A macroscopic system with *undamped* periodic compressional oscillations

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Abstract

A class of macroscopic systems is described which have the remarkable feature that they can sustain *undamped* compressional radial oscillations. They consist of an arbitrary number of particles confined by a harmonic potential and interacting among themselves through conservative repulsive forces scaling as the inverse cube of distances (but being otherwise arbitrary). The radial oscillation leads to a variation of some of the thermodynamic quantities characterizing these systems, which therefore do not tend to thermodynamic equilibrium, since the (macroscopic) amplitude of the oscillation does not decrease over time. The oscillation is harmonic and isochronous, that is, its frequency is fixed and independent of the initial condition. These results hold independently of the dimension of the system and are also valid in the quantal context.

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A class of macroscopic systems has the remarkable feature that they sustain *undamped* compressional radial oscillations. These systems consist of an arbitrary number of particles confined by a harmonic potential and interacting among themselves through conservative forces *scaling as the inverse cube of distances*. The radial oscillation leads to a variation of some of the thermodynamic quantities characterizing these systems (for instance, their density), which therefore do *not* tend to thermodynamic equilibrium, since the (macroscopic) amplitude of the oscillation does *not* decrease over time. The oscillation is *isochronous*, that is, its frequency is fixed and independent of the initial condition. These results hold independently of the dimension of the system and are valid in the context of standard nonrelativistic mechanics (both *classical* and *quantal*).

Let me emphasize that this finding refers to the behavior of a many-body system (with special, but not too peculiar, forces), in the context of quite standard (*classical* or *quantal*) mechanics, in a space with arbitrary dimensions d (including $d=3$).

The system under consideration is characterized by the standard Hamiltonian

[H] $$H(\vec{r},\vec{p}) = \frac{1}{2}\sum_{n=1}^{N}(p_n^2 + \Omega^2 r_n^2) + V^{(-2)}(R) \, ,$$

entailing the Hamiltonian equations of motion

[HamEq] $$\dot{\vec{r}}_n = \vec{p}_n \, , \qquad \dot{\vec{p}}_n = -\Omega^2 \, \vec{r}_n + \vec{F}_n^{(-3)}(R) \, ,$$

and the Newtonian equations of motion

$$\ddot{\vec{r}}_n = -\Omega^2 \, \vec{r}_n + \vec{F}_n^{(-3)}(R) \, ,$$

with the forces $\vec{F}_n^{(-3)}(R)$ defined of course as follows:

$$\vec{F}_n^{(-3)}(R) = -\frac{\partial \, V^{(-2)}(R)}{\partial \, \vec{r}_n} \, .$$

Notation: $\vec{r}_n$ and $\vec{p}_n$ are *d*-vectors, $\vec{r}_n \equiv (x_{n1}, \dots, x_{nd})$ and $\vec{p}_n \equiv (p_{n1}, \dots, p_{nd})$, while *R* denotes the $(N \times d)$-matrix of components $x_{nj}, n = 1, \dots, N, \ \ j = 1, \dots, d$. The harmonic oscillator potential acts as a container (with *soft* walls). Note that for simplicity we assume here all particles to have unit mass (the generalization to different masses is discussed in the paper).

Throughout this talk we assume the potential $V^{(-2)}(R)$ to be *nonnegative* for all values of its *Nd* arguments, and (most importantly) to scale as the inverse-square of its arguments,

$$V^{(-2)}(R) \geq 0 \ , \qquad V^{(-2)}(a\,R) = a^{-2}\, V^{(-2)}(R) \, ,$$

entailing of course that the corresponding forces $\vec{F}_n^{(-3)}(R)$ scale as the inverse-cube of their arguments,

$$\vec{F}^{(-3)}(a\,R) = a^{-3}\, \vec{F}^{(-3)}(R) \, .$$

Note (again, most importantly) that these scaling properties imply the relation

[Scale] $$\sum_{n=1}^{N} \left[\vec{r}_n \cdot \vec{F}^{(-3)}(R) \right] =$$

$$= -\sum_{n=1}^{N} \left[\vec{r}_n \cdot \frac{\partial\, V^{(-2)}(R)}{\partial\, \vec{r}_n} \right] = 2\, V^{(-2)}(R) \, .$$

These properties of the forces and the potential energy entail that they *diverge* when the particle positions all tend to zero. Moreover, whenever the potential energy has a singularity, it *diverges* to *positive infinity*; hence, for any initial conditions, the moving particles are kept away by a *finite* amount from any singularity. Hence, as long as attention is restricted to the real domain (corresponding to the physical situation), all solutions of the Hamiltonian and Newtonian equations written above are *uniquely defined* and *finite* for *all* time.

A (possibly particularly interesting) subcase of the class of models we just described is characterized by the potential function

$$V^{(-2)}(R) = \frac{1}{4} \sum_{n,m=1;n\neq m}^{N} \frac{g_{nm}^2}{(\vec{r}_n - \vec{r}_m)^2}$$

entailing

$$\vec{F}^{(-3)}(R) = \sum_{m=1;n\neq m}^{N} \frac{g_{nm}^2\,(\vec{r}_n - \vec{r}_m)}{(\vec{r}_n - \vec{r}_m)^4} \quad .$$

Derivation of the main result (in a classical context)

Let us introduce the following two (scalar) collective coordinates:

[Q] $$Q=\sum_{n=1}^{N} r_n^2=\sum_{n=1}^{N}\sum_{j=1}^{d} x_{nj}^2 \ ,$$

[D] $$D=\sum_{n=1}^{N} \vec{r}_n\cdot\vec{p}_n \ .$$

Remark: the quantity $Q\equiv Q(t)$ is a measure of the volume occupied by the *N*-body system at time *t*.

The Hamiltonian equations of motion [HamEq] imply the following time-evolution of these two quantities, $Q\equiv Q(t)$ and $D\equiv D(t)$:

[Qdot] $$\dot{Q}=2D \ ,$$

$$\dot{D}=\sum_{n=1}^{N}\left[p_n^2-\Omega^2 r_n^2+\vec{r}_n\cdot F^{(-3)}(R)\right] .$$

But via [Scale] the second of these equations becomes

$$\dot{D}=\sum_{n=1}^{N}(p_n^2-\Omega^2 r_n^2)+2V^{(-2)}(R) \ ,$$

hence, via [Q] and [H],

[Ddot] $$\dot{D}=2H-2\Omega^2 Q \ .$$

This system of two ODEs, [Qdot] and [Ddot], is closed and easily solved (note that the Hamiltonian *H* is of course time-independent), yielding

$$Q(t) = \frac{H}{\Omega^2} + \left[Q(0) - \frac{H}{\Omega^2}\right]\cos(2\Omega\mathrm{t}) + \frac{D(0)\sin(2\Omega\mathrm{t})}{\Omega} ,$$

$$D(t) = D(0)\cos(2\Omega\mathrm{t}) - \left[Q(0) - \frac{H}{\Omega^2}\right]\Omega\sin(2\Omega\mathrm{t}) .$$

The first of these equations demonstrates the validity of our main finding: indeed, it clearly implies that any function $F(t) \equiv F[Q(t)]$ of the collective coordinate $Q \equiv Q(t)$ evolves periodically with the period (independent of the initial data) $T = \pi/\Omega$,

$$F(t+T) = F(t) , \ \ T = \frac{\pi}{\Omega} ,$$

including of course in particular $Q(t)$ itself,

$$Q(t+T) = Q(t) .$$

This proves our result in the context of classical mechanics.

In our paper we have studied a broad class of many-body systems in which particles interact with each other according to a potential of degree of homogeneity -2. If these particles are additionally confined by a harmonic external potential, they have the remarkable property of showing *undamped* compressional oscillations whenever the initial conditions deviate from thermodynamic equilibrium. These oscillations are adiabatic, and lead therefore to variations in temperature and pressure. No irreversible processes arise, meaning that no temperature gradients can be generated by the oscillation, nor can there be damping due to bulk viscosity. We also generalized the approach to include a fairly general time-dependent Hamiltonian, so that we could model mechanical manipulation on the system. This allowed to show, for example, that it is possible to reverse a free expansion made by suddenly decreasing the strength of the confining harmonic potential. It was not possible, however, to display any counterexamples to the second law of thermodynamics.

The reason for the phenomenon of *undamped* oscillations is that three collective coordinates, including the squared radius of gyration Q of the system, form a closed algebra under Poisson brackets, and that the Hamiltonian is a linear combination of these 3 collective coordinates.

Finally, it was shown that the main results remain valid in the *quantal* context.

The quantum case

We now show that our main finding remains true in the *quantal* context; i. e., also in that context the time evolution of the effective volume occupied by the *N*-body system whose dynamics is determined by the Hamiltonian [H] *oscillates isochronously without any damping*.

A fairly straightforward way to show this is by working in the Heisenberg representation. The Heisenberg equations of motion for the observables corresponding to the classical quantities *Q* and *D* then reduce to [Qdot] and [Ddot], which must, however, be interpreted as operator equations. Since they are linear, their solution in terms of noncommuting operators is essentially trivial (but not very enlightening).

We now provide instead a proof which is based on the Schrödinger representation and has the advantage of displaying explicitly the relevant Schrödinger wave function.

Our point of departure is the time-dependent Schrödinger equation

$$\mathrm{i}(\partial/(\partial t))\Psi(R;t) = H\,\Psi(R;t)\ .$$

Of course here and hereafter we use units such that $h = 2\pi$.

The wave function Ψ depends on the time *t* and on the *Nd* coordinates x_{nj} , and the Hamiltonian operator *H* is obtained from the Hamiltonian [H] via the standard replacement $\vec{p}_n \Rightarrow -\frac{i\partial}{\partial \vec{r}_n}$, so that

$$H = (\Delta + \Omega^2 N^2 \rho^2)/2 + V^{(-2)}(R)$$

where Δ is the Laplace operator in the space spanned by the *Nd* Cartesian coordinates x_{nj} ,

$$\Delta = \sum_{n=1}^{N} \sum_{j=1}^{d} \frac{\partial^2}{\partial {x_{nj}}^2}$$

and ρ is the hyperradial coordinate,

$$\rho = \frac{\sqrt{Q}}{N} = \frac{1}{N}\sqrt{\sum_{n=1}^{N} \sum_{j=1}^{d} {x_{nj}}^2}\ .$$

It is now convenient to replace the *Nd* Cartesian coordinates x_{nj} with the hyperspherical coordinates ρ and θ_k, where we denote as θ_k the *Nd-1* "angles" which complement the hyperradial coordinate ρ in the system of hyperspherical coordinates providing an alternative reference frame in the *Nd*-dimensional space spanned by the *Nd* Cartesian coordinates x_{nj}. The scaling property [Scale] then implies

$$V^{(-2)}(R) = \rho^{-2} U(\vartheta) \ ,$$

where we denote by ϑ the set of the *Nd-1* angles θ_k. Then in the Schrödinger equation the Hamiltonian operator reads

$$H = \frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2} + \frac{Nd-1}{\rho}\frac{\partial}{\partial\rho} \ + \frac{\Lambda}{\rho^2}\right),$$

where Λ is now an operator acting only on the *Nd-1* angles θ_k. This operator has an infinity of eigenvalues $\lambda_\sigma, \sigma = 0,1,2,...$, and, correspondingly, a complete set of eigenfunctions $\phi_\sigma(\vartheta)$,

$$\Lambda\, \phi_\sigma(\vartheta) = \lambda_\sigma\, \phi_\sigma(\vartheta)\ , \ \ \sigma = 0,1,2, ... \, .$$

The eigenvalues λ_σ are not known (they depend on the interparticle potentials), but clearly they are *all positive*, due to the assumed repulsive character of the interparticle forces and to the intrinsically repulsive character of the "hypercentrifugal" potential.

The foundation has thus been established to solve the Schrödinger equation by separation of variables, namely by setting

$$\Psi(\rho,\vartheta;t)=\sum_{s,\sigma=0}^{\infty}\psi_{s\sigma}(\rho)\,\phi_\sigma(\vartheta)\exp(-\mathrm{i}\Omega_{s\sigma}t)\ ,$$

where for convenience we can assume the eigenfunctions $\phi_\sigma(\vartheta)$ to be orthonormal, say

$$<\phi_\sigma,\phi_{\sigma'}>\equiv\int \mathrm{d}\vartheta\ \phi_\sigma(\vartheta)\phi_{\sigma'}(\vartheta)=\delta_{\sigma,\sigma'}\ ,$$

with an obvious significance of the symbol $\int \mathrm{d}\vartheta$ and with $\delta_{\sigma,\sigma'}$ the standard Kronecker symbol. As for the *hyperradial eigenfunctions* $\psi_{s\sigma}(\rho)$ and the corresponding *eigenvalues* $\Omega_{s\sigma}$, they correspond to the normalizable solutions (in the interval $0\leq\rho<\infty$) of the ODE

$$-\psi''_{s\sigma}(\rho) - \frac{Nd-1}{\rho}\psi'_{s\sigma}(\rho) + \left(\Omega^2\rho^2 + \frac{2\lambda_\sigma}{\rho^2}\right)\psi_{s\sigma}(\rho)$$
$$= -2\mathit{\Omega}_{s\sigma}\psi_{s\sigma}(\rho)\,,$$

hence they read as follows (with the *integer* indices *s* and *σ* ranging from *0* to ∞):

$$\psi_{s\sigma}(\rho) = x^{1-\frac{Nd}{2}+\alpha_\sigma}\exp(-x^2/2))\,L_s^{\alpha_\sigma}(x^2)\,,$$

$$x^2 = \Omega\,\rho^2\,, \qquad \alpha_\sigma = \sqrt{2\lambda_\sigma + (Nd/2-1)^2}\,,$$

[Eigen] $$\mathit{\Omega}_{s\sigma} = (2s+1+\alpha_\sigma)\,\Omega\,.$$

Here $L_s^\alpha(z)$ is the generalized Laguerre polynomial of upper index *α* and of degree *s* in its argument *z* .

We are now able to look at the time evolution *F(t)* predicted in this *quantal* context for an arbitrary function *F(ρ)* of the collective coordinate ρ. It is clearly given by the formula

$$F(t) =< \Psi(\rho,\vartheta;t), F(\rho)\,\Psi(\rho,\vartheta;t) >$$

whose significance is, we trust, clear.

Inserting in this formula the above expression of the Schrödinger wave function $\Psi(\rho,\vartheta;t)$ this reads

$$F(t) = \sum_{s,s',\sigma,\sigma'=0}^{\infty} < \psi_{s\sigma}(\rho), F(\rho)\psi_{s'\sigma'}(\rho) >< \phi_{\sigma}(\vartheta), \phi_{\sigma'}(\vartheta) > \exp[\mathrm{i}(\Omega_{s\sigma} - \Omega_{s'\sigma'}\,)t] \;,$$

again with a self-evident meaning of the notation employed. And clearly this formula becomes (via [Eigen])

$$F(t) = \sum_{s,s';\sigma=0}^{\infty} < \psi_{s\sigma}(\rho), F(\rho)\psi_{s'\sigma}(\rho) > \exp[2\mathrm{i}(s-s')\,\Omega\, t]\,.$$

It is then plain---given the integer character of the indices s and s'---that this formula entails the periodicity property

$$F(t+T) = F(t)\;,\;\; T = \frac{\pi}{\Omega}\,.$$

QED.