm}|^2\int_0^1 e^{-\frac{\tau}{\theta}E_m}e^{-\frac{(1-\tau)}{\theta}E_n}d\tau\cdot Q^{-1}\\ &= \frac{V}{\theta}\frac{1}{Q}\sum_{(n,m)}\frac{|D_{nm}|^2}{E_n - E_m}\left(e^{-\frac{E_m}{\theta}} - e^{-\frac{E_n}{\theta}}\right).\end{aligned}$$

Thus, we see that

$$-\frac{\partial^2 f}{\partial\nu\partial\nu^*} = V\sum_{(n,m)}\frac{|D_{nm}|^2}{Q}\frac{e^{-\frac{E_m}{\theta}} - e^{-\frac{E_n}{\theta}}}{E_n - E_m} \ge 0. \tag{1.15}$$

Let us apply the Hölder inequality.[2] In our case, it is convenient to write this inequality as

$$\sum_{(k)} |u_k|^2 \le \left(\sum_{(k)} \frac{|u_k|^2}{p_k}\right)^{2/3} \left(\sum_{(k)} |u_k|^2 p_k^2\right)^{1/3} \tag{1.16}$$

$$\left(p_k \ge 0, \quad \left|\frac{u_k}{\sqrt{p_k}}\right| \text{ is finite}\right),$$

$$\sum_{(k)} |u_k|^2 = \sum_{(k)} \left(\frac{|u_k|^{4/3}}{p_k^{2/3}}\right) \left(|u_k|^{2/3} \cdot p_k^{2/3}\right), \qquad p_k = |E_n - E_m|,$$

$$|u_k|^2 = |D_{nm}|^2 \cdot \left|e^{-\frac{E_m}{\theta}} - e^{-\frac{E_n}{\theta}}\right| VQ^{-1}.$$

Substituting the two last expressions for p_k and $|u_k|^2$ into (1.16), we obtain

$$\frac{V}{Q} \sum_{(n,m)} |D_{nm}|^2 \cdot \left|e^{-\frac{E_m}{\theta}} - e^{-\frac{E_n}{\theta}}\right|$$

$$\le \left(\frac{V}{Q} \sum_{(n,m)} \frac{|D_{nm}|^2 \cdot |e^{-\frac{E_m}{\theta}} - e^{-\frac{E_n}{\theta}}|}{|E_n - E_m|}\right)^{2/3} \left(\frac{V}{Q} \sum_{(n,m)} |D_{nm}|^2 \cdot |E_n - E_m|^2 \cdot \left|e^{-\frac{E_m}{\theta}} - e^{-\frac{E_n}{\theta}}\right|\right)^{1/3}.$$

By (1.15), we have

$$\frac{V}{Q} \sum_{(n,m)} |D_{nm}|^2 \cdot \left|e^{-\frac{E_m}{\theta}} - e^{-\frac{E_n}{\theta}}\right|$$

$$\le \left(-\frac{\partial^2 f}{\partial \nu \partial \nu^*}\right)^{2/3} \left(\frac{V}{Q} \sum_{(n,m)} |D_{nm}|^2 \cdot |E_n - E_m|^2 \left(e^{-\frac{E_m}{\theta}} + e^{-\frac{E_n}{\theta}}\right)\right)^{1/3}.$$

Applying the simple transformation

$$\frac{V}{Q} \sum_{(n,m)} |D_{nm}|^2 \cdot |E_n - E_m| \left(e^{-\frac{E_m}{\theta}} + e^{-\frac{E_n}{\theta}}\right)$$

$$= \frac{V}{Q} \operatorname{Sp} e^{-\frac{\Gamma}{\theta}} \left\{(\Gamma D - D\Gamma)(D^\dagger \Gamma - \Gamma D^\dagger) + (D^\dagger \Gamma - \Gamma D^\dagger)(\Gamma D - D\Gamma)\right\}$$

$$= V \left\langle (\Gamma J - J\Gamma)(\Gamma J - J\Gamma)^\dagger + (\Gamma J - J\Gamma)^\dagger (\Gamma J - J\Gamma) \right\rangle \le 2VK^2,$$

we can prove that

$$\frac{V}{Q} \sum_{(n,m)} |D_{nm}|^2 \cdot |E_n - E_m| \cdot \left|e^{-\frac{E_m}{\theta}} - e^{-\frac{E_n}{\theta}}\right| \le \left(-\frac{\partial^2 f}{\partial \nu \partial \nu^*}\right)^{2/3} (2VK^2)^{1/3}.$$

Hence,

E_n

$$\frac{V}{Q} \sum_{(n,m)} |D_{nm}|^2 e^{-\frac{\Gamma}{\theta}} = \theta \frac{V}{Q} \sum_{(n,m)} \frac{|D_{nm}|^2}{E_n - E_m} \left(e^{-\frac{E_m}{\theta}} - e^{-\frac{E_n}{\theta}}\right) + \sum_{(n,m)} |D_{nm}|^2 \cdot \left|e^{-\frac{E_m}{\theta}} - e^{-\frac{E_n}{\theta}}\right|,$$

[2]The Hölder inequality is

$$\left|\sum ab\right| \le \left(\sum |a|^p\right)^{1/p} \left(\sum |b|^q\right)^{1/q},$$

where $p > 0$, $q > 0$, and $1/p + 1/q = 1$. Hence, $p > 1$ and $q > 1$. We choose $p = 3/2$ and $q = 3$.

where

$$\frac{V}{Q}\sum_{(n,m)}|D_{nm}|^2 e^{-\frac{\Gamma}{\theta}} = \frac{V}{Q}\,\mathrm{Sp}\,D\cdot D^\dagger e^{-\frac{\Gamma}{\theta}} = V\langle D\cdot D^\dagger\rangle$$
$$= V\langle (J-\langle J\rangle)(J^\dagger - \langle J^\dagger\rangle)\rangle.$$

As a result, we arrive at the inequality

$$\langle (J-\langle J\rangle)(J^\dagger - \langle J^\dagger\rangle)\rangle \le -\frac{\partial^2 f}{\partial\nu\partial\nu^*}\frac{\theta}{V} + \frac{(2K^2)^{1/3}}{V^{2/3}}\left(-\frac{\partial^2 f}{\partial\nu\partial\nu^*}\right)^{2/3}. \tag{1.17}$$

Thus, we can see that our problem would have been solved if we could show that the second-order derivatives were bounded in the limit as $V\to\infty$. Unfortunately, we do not have a direct proof of this assertion; that is why we have to rely solely on the boundedness of the first-order derivatives. Therefore, we should develop a method that is not based on the boundedness of the second-order derivatives and by which we would be able to prove that the difference

$$f_{\Gamma^0} - f_\Gamma$$

between free energies per unit volume is asymptotically small. Note that $f(\nu,\nu^*)$ depends only on the absolute value $r=|\nu|$ and is independent of the phase factor of the parameter ν, so that $f(\nu,\nu^*) = f(\sqrt{\nu\nu^*}) = f(r)$. Differentiating f with respect to ν and ν^*, we obtain

$$\frac{\partial f}{\partial\nu^*} = \frac{1}{2}\sqrt{\frac{\nu}{\nu^*}}f'_r(r), \qquad \frac{\partial^2 f}{\partial\nu\partial\nu^*} = \frac{1}{4r}(f'_r + f''_r r) = \frac{1}{4r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) \le 0.$$

Since $|\Gamma J - J\Gamma| \le K = \text{const}$, we can rewrite inequality (1.17) as

$$D(r) \le \frac{\theta}{4V}\left(-f''_r - \frac{1}{r}f'_r\right) + \left(-f''_r - \frac{1}{r}f'_r\right)^{2/3}\frac{K^{2/3}}{2V^{2/3}},$$

where we introduced the following notation:

$$D(r) = \langle (J-\langle J\rangle)(J^\dagger - \langle J^\dagger\rangle)\rangle.$$

Let us integrate (1.17) with respect to r and show that

$$\int_{r_0}^{r_1} rD(r)\,dr \to 0 \qquad \text{as} \quad V\to\infty.$$

In fact, we have

$$\int_{r_0}^{r_1} rD(r)\,dr \le \frac{\theta}{4V}r\frac{\partial f}{\partial r}\bigg|_{r_1}^{r_0} + \frac{K^{2/3}}{2V^{2/3}}\int_{r_0}^{r_1} r^{1/3}\left(-\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)\right)^{2/3} dr$$
$$\equiv \frac{\theta}{4V}r\frac{\partial f}{\partial r}\bigg|_{r_1}^{r_0} + \frac{K^{2/3}}{2V^{2/3}}\int_{r_0}^{r_1} u(r)v(r)\,dr,$$

where

$$u(r) = r^{1/3}, \qquad v(r) = \left(-\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)\right)^{2/3}.$$

Let us apply the Hölder inequality expressed in the form

$$\int |uv| dr \leq \left(\int |u|^3 dr\right)^{1/3} \left(\int |v|^{3/2} dr\right)^{2/3}$$

in order to transform the right-hand side of the last inequality. Noting that

$$\left|\frac{\partial f}{\partial r}\right| \leq 2k_1, \tag{1.18}$$

we obtain

$$\int_{r_0}^{r_1} rD(r)\, dr \leq \frac{\theta}{2V} k_1(r_0 + r_1) + \frac{1}{2V^{2/3}} (2k_1 K(r_0 + r_1))^{2/3} \left(\frac{r_1^2 - r_0^2}{2}\right)^{1/3}; \tag{1.19}$$

this implies that the latter integral asymptotically decreases as $V \to \infty$.

Let us return to inequality (1.12):

$$0 \leq f_{\Gamma^0} - f_\Gamma \leq 2\langle (J - \langle J \rangle)(J^\dagger - \langle J^\dagger \rangle)\rangle.$$

Denoting $a = f_{\Gamma^0} - f_\Gamma$ and using (1.19), we obtain

$$\int_{r_0}^{r_1} ra(r)\, dr \leq \frac{\theta k_1(r_0 + r_1)}{V} + \frac{(2k_1 K(r_0 + r_1))^{2/3} \left(\frac{r_1^2 - r_0^2}{2}\right)^{1/3}}{V^{2/3}}.$$

Recall that the first derivatives $\frac{\partial f}{\partial r}$ are bounded (see (1.18)) and that $|a'_r(r)| \leq 4k_1$. Let us set $r_0 = r + l$ and $r_1 = r + 2l$ and make use of the following equality:

$$a(\xi) \int_{r+l}^{r+2l} r\, dr = \int_{r+l}^{r+2l} ra(r)\, dr,$$

where $r + l \leq \xi \leq r + 2l$. By the obvious identity

$$a(r) = a(\xi) - \int_r^\xi a'_r\, dr,$$

we can show that

$$a(r) \leq \frac{\int_{r+l}^{r+2l} ra(r)\, dr}{\frac{1}{2}[(r + 2l)^2 - (r + l)^2]} + 4k_1 2l \leq 8k_1 l + \frac{2\theta k_1}{Vl} + \frac{(4k_1 K)^{2/3}}{l^{2/3} V^{2/3}}.$$

We now choose l, which is an arbitrary positive quantity, such that

$$8k_1 l = \frac{(4k_1 K)^{2/3}}{l^{2/3} V^{2/3}}; \qquad \text{then} \qquad l = \frac{K^{2/5}}{2V^{2/5} k_1^{1/5}}.$$

Substituting this expression for l into the above inequality, we finally obtain

$$0 \leq f_{\Gamma^0} - f_\Gamma \leq \frac{8(k_1^2 K)^{2/5}}{V^{2/5}} + \frac{4\theta \left(\frac{k_1^3}{K}\right)^{2/5}}{V^{3/5}} \leq \frac{L}{V^{2/5}}, \qquad L = \text{const}. \tag{1.20}$$

Here, L is a simple combination of the original constants. Since the difference $f_{\Gamma^0} - f_\Gamma$ tends to zero as $V \to \infty$, we can pass to the limit as $r = |\nu| \to 0$ in inequality (1.20). In this case,

$$0 \le f_{\Gamma^0} - f_\Gamma \le \frac{L}{V^{2/5}}, \qquad L = \text{const}.$$

It is clear that this estimate is uniform with respect to $\theta \ge 0$ and, hence, is valid for $\theta = 0$.

In conclusion, we note that, within the framework of the method of approximating Hamiltonians, it is also possible to calculate asymptotically exactly the correlation functions and the Green functions for this model. In particular, we can show that

$$|\langle A(t)B(\tau)\rangle_\Gamma - \langle A(t)B(\tau)\rangle_{\Gamma^0}| \le \eta\left(\frac{1}{V},\delta\right)|t-\tau| + \eta'\left(\frac{1}{V},\delta\right), \tag{1.21}$$

where $A, B = a_f, a_f^\dagger, a_{-f}, a_{-f}^\dagger$ and

$$\eta\left(\frac{1}{V},\delta\right) \to 0, \qquad \eta'\left(\frac{1}{V},\delta\right) \to 0 \qquad \text{as} \quad V \to \infty$$

for any fixed $\delta \ge 0$. We emphasize that these inequalities hold for $r \ge \delta$. The average $\langle A(t)B(\tau)\rangle_{\Gamma^0}$ can be readily calculated. and we see that

$$\lim_{r\to 0}\left\{\lim_{V\to\infty}\langle A(t)B(\tau)\rangle_{\Gamma^0}\right\} = \lim_{V\to\infty}\langle A(t)B(\tau)\rangle_{H^0}.$$

In [4, 5], we considered a wider class of model systems in which certain special conditions were imposed on the operators. As an example, consider a model system with negative interaction described by the Hamiltonian

$$H = T - 2V \sum_{(1\le\alpha\le s)} g_\alpha J_\alpha J_\alpha^\dagger, \tag{1.22}$$

where all the parameters g_α are positive. If we choose the operators T and J_α in the form

$$J_\alpha = \frac{1}{2V}\sum_{(f)} \lambda_\alpha a_f^\dagger a_{-f}^\dagger, \qquad T = \sum_{(f)} a_f^\dagger a_f, \tag{1.23}$$

we arrive at the conventional Hamiltonian of the BCS model [6, 7]. In fact, as we will see below, it is not necessary to define the operators T and J_α strictly in the form (1.23).

Theorem 1. *Suppose that the operators T and J_α in the Hamiltonian (1.22) satisfy the conditions*

$$\begin{gathered} |J_\alpha| \le M_1, \qquad T = T^\dagger, \\ |TJ_\alpha - J_\alpha T| \le M_2, \qquad |J_\alpha J_\beta - J_\beta J_\alpha| \le \frac{M_3}{V}, \qquad |J_\alpha^\dagger J_\beta - J_\beta J_\alpha^\dagger| \le \frac{M_3}{V}, \end{gathered} \tag{1.24}$$

M_1, M_2, and M_3 are constants in the limit as $V \to \infty$, where $1 \le \alpha \le s$ and $1 \le \beta \le s$. Let, in addition, the free energy per unit volume calculated for the Hamiltonian T be bounded by a certain constant:

$$|f(T)| \le M_0. \tag{1.25}$$

Then, if we construct a trial Hamiltonian in the form

$$H(C) = T - 2V\sum_{(\alpha)} g_\alpha (C_\alpha J_\alpha^\dagger + C^* J_\alpha - C_\alpha C_\alpha^*), \tag{1.26}$$

where $C_1, \dots, C_s$ are complex numbers, the following inequalities hold:

$$0 \le \min_{(C)} f(H_0(C)) - f(H) \le \mathcal{E}\left(\frac{1}{V}\right), \tag{1.27}$$

and $\mathcal{E}(\frac{1}{V}) \to 0$ as $V \to \infty$ uniformly with respect to θ on any interval $0 < \theta \le \theta_0$, where θ_0 is an arbitrary fixed temperature.[3]

This theorem has found numerous applications. For example, using this theorem, Hertel and Thirring calculated the free energy in the thermodynamic limit for the model describing a system of attracting fermions [8].

We also should note that the existence of the limit of the free energy calculated for the Hamiltonian (1.22),

$$\lim_{V\to\infty} f(H), \tag{1.28}$$

does not follow from the aforementioned inequalities.

Now, consider the case when the operators T and J_α in (1.22) are given by (1.23). Then, the hypotheses of the theorem are fulfilled if

$$\begin{gathered} \frac{1}{V}\sum_{(p)} |T(p)\lambda_\alpha(p,\sigma)| \le Q_0 = \text{const}, \\ |\lambda_\alpha(p,\sigma)| \le \bar{Q} = \text{const}, \qquad \alpha = 1, 2, \dots, s, \quad \sigma = \pm 1/2. \end{gathered} \tag{1.29}$$

Next, we formulate a theorem that allows us to analyze in greater detail the properties of the free energies corresponding to the Hamiltonians (1.22) and (1.26) and prove the existence of the limit (1.28).

Theorem 2. *Let the operators T and J_α in the Hamiltonian* (1.22) *be given by* (1.23) *and the functions $T(f)$ and $\lambda(p,\sigma)$ satisfy conditions* (1.29). *Suppose that the functions $\lambda(p,\sigma)$ are continuous in the space E except, possibly, a set of zero measure. Then,*

$$|f_V\{H(C)\} - f_\infty\{H(C)\}| \le \delta_V$$

for $|C_\alpha| \le 2M_1$, $\alpha = 1, 2, \dots, s$, and this inequality is uniform with respect to θ on any interval of the form $0 < \theta < \theta_0$. The function $f_\infty\{H(C)\}$ is defined as usually and has continuous partial

[3] We denote the free energy per unit volume for an arbitrary Hamiltonian H by $f(H)$ or, if we want to stress the fact that it depends on the volume V, by $f_V(H)$. By $\min_{(C)} f(C)$, we always mean the absolute minimum of the function $f(C)$ in the space of complex parameters C.

Inequalities (1.24) also imply that

$$\frac{1}{V}\sum_{(p)} |\lambda_\alpha(p,\sigma)| \le Q_1, \qquad \frac{1}{V}\sum_{(p)} |\lambda_\alpha(p,\sigma)|^2 \le Q_2,$$

where Q_1 and Q_2 are certain constants. We can correlate the choice of these constants with the corresponding constants in inequality (1.24):

$$M_1 = Q_1, \qquad M_2 = 2Q_0, \qquad M_3 = Q_2.$$

Obviously, conditions (1.24) are fulfilled if

$$|\lambda_\alpha(p,\sigma)| \le \frac{A}{(p^2+B)^3},$$

where A and B are certain positive constants.

derivatives of arbitrary order with respect to the complex variables $C_1, \dots, C_s, C_1^*, \dots, C_s^*$. Moreover, it can be shown the following.

1. *These functions attain the absolute minimum in the space of complex numbers (C) at certain points $C = \bar{C}$; i.e.,*

$$\min_{(C)} f_\infty\{H(C)\} = f_\infty\{H(\bar{C})\}.$$

2. *The inequality*

$$|f_V(H) - f_\infty\{H(\bar{C})\}| \le \bar{\delta}_V, \tag{1.30}$$

where

$$\bar{\delta} = \mathcal{E}\left(\frac{1}{V}\right) + \delta_V \to 0,$$

holds uniformly with respect to θ on any interval $0 < \theta \le \theta_0$.

This theorem was proved for the first time in [4].

For a specific choice of operators in the form (1.23), the approximating Hamiltonian mentioned in Theorem 2 is expressed as

$$H_0(C) = \sum_{(f)} T(f) a_f^\dagger a_f - \frac{1}{2} \sum_{(f)} \left\{ \Lambda^*(f) a_{-f} a_f + \Lambda(f) a_f^\dagger a_{-f}^\dagger \right\} + 2V \sum_{(\alpha)} g_\alpha C_\alpha C_\alpha^*, \tag{1.31}$$

where $\Lambda^*(f) = 2\sum_{(\alpha)} C_\alpha \lambda_\alpha^*(f)$. Introducing new Fermi operators α_f and $\alpha_f^\dagger$ such that

$$a_f = u_f \alpha_f - v_f \alpha_{-f}^\dagger,$$

$$u_f = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{T_f}{E_f}}, \qquad v_f = -\frac{\Lambda(f)}{\sqrt{2}|\Lambda(f)|} \sqrt{1 - \frac{T_f}{E_f}}, \qquad E_f = \sqrt{T^2(f) + |\Lambda(f)|^2},$$

we rewrite (1.31) as

$$H_0(C) = \sum_{(f)} E_f \alpha_f^\dagger \alpha_f + V \left\{ 2 \sum_{(\alpha)} g_\alpha C_\alpha^* C_\alpha - \frac{1}{2V} \sum_{(f)} (E_f - T_f) \right\}.$$

The free energy per unit volume corresponding to this Hamiltonian can be represented as

$$f_V = 2 \sum_{(\alpha)} g_\alpha C_\alpha C_\alpha^* - \frac{1}{2V} \sum_{(f)} (E(f) - T(f)) + \frac{\theta}{V} \sum_{(f)} \ln\left(1 + e^{-\frac{E(f)}{\theta}}\right). \tag{1.32}$$

As follows from Theorem 2, f_V is approximated by the limit free energy[4]

$$f_\infty\{H_0(C)\} = 2 \sum_{(\alpha)} g_\alpha C_\alpha^* C_\alpha - \frac{1}{2(2\pi)^3} \int \left\{ E(f) - T(f) - 2\theta \ln\left(1 + e^{-\frac{E(f)}{\theta}}\right) \right\} d^3 f \tag{1.33}$$

in the limit as $V \to \infty$. Here, the integration $\int \dots df$ implies the operation $\sum_\sigma \int \dots d\bar{p}$.

[4] An alternative approach in which, in order to avoid the passage to the limit as $V \to \infty$, the volume V is assumed to be infinite from the very beginning was developed in [9].

2. A GENERAL MODEL

Now, consider a general model of four-fermion interaction [10],

$$H = \sum_{(f,f')} \Omega(f',f) a_f^\dagger a_{f'} + \frac{1}{2} \sum_{(f_1,f_2,f_2',f_1')} U(f_1,f_2;f_2',f_1') a_{f_1}^\dagger a_{f_2}^\dagger a_{f_2'} a_{f_1'}$$
$$+ \frac{1}{2} \sum_{(f,f')} j_-(f',f,t) a_f^\dagger a_{f'}^\dagger + \frac{1}{2} \sum_{f,f'} j_+(f',f,t) a_f a_{f'}, \tag{2.1}$$

where $U(f_1, f_2; f_2', f_1')$ are symmetric functions with respect to the permutation of arguments

$$(1 \leftrightarrow 2): \quad \{f_1 \leftrightarrow f_2,\ f_1' \leftrightarrow f_2'\}$$

and $\Omega(f', f) = \Omega_0(f', f) + j(f', f, t)$. This model includes, as a particular case, the model considered above. In (2.1), we introduced the auxiliary sources

$$\frac{1}{2} \sum_{(f,f')} j_-(f',f,t) a_f^\dagger a_{f'}^\dagger, \qquad \frac{1}{2} \sum_{(f,f')} j_+(f',f,t) a_f a_{f'}, \quad \text{and} \quad \sum_{(f,f')} j(f',f,t) a_f^\dagger a_{f'},$$

which are chosen so that the conservation law of the total momentum is fulfilled and, at the same time, the conservation law of the number of particles is violated.

For model (2.1), we introduce a certain approximating Hamiltonian, which is constructed by analogy with the approximating Hamiltonian in the simplified model considered above. This construction is based on the following approximation:

$$a_{f_1}^\dagger a_{f_2}^\dagger a_{f_2'} a_{f_1'} \to \langle a_{f_1}^\dagger a_{f_1'} \rangle a_{f_2}^\dagger a_{f_2'} - \langle a_{f_1}^\dagger a_{f_2'} \rangle a_{f_2}^\dagger a_{f_1'} + \langle a_{f_2}^\dagger a_{f_2'} \rangle a_{f_1}^\dagger a_{f_1'}$$
$$- \langle a_{f_2}^\dagger a_{f_1'} \rangle a_{f_1}^\dagger a_{f_2'} + \langle a_{f_1}^\dagger a_{f_2}^\dagger \rangle a_{f_2'} a_{f_1'} + \langle a_{f_2'} a_{f_1'} \rangle a_{f_1}^\dagger a_{f_2}^\dagger. \tag{2.2}$$

Define the function

$$W(f_1, f_2; f_2', f_1') = U(f_1, f_2; f_2', f_1') - U(f_1, f_2; f_1', f_2'),$$

which is antisymmetric in the sense that

$$W(f_1, f_2; f_2', f_1') = -W(f_2, f_1; f_2', f_1'), \qquad W(f_1, f_2; f_2', f_1') = -W(f_1, f_2; f_1', f_2').$$

Then, the approximating Hamiltonian for model (2.1) is expressed as

$$H_{\text{app}} = \sum_{(f,f')} K(f',f) a_f^\dagger a_{f'} + \frac{1}{2} \sum_{(f,f')} K_-(f',f) a_f^\dagger a_{f'}^\dagger + \frac{1}{2} \sum_{(f,f')} K_+(f',f) a_f a_{f'}, \tag{2.3}$$
$$K(f',f) = j(f',f,t) + \Omega_0(f',f) + \sum_{(f_1,f_2)} W(f_1,f;f_1',f_2) \langle a_{f_1}^\dagger a_{f_2} \rangle,$$
$$K_+(f',f) = j_+(f',f,t) + \frac{1}{2} \sum_{(f_1,f_2)} W(f_1,f_2;f,f') \langle a_{f_1}^\dagger a_{f_2}^\dagger \rangle$$
$$= j_+(f',f,t) + \frac{1}{2} \sum_{(f_1,f_2)} U(f_1,f_2;f,f') \langle a_{f_1}^\dagger a_{f_2}^\dagger \rangle$$

$$E\langle\langle a_f a_g; A_\beta\rangle\rangle_E$$

$$\begin{aligned}
= \sum_{(f_1,f_2,f')} \Big\{ & W(f_1,f;f',f_2)\langle a^\dagger_{f_1} a_{f_2}\rangle_0 \langle\langle a_{f'} a_g; A_\beta\rangle\rangle_E + W(f_1,f;f',f_2)\langle a_{f'} a_g\rangle_0 \langle\langle a^\dagger_{f_1} a_{f_2}; A_\beta\rangle\rangle_E \\
& + U(f,f';f_1,f_2)\langle a_{f_1} a_{f_2}\rangle_0 \langle\langle a^\dagger_{f'} a_g; A_\beta\rangle\rangle_E + U(f,f';f_1,f_2)\langle a^\dagger_{f_1} a_g\rangle_0 \langle\langle a_{f_1} a_{f_2}; A_\beta\rangle\rangle_E \\
& + W(f_1,g;f',f_2)\langle a^\dagger_{f_1} a_{f_2}\rangle_0 \langle\langle a_f a_{f'}; A_\beta\rangle\rangle_E + W(f_1,g;f',f_2)\langle a_f a_{f'}\rangle_0 \langle\langle a^\dagger_{f_1} a_{f_2}; A_\beta\rangle\rangle_E \\
& + U(g,f';f_1,f_2)(\delta_{f,f'} - \langle a^\dagger_f a_{f'}\rangle_0)\langle\langle a_{f_1} a_{f_2}; A_\beta\rangle\rangle_E - U(g,f';f_1,f_2)\langle a_{f_1} a_{f_2}\rangle_0 \langle\langle a^\dagger_f a_{f'}; A_\beta\rangle\rangle_E \Big\} \\
& + I_{3,\beta} + \sum_{(f')} \Omega_0(f',f)\langle\langle a_{f'} a_g; A_\beta\rangle\rangle_E + \sum_{(f)} \Omega_0(f',g)\langle\langle a_f a_{f'}; A_\beta\rangle\rangle_E,
\end{aligned} \tag{2.10b}$$

$$E\langle\langle a^\dagger_f a^\dagger_g; A_\beta\rangle\rangle_E = -\sum_{(f')} \Big\{ \Omega_0(f,f')\langle\langle a^\dagger_{f'} a^\dagger_g; A_\beta\rangle\rangle_E + \Omega_0(g,f')\langle\langle a^\dagger_f a^\dagger_{f'}; A_\beta\rangle\rangle_E \Big\}$$

$$\begin{aligned}
- \sum_{(f_1,f_2,f')} \Big\{ & W(f_1,f';f,f_2)\langle a^\dagger_{f_1} a_{f_2}\rangle_0 \langle\langle a^\dagger_{f'} a^\dagger_g; A_\beta\rangle\rangle_E + W(f_1,f';f,f_2)\langle a^\dagger_{f'} a^\dagger_g\rangle_0 \langle\langle a^\dagger_{f_1} a_{f_2}; A_\beta\rangle\rangle_E \\
& + U(f_1,f_2;f',f)[\delta_{g,f'} - \langle a^\dagger_g a_{f'}\rangle_0]\langle\langle a^\dagger_{f_1} a^\dagger_{f_2}; A_\beta\rangle\rangle_E - U(f_1,f_2;f',f_2)\langle a^\dagger_{f_1} a^\dagger_{f_2}\rangle_0 \langle\langle a^\dagger_g a_{f'}; A_\beta\rangle\rangle_E \\
& + W(f_1,f';g,f_2)\langle a^\dagger_f a^\dagger_{f'}\rangle_0 \langle\langle a^\dagger_{f_1} a_{f_2}; A_\beta\rangle\rangle_E + W(f_1,f';g,f_2)\langle a^\dagger_{f_1} a_{f_2}\rangle_0 \langle\langle a^\dagger_f a^\dagger_{f'}; A_\beta\rangle\rangle_E \\
& + U(f_1,f_2;f',g)\langle a^\dagger_{f_1} a^\dagger_{f_2}\rangle_0 \langle\langle a^\dagger_f a_{f'}; A_\beta\rangle\rangle_E + U(f_1,f_2;f',g)\langle a^\dagger_f a_{f'}\rangle_0 \langle\langle a^\dagger_{f_1} a^\dagger_{f_2}; A_\beta\rangle\rangle_E \Big\} + I_{2,\beta}.
\end{aligned} \tag{2.10c}$$

Here, $A_\beta = A_\beta(g_2, g_1)$ according to definitions (2.6); hence,

$$\begin{aligned}
I_{1,1} &= -i\langle a^\dagger_{g_2} a_g\rangle_0 \frac{\delta(f-g_1)}{2\pi} + i\langle a^\dagger_f a_{g_1}\rangle_0 \frac{\delta(g-g_2)}{2\pi}, \\
I_{3,1} &= i\langle a_{g_1} a_g\rangle_0 \frac{\delta(f-g_2)}{2\pi} + i\langle a^\dagger_f a_{g_1}\rangle_0 \frac{\delta(g-g_2)}{2\pi}, \\
I_{2,1} &= i\langle a^\dagger_{g_2} a^\dagger_g\rangle_0 \frac{\delta(f-g_1)}{2\pi} - i\langle a^\dagger_f a_{g_2}\rangle_0 \frac{\delta(g-g_1)}{2\pi}, \\
I_{1,2} &= -i\langle a_{g_2} a_g\rangle \frac{\delta(f-g_1)}{2\pi}, \qquad I_{2,2} = 0, \\
I_{3,2} &= -\frac{i}{2\pi}[\delta(g-g_2) - \langle a^\dagger_g a_{g_2}\rangle_0]\delta(f-g_1) - \frac{i}{2\pi}\langle a^\dagger_f a_{g_2}\rangle_0 \delta(g-g_1), \\
I_{1,3} &= \frac{i}{2\pi}\langle a^\dagger_f a^\dagger_{g_1}\rangle_0 \delta(g-g_2), \\
I_{2,3} &= \frac{i}{2\pi}\langle a^\dagger_{g_1} a_g\rangle_0 \delta(f-g_2) + \frac{i}{2\pi}[\delta(f-g_1) - \langle a^\dagger_{g_1} a_f\rangle_0]\delta(g-g_2), \qquad I_{3,3} = 0.
\end{aligned}$$

The Hartree–Fock–Bogolyubov equations without sources do not allow one to calculate correctly the so-called zero or anomalous averages

$$\langle a^\dagger_f a_g\rangle_0, \qquad \langle a^\dagger_f a^\dagger_g\rangle_0, \qquad \langle a_f a_g\rangle_0.$$

Hence, it is necessary to give a different interpretation to the very procedure of calculation of such averages. For example, we can use the approximating Hamiltonian in the form (2.3) with $\eta = 0$:

$$H^0_{\rm app} = \sum_{(f,f')} K^{(0)}(f,f') a^\dagger_f a_{f'} a^\dagger_f a_{f'} + \frac{1}{2}\sum_{(f,f')} K^{(0)}_-(f,f') a^\dagger_f a^\dagger_{f'} + \frac{1}{2}\sum_{(f,f')} K^{(0)}_+(f',f) a_f a_{f'} + \text{const},$$

where

$$K^{(0)}(f',f) = \Omega_0(f',f) + \sum_{(f,f_2)} W(f_1,f;f',f_2)\langle a^{\dagger}_{f_1} a_{f_2}\rangle_0,$$

$$K^{(0)}_{+}(f',f) = \sum_{(f,f_2)} U(f_1,f_2;f,f')\langle a^{\dagger}_{f_1} a^{\dagger}_{f_2}\rangle_0,$$

$$K^{(0)}_{-}(f',f) = \sum_{(f,f_2)} U(f,f';f_1,f_2)\langle a_{f_1} a_{f_2}\rangle_0.$$

As a result, the approximate self-consistency equations for calculating the anomalous averages are rewritten as

$$\frac{\mathrm{Sp}\left\{a^{\dagger}_f a_g e^{-\beta H^0_{\mathrm{app}}}\right\}}{\mathrm{Sp}\, e^{-\beta H^0_{\mathrm{app}}}} = \langle a^{\dagger}_f a_g\rangle_0,$$

$$\frac{\mathrm{Sp}\left\{a^{\dagger}_f a^{\dagger}_g e^{-\beta H^0_{\mathrm{app}}}\right\}}{\mathrm{Sp}\, e^{-\beta H^0_{\mathrm{app}}}} = \langle a^{\dagger}_f a^{\dagger}_g\rangle_0,$$

$$\frac{\mathrm{Sp}\left\{a_f a_g e^{-\beta H^0_{\mathrm{app}}}\right\}}{\mathrm{Sp}\, e^{-\beta H^0_{\mathrm{app}}}} = \langle a_f a_g\rangle_0.$$

However, one can also apply an alternative approach based on the Green functions technique. First, consider the case $\eta \neq 0$ and write out the dynamic equations for the following retarded and advanced Green functions constructed on the basis of the Fermi operators:[5]

$$G_1(f,t;f',\tau) = \langle\langle a^{\dagger}_f(t) a_{f'}(\tau)\rangle\rangle = \theta(t-\tau)\langle a^{\dagger}_f(t)a_{f'}(\tau) + a_{f'}(\tau)a^{\dagger}_f(t)\rangle,$$

$$G_2(f,t;f',\tau) = \langle\langle a_f(t) a_{f'}(\tau)\rangle\rangle = \theta(t-\tau)\langle a^{\dagger}_f(t)a_{f'}(\tau) + a_{f'}(\tau)a_f(t)\rangle.$$

We obtain the following system of equations for these Green functions:

$$i\frac{\partial}{\partial t}G_{(1)} = -\left(\sum_{(g)} G_{(1)}K(f,g) + \sum_{(g)} G_{(2)}K(f,g)\right) + \delta(t-\tau)\delta_{ff'},$$

$$i\frac{\partial}{\partial t}G_{(2)} = -\sum_{(g)} G_{(2)}K(g,f) + \sum_{(g)} G_{(1)}K_{-}(g,f).$$

Here, the function K depends on t in the general case, and, therefore, it is relevant to denote it by $K(g,f,t)$. Now, let us analyze the case when the sources are equal to zero, i.e., $\eta = 0$. Then. $G_{(\alpha)}$ depends only on the difference of the variables $t-\tau$, which is a consequence of the time uniformity:

$$G_{(\alpha)}(f,t;f',\tau) = G_{(\alpha)}(f,f',t-\tau).$$

[5]It is equally admissible to consider the causal Green functions

$$G(f,t;f',\tau) = \langle T\{a^{\dagger}_f(t)a_{f'}(\tau)\}\rangle = \theta(t-\tau)\langle a^{\dagger}_f(t)a_{f'}(\tau)\rangle - \theta(\tau-t)\langle a_{f'}(\tau)a^{\dagger}_f(t)\rangle$$

instead of the advanced and retarded Green functions.

Hence, it is convenient to employ the energy E-representation of the Green functions considered,

$$\langle\langle a_f^\dagger a_{f'}\rangle\rangle_E = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} G_{(1)}(f,f',t)e^{iEt}\,dt,$$

$$\langle\langle a_f a_{f'}\rangle\rangle_E = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} G_{(2)}(f,f',t)e^{iEt}\,dt,$$

and appropriately transform the dynamic equations,

$$E\langle\langle a_f^\dagger a_{f'}\rangle\rangle_E + \sum_{(g)}\left\{K^{(0)}(f,g)\langle\langle a_g^\dagger a_{f'}\rangle\rangle_E + K_+^{(0)}(f,g)\langle\langle a_g a_{f'}\rangle\rangle_E\right\} = \frac{i}{2\pi}\delta(f-f'),$$

$$E\langle\langle a_f a_{f'}\rangle\rangle_E + \sum_{(g)}\left\{K^{(0)}(f,g)\langle\langle a_g a_{f'}\rangle\rangle_E + K_-^{(0)}(f,g)\langle\langle a_g^\dagger a_{f'}\rangle\rangle_E\right\} = \frac{i}{2\pi}\delta(f-f').$$

Consider, as an example, the BCS Hamiltonian. We use the following spectral E-representation for the correlation functions and the Green functions, consisting of pairs of Fermi operators a_f, $a_{f'}^\dagger$, and $a_g^\dagger$:

$$\langle A(t)B(\tau)\rangle = \int\limits_{-\infty}^{+\infty} J_{AB}(\omega)e^{\frac{\omega}{\theta}}e^{-i\omega(t-\tau)}d\omega,$$

$$\langle B(\tau)A(t)\rangle = \int\limits_{-\infty}^{+\infty} J_{AB}(\omega)e^{-i\omega(t-\tau)}d\omega,$$

$$\langle AB\rangle = \int\limits_{-\infty}^{+\infty} J_{AB}(\omega)e^{\frac{\omega}{\theta}}d\omega,$$

$$\langle\langle A_\alpha B_\beta\rangle\rangle = \frac{i}{2\pi}\int\limits_{-\infty}^{+\infty} J_{AB}(\omega)\frac{e^{\beta\omega}+1}{E-\omega}d\omega.$$

The BCS Hamiltonian is given by

$$H = \sum_{(f)} T(p)a_f^\dagger a_f - \frac{1}{2V}\sum_{(f,f')} J(f)J(f')a_f^\dagger a_{-f}^\dagger a_{-f'}a_{f'},$$

where

$$f=(p,\sigma),\quad \sigma=\pm\frac{1}{2},\quad p=\left(\frac{2\pi n^{(1)}}{L},\frac{2\pi n^{(2)}}{L},\frac{2\pi n^{(3)}}{L}\right),\quad T(p)=\frac{p^2}{2m}-\lambda,\quad \lambda>0,$$

$$J(f)=\varepsilon(\sigma_1-\sigma_2)J(p),\qquad \varepsilon(\sigma)=\begin{cases}+1, & \sigma>0,\\ -1, & \sigma<0.\end{cases}$$

It is assumed that $J(p)$ is a symmetric function, so that $J(-f)=-J(f)$.

The conservation laws of momentum and spin projection lead to the following rules of selection for the averages:

$$\langle a_f^\dagger a_{f'}\rangle = \delta(f-f')\langle a_f^\dagger a_f\rangle,$$

$$\langle a_f^\dagger a_{f'}^\dagger\rangle = \delta(f+f')\langle a_f^\dagger a_{-f}^\dagger\rangle,$$

$$\langle a_f a_{f'}\rangle = \delta(f+f')\langle a_{-f}a_f\rangle$$

and similar rules of selection for the Green functions $\langle\langle a_f^\dagger a_{f'}\rangle\rangle_E$ and $\langle\langle a_f a_{f'}\rangle\rangle_E$.

In the case of the BCS model, we have, in particular,

$$U(f_1, f_2; f_2', f_1') = -\frac{1}{V} J(f_1) J(f_1') \delta(f_1 + f_2) \delta(f_1' + f_2'),$$
$$W(f_1, f_2; f_2', f_1') = -\frac{2}{V} J(f_1) J(f_1') \delta(f_1 + f_2) \delta(f_1' + f_2')$$

and

$$K^{(0)}(f, f') = T(p)\delta(f - f') + \sum_{(f_1)} W(f_1, f; f', f_1) \langle a^\dagger_{f_1} a_{f_1} \rangle_0$$
$$= T(p)\delta(f - f') + \delta(f - f') \left(-\frac{2}{V} |J(f)|^2 \langle a^\dagger_{-f} a_{-f} \rangle_0 \right) = \delta(f - f') T(p),$$
$$K_+^{(0)}(f', f) = \delta(f + f') \left(\frac{1}{V} \sum_{(f_1)} J(f_1) \langle a^\dagger_{f_1} a^\dagger_{-f_1} \rangle_0 \right) J(f) = -\delta(f + f') C^* J(f'),$$
$$K_-^{(0)}(f', f) = \sum_{(f_1)} U(f, f'; -f_1, f_1) \langle a_{-f_1} a_{f_1} \rangle$$
$$= \delta(f + f') \left(-\frac{1}{V} \sum_{(f_1)} J(f_1) \langle a_{-f_1} a_{f_1} \rangle \right) J(f) = -\delta(f + f') C J(f).$$

The corresponding system of equations for the BCS model has the form

$$\{E + T(p)\} \langle\langle a^\dagger_f a_f \rangle\rangle_E - C^* J(f) \langle\langle a_{-f} a_f \rangle\rangle_E = \frac{i}{2\pi},$$
$$-C J(f) \langle\langle a^\dagger_f a_f \rangle\rangle_E + \{E - T(p)\} \langle\langle a_{-f} a_f \rangle\rangle_E = 0;$$

this implies that

$$\langle\langle a^\dagger_f a_f \rangle\rangle_E = \frac{i}{2\pi} \frac{E - T(p)}{E^2 - T^2(p) - |C|^2 J^2},$$
$$\langle\langle a_{-f} a_f \rangle\rangle_E = \frac{i}{2\pi} \frac{C J(f)}{E^2 - T^2(p) - |C|^2 J^2}.$$

Let

$$E(p) = \sqrt{T^2(p) + |C|^2 |J|^2}.$$

Hence,

$$\frac{1}{E^2 - E^2(p)} = \frac{1}{2E(p)} \left\{ \frac{1}{E - E(p)} - \frac{1}{E + E(p)} \right\},$$
$$\langle\langle a^\dagger_f a_f \rangle\rangle_E = \frac{i}{2\pi} \frac{E - T(p)}{2E(p)} \left\{ \frac{1}{E - E(p)} - \frac{1}{E + E(p)} \right\},$$
$$\langle\langle a_{-f} a_f \rangle\rangle_E = \frac{i}{2\pi} \frac{C J(f)}{2E(p)} \left\{ \frac{1}{E - E(p)} - \frac{1}{E + E(p)} \right\}.$$

After obvious transformations, we obtain the expressions for the spectral densities,

$$J_{a^\dagger a} = \frac{\omega - T(p)}{2E(p)} \frac{1}{1 + e^{\beta\omega}} \{\delta(\omega - E(p)) - \delta(\omega + E(p))\},$$
$$J_{a_{-f} a_f} = \frac{C J(f)}{2E(p)(1 + e^{\beta\omega})} \{\delta(\omega - E(p)) - \delta(\omega + E(p))\},$$

since $\langle a^\dagger_{f_1} a^\dagger_{f_2}\rangle = -\langle a^\dagger_{f_2} a^\dagger_{f_1}\rangle$ and

$$\begin{aligned} K_-(f',f) &= j_-(f',f,t) + \frac{1}{2}\sum_{(f_1,f_2)} W(f,f';f_1,f_2)\langle a_{f_1}a_{f_2}\rangle \\ &= j_-(f',f,t) + \frac{1}{2}\sum_{(f_1,f_2)} U(f,f';f_1,f_2)\langle a_{f_1}a_{f_2}\rangle. \end{aligned}$$

The equations of motion for Hamiltonian (2.3) can be represented as

$$\begin{aligned} i\frac{da^\dagger_f}{dt} &= -\sum_{(f')} K(f,f')a^\dagger_{f'} + K_+(f,f')a_{f'}, \\ i\frac{da_f}{dt} &= \sum_{(f')} K(f',f)a_{f'} + K_-(f',f)a^\dagger_{f'}. \end{aligned} \tag{2.4}$$

Equations (2.4) allow us to write out the appropriate equations for the correlation functions $\langle a^\dagger_f a_g\rangle$, $\langle a_f a_g\rangle$, and $\langle a^\dagger_f a^\dagger_g\rangle$:

$$\begin{aligned} i\frac{d\langle a^\dagger_f a_g\rangle}{dt} &= -\sum_{(f')}\left\{K(f,f')\langle a^\dagger_{f'}a_g\rangle + K_+(f,f')\langle a_{f'}a_g\rangle\right\} \\ &\quad + \sum_{(f')}\left\{K(f',g)\langle a^\dagger_f a_{f'}\rangle + K_-(f',g)\langle a^\dagger_f a^\dagger_{f'}\rangle\right\}, \\ i\frac{d\langle a_f a_g\rangle}{dt} &= \sum_{(f')}\left\{K(f',f)\langle a_{f'}a_g\rangle + K_-(f',f)\langle a^\dagger_{f'}a_g\rangle\right\} \\ &\quad + \sum_{(f')}\left\{K(f',g)\langle a_f a_{f'}\rangle + K_-(f',g)[\delta_{f,f'} - \langle a^\dagger_{f'}a_f\rangle]\right\}, \\ i\frac{d\langle a^\dagger_f a^\dagger_g\rangle}{dt} &= -\sum_{(f')}\left\{K(f,f')\langle a_{f'}a^\dagger_g\rangle + K_+(f,f')[\delta_{g,f'} - \langle a^\dagger_g a_{f'}\rangle]\right\} \\ &\quad - \sum_{(f')}\left\{K(g,f')\langle a^\dagger_f a^\dagger_{f'}\rangle + K_+(g,f')\langle a^\dagger_f a_{f'}\rangle]\right\}. \end{aligned} \tag{2.5}$$

Henceforth, we will call equations (2.5) the Hartree–Fock–Bogolyubov system of equations. Denote

$$\begin{aligned} a^\dagger_f(t)a_g(t) &= A_1(f,g,t), & j(f,g,t) &= \eta_1(f,g,t), \\ a^\dagger_f(t)a^\dagger_g(t) &= A_2(f,g,t), & j_+(f,g,t) &= \eta_2(f,g,t), \\ a_f(t)a_g(t) &= A_3(f,g,t), & j_-(f,g,t) &= \eta_3(f,g,t) \end{aligned} \tag{2.6}$$

and introduce the Green functions in the form

$$\left\{\frac{\delta\langle A(f,g,t)\rangle}{\delta\eta_\beta(g_1,g_2,\tau)}\right\}_{\eta=0} = \langle\langle A_\alpha(f,g,t)A_\beta(g_2,g_1,\tau)\rangle\rangle. \tag{2.7}$$

The retarded and advanced Green functions are defined conventionally [11, 12]:

$$
\begin{aligned}
\langle\langle A_\alpha(t)A_\beta(\tau)\rangle\rangle^{\mathrm{ret}} &= \theta(t-\tau)\langle A_\alpha(t)A_\beta(\tau)+A_\beta(\tau)A_\alpha(t)\rangle, \\
\langle\langle A_\alpha(t)A_\beta(\tau)\rangle\rangle^{\mathrm{adv}} &= -\theta(\tau-t)\langle A_\alpha(t)A_\beta(\tau)+A_\beta(\tau)A_\alpha(t)\rangle,
\end{aligned}
\tag{2.8}
$$

$$
\langle\langle A_\alpha(t)A_\beta(\tau)\rangle\rangle = \int\limits_{-\infty}^{+\infty} \langle\langle A_\alpha A_\beta\rangle\rangle_E e^{-iE(t-\tau)}dE,
$$

and the spectral representation of the two-time correlation functions is given by

$$
\begin{aligned}
\langle A_\beta(\tau)A_\alpha(t)\rangle &= \int\limits_{-\infty}^{+\infty} J_{\alpha\beta}(\omega)e^{-i\omega(t-\tau)}d\omega, \\
\langle A_\alpha(t)A_\beta(\tau)\rangle &= \int\limits_{-\infty}^{+\infty} J_{\alpha\beta}(\omega)e^{\beta\omega}e^{-i\omega(t-\tau)}d\omega.
\end{aligned}
\tag{2.9}
$$

If we define a function

$$
\langle\langle A_\alpha A_\beta\rangle\rangle_E = \frac{i}{2\pi}\int\limits_{-\infty}^{+\infty} J_{\alpha\beta}\frac{e^{\beta\omega}+1}{E-\omega}\,d\omega
$$

on the complex plane E, then

$$
\begin{aligned}
\langle\langle A_\alpha A_\beta\rangle\rangle_E^{\mathrm{ret}} &= \langle\langle A_\alpha A_\beta\rangle\rangle_{E+i0}, \\
\langle\langle A_\alpha A_\beta\rangle\rangle_E^{\mathrm{adv}} &= \langle\langle A_\alpha A_\beta\rangle\rangle_{E-i0}.
\end{aligned}
$$

The variation of the Hartree–Fock–Bogolyubov equation with respect to the sources $\eta_\beta(t)$ (see (2.6)) followed by the setting of all the sources to zero yield the following system of equations for the Green functions:

$$
\begin{aligned}
E\langle\langle a_f^\dagger a_g; A_\beta\rangle\rangle_E &= -\sum_{(f')}\Omega_0(f,f')\langle\langle a_{f'}^\dagger a_g; A_\beta\rangle\rangle_E \\
&- \sum_{(f_1,f_2,f')}\Big\{W(f_1,f';f,f_2)\langle a_{f_1}^\dagger a_{f_2}\rangle_0\langle\langle a_{f'}^\dagger a_g; A_\beta\rangle\rangle_E + W(f_1,f';f,f_2)\langle a_{f'}^\dagger a_g\rangle_0\langle\langle a_{f_1}^\dagger a_{f_2}^\dagger; A_\beta\rangle\rangle_E \\
&+ U(f_1,f_2;f',f)\langle a_{f_1}^\dagger a_{f_2}^\dagger\rangle_0\langle\langle a_{f'} a_g; A_\beta\rangle\rangle_E + U(f_1,f_2;f',f)\langle a_{f_1} a_g\rangle_0\langle\langle a_{f_1}^\dagger a_{f-2}^\dagger; A_\beta\rangle\rangle_E\Big\} \\
&+ \sum_{(f')}\langle\langle a_f^\dagger a_{f'}; A_\beta\rangle\rangle\Omega_0(f',g) \\
&+ \sum_{(f_1,f_2,f')}\Big\{W(f,g;f',f_2)\langle a_{f_1}^\dagger a_{f_2}\rangle_0\langle\langle a_f^\dagger a_{f'}; A_\beta\rangle\rangle_E + W(f_1,g;f',f_2)\langle a_f^\dagger a_{f'}\rangle_0\langle\langle a_{f_1}^\dagger a_{f_2}; A_\beta\rangle\rangle_E \\
&+ U(g,f';f_1,f_2)\langle a_{f_1} a_{f_2}\rangle_0\langle\langle a_f^\dagger a_{f'}^\dagger; A_\beta\rangle\rangle_E + U(g,f';f_1,f_2)\langle a_f^\dagger a_{f'}^\dagger\rangle_0\langle\langle a_{f_1} a_{f_2}; A_\beta\rangle\rangle_E\Big\} + I_{1,\beta},
\end{aligned}
\tag{2.10a}
$$

and the appropriate expressions for the averages.

$$\langle a_f^\dagger a_f\rangle_0 = \frac{E(p)-T(p)}{2E(p)}\frac{e^{\beta E(p)}}{1+e^{\beta E(p)}} + \frac{E(p)+T(p)}{2E(p)}\frac{e^{-\beta E(p)}}{1+e^{-\beta E(p)}},$$

$$\langle a_{-f}a_f\rangle = \frac{CJ(f)}{2E(p)}\left\{\frac{e^{\beta E(p)}}{1+e^{\beta E(p)}} - \frac{e^{-\beta E(p)}}{1+e^{-\beta E(p)}}\right\} = \frac{CJ(f)}{2E(p)}\tanh\frac{\beta E(p)}{2},$$

$$\tanh x = \frac{1-e^{-2x}}{1+e^{-2x}}.$$

In all the expressions above, we introduced the notation

$$C = \frac{1}{V}\sum_{(f)} J(f)\langle a_{-f}a_f\rangle.$$

Thus, we have arrived at the known gap equation:

$$C = C\frac{1}{V}\sum_{(f)}|J(f)|^2\frac{1}{2E(p)}\tanh\frac{\beta E(p)}{2}.$$

The equation

$$1 = \frac{1}{(2\pi)^3}\int\frac{|J(p)|^2}{E(p)}\tanh\frac{\beta E(p)}{2}\,d^3p$$

has a unique solution if $\beta > \beta_0$, where the inverse of the critical temperature, β_0, is determined from the equation

$$1 = \frac{1}{(2\pi)^3}\int\frac{|J(p)|^2}{E(p)}\tanh\frac{\beta_0 T(p)}{2}\,d^3p.$$

It should also be noted that the general expressions (2.10) for the Green functions not only represent certain formal relations but also allow one to calculate a correction term for the gap equation derived in the analysis of a correlation function of the form $\langle\langle a_f a_{-f}; a_f^\dagger a_{-f}^\dagger\rangle\rangle_E$ in the theory of superconductivity based on the BCS model (see [10]).

REFERENCES

1. Bogolyubov, N.N., Zubarev, D.N., and Tserkovnikov, Yu.A., On the Phase Transition Theory, *Dokl. Akad. Nauk SSSR*, 1957, vol. 117, pp. 788–791.
2. Bogolyubov, N.N., Zubarev, D.N., and Tserkovnikov, Yu.A., Asymptotically Exact Solution for a Model Hamiltonian in the Theory of Superconductivity, *Zh. Eksp. Teor. Fiz.*, 1960, vol. 39, no. 1, pp. 120–129.
3. Bogolyubov, N.N., On the Model Hamiltonian in the Theory of Superconductivity, *Preprint of Joint Inst. of Nuclear Research*, Dubna, 1960, no. R-511.
4. Bogolyubov, N.N., Jr., On Model Dynamical Systems in Statistical Mechanics, *Physica*, 1966, vol. 32, no. 5, pp. 933–944; see also Bogolyubov, N.N., Jr., *Method for Studying Model Hamiltonians*, Oxford: Pergamon, 1972.
5. Bogolyubov, N.N., Jr., Construction of Limit Relations for Many-Time Means, *Teor. Mat. Fiz.*, 1970, vol. 4, no. 3, pp. 412–419.
6. Bardeen, J., Cooper, L.N., and Schrieffer, J.R., Theory of Superconductivity, *Phys. Rev.*, 1957, vol. 108, pp. 1175–1204.

7. Bogolyubov, N.N., Tolmachev, V.V., and Shirkov, D.V., *Novyi metod v teorii sverkhprovodimosti* (A New Method in the Theory of Superconductivity), Moscow: Akad. Nauk SSSR, 1958; see also Bogolyubov, N.N., Tolmachev, V.V., and Shirkov, D.V., *A New Method in the Theory of Superconductivity*, New York: Consultants Bureau, 1959.
8. Hertel, P. and Thirring. W., Free Energy of Gravitating Fermions, *Commun. Math. Phys.*, 1971, vol. 24. pp. 22–36.
9. Bogolyubov, N.N., Jr. and Petrina, D.Ya., A Class of Model Systems Admitting a Reduction of the Degree of the Hamiltonian in the Thermodynamic Limit: I, *Teor. Mat. Fiz.*, 1977, vol. 33, no. 2, pp. 231–245.
10. Bogolyubov, N.N., Jr. and Soldatov, A.V., Hartree–Fock–Bogolyubov Approximation in the Models with General Four-Fermion Interaction, *Int. J. Mod. Phys. B*, 1996, vol. 10, no. 5/6, pp. 579–597.
11. Bogolyubov, N.N. and Bogolyubov, N.N., Jr., *Vvedenie v kvantovuyu statisticheskuyu mekhaniku* (An Introduction to Quantum Statistical Mechanics), Moscow: Nauka, 1984, pp. 92–104; see also Bogolyubov, N.N. and Bogolyubov, N.N., Jr., *An Introduction to Quantum Statistical Mechanics*, New York: Gordon & Breach, 1994.
12. Bogolyubov, N.N. and Bogolyubov, N.N., Jr., *Model Problems of Polaron Theory*, London: Gordon and Breach Sci. Publ., 2000; ISBN 90-5699-162-0.

Bogolubov NN, JR, J. G Brankov, V. A. Zagrebnov, A M. Kurbatov, Tonchev
Method approximation Gamiltonian in Statystical physic
Sovia 1981. Bolgaria Akad. of Scienc, (book)

N. Bogolubov JR

N. Bogolubov JR

19 MAY 2013

Dopolnena

[Bogolyubov Jr. (1970)] Bogolyubov Jr., N.N. (1970) "On a minimax principle for some model problems of statistical physics", *Soviet J. Nuclear Phys.* **10**, 243-245.

[Bogolyubov Jr. (1972)] Bogolyubov Jr., N.N. (1972) *A Method for Studying Model Hamiltonians*, Pergamon Press, Oxford.

[Bogolyubov Jr. and Sadovnikov (1975)] Bogolyubov Jr., N.N. and Sadovnikov, B.I. (1975) *Some problems of statistical mechanics* (in Russian) (Izdat. Vysshaya Shkola, Moscow, 1975), Pt. II.

[Bogolyubov Jr. et. al. (1981)] Bogolyubov Jr., N.N., Brankov, J.G., Zagrebnov, V.A., Kurbatov, A.M. and Tonchev, N.S. (1981) *The Approximating Hamiltonian Method in Statistical Physics* (in Russian), Publ. House Bulg. Acad. Sci., Sofia.

[Bogolyubov Jr. et. al. (1984)] Bogolyubov Jr., N.N., Brankov, J.G., Zagrebnov, V.A., Kurbatov, A.M. and Tonchev, N.S. (1984) "Some classes of exactly soluble models of problems in quantum statistical mechanics", *Russian Math. Surveys* **39**, No. 6, 1-50.

[Brankov et. al. (1975a)] Brankov, J.G., Zagrebnov, V.A. and Tonchev N.S. (1975) "An asymptotically exact solution of the generalized Dicke model", *Theoret. and Math. Phys.* **22**, 13-20.

[Brankov et. al. (1975b)] Brankov, J.G., Tonchev N.S. and Zagrebnov, V.A. (1975) "An Exactly Solvable Model for Metal-Insulator Phase Transition", *Physica* **79A**, 125-127.

[Brankov et. al. (1977)] Brankov, J.G., Tonchev N.S. and Zagrebnov, V.A. (1977) "A Nonpolynomial Generalization of Exactly Soluble Models in Statistical Mechanics", *Ann. Phys. (N.Y.)* **107**, 82-94.

[Brankov et. al. (1979)] Brankov, J.G., Tonchev N.S. and Zagrebnov, V.A. (1979) "On a Class of Exactly Soluble Statistical Mechanical Models with Nonpolynomial Interactions", *J. Stat. Phys.* **20**, 317.

[Brankov and Zagrebnov (1983)] Brankov J. G. and Zagrebnov V. A. (1983) "On the description of the phase transition in the Husimi-Temperley model", J. Phys. A **16**, 2217.