Flag structures on smooth manifolds: equivalence and applications

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Outline

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- *Questions: which geometric structures? what does "natural" mean?*

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- M. Dunajski, M. Godlinski (2012): G_2-structures on the solution space of 7-th order
- S. Casey, M. Dunajski, P. Tod (2012): ASD conformal on the solution space of a pair of 2nd order ODEs

Admissible curves in flag varieties

- Let $F_\alpha(V)$, $\alpha = (\alpha_1, \dots, \alpha_r)$, be a flag variety:

$$V_0 = 0 \subset V_1 \subset \cdots \subset V_r \subset V, \quad \dim V_i = \alpha_i.$$

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- Let $W(t)$ be any curve of subspaces (of fixed dimension k). Then we define $W'(t)$ as:

$$W'(t) = \langle e_1(t), \dots, e_k(t), e_1'(t), \dots, e_k'(t) \rangle,$$

where $\{e_1(t), \dots, e_k(t)\}$ is any frame in $W(t)$. This definition does not depend on the choice of the frame and is invariant under reparametrizations.

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- An (unparametrized) curve $\gamma \subset F_\alpha$ is called *admissible*, if for any (local) parametrization $\gamma(t) = \{V_i(t)\}$ we have:

$$V_i'(t) \subset V_{i+1}(t).$$

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- Examples: osculating flags of projective curves (or curves in Grassmanians).

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- Each non-linear system of ODEs (of order ≥ 2) defines a flag structure on the solution manifold!

Flag structures via jet spaces

- An ordinary differential equation of order n is a hypersurface $\mathcal{E} \subset J^n(\mathbb{R}, \mathbb{R})$ foliated by solutions. Contact geometry of J^n defines a natural double fibration (**Lie theorem**):

$$\pi_1 \colon \mathcal{E} \to \mathcal{S}, \quad \pi_2 \colon \mathcal{E} \to J^{(n-2)}(\mathbb{R}, \mathbb{R}).$$

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- For any solution $y(x) \in \mathcal{S}$ we have a family of curves $\mathcal{L}(x_0) \subset \mathcal{S}$ through $y(x)$:

$$\mathcal{L}(x_0) = \pi_1 \pi_2^{-1}(j_{x_0}^{n-2} y).$$

That is: for any given solution $y(x)$ and any fixed $x_0 \in \mathbb{R}$ the curve $\mathcal{L}(x_0) \subset \mathcal{S}$ consists of all solutions having contact of order $(n-2)$ with $y(x)$ at $x = x_0$.

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- Tangent spaces to curves $\mathcal{L}(x_0)$ define a curve γ of 1-dimensional subspaces in $T_{y(x)}\mathcal{S}$. Osculating flags of this curve provide a natural flag structure on $\mathcal{S}$.

Flag structures via linearization

- Tangent space to the solution space $\mathcal{S}$ at $y(x)$ is defined via its linearization:

$$z^{(n)}(x) + p_1(x)z^{(n-1)}(x) + \cdots + p_n(x)z(x) = 0,$$

where $y(x) + \varepsilon z(x)$ is a first order variation (satisfies the given ODE up to $o(\varepsilon)$).

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- **These two definitions are equivalent and produce the same flag structure on the solution space $\mathcal{S}$ for any non-linear ODE of order ≥ 3.**

Projective invariants of curves

- Local differential (relative) invariants of unparametrized curves in P^{n-1}:

$$\theta_3, \theta_4, \ldots, \theta_n.$$

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- For any non-linear n-th order ODE $\mathcal{E}$ the Wilczynski invariants θ_i of all linearizations of $\mathcal{E}$ define (relative) contact invariants of $\mathcal{E}$:

$$\Theta_3, \Theta_4, \ldots, \Theta_n.$$

Projective invariants of curves

- Wilczynski invariant θ_3 for $y''' + p_2 y'' + p_1 y' + p_0 y = 0$:

$$\theta_3 = \frac{1}{6}p_2'' + \frac{1}{3}p_2 p_2' + \frac{2}{27}p_2^3 - \frac{1}{2}p_1' - \frac{1}{3}p_1 p_2 + p_0.$$

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- Generalized Wilczynski invariant for $y''' = F(x, y, y', y'')$ coincides with Wunschmann invariant:

$$\Theta_3 = -\frac{1}{6}F_{2xx} + \frac{1}{3}F_2 F_{2x} - \frac{2}{27}F_2^3 + \frac{1}{2}F_{1x} - \frac{1}{3}F_1 F_2 - F_0.$$

Conformal and symplectic structures

- We say that a curve $\gamma(t) = \{V_i(t)\}$ in the flag variety $F_\alpha(V)$ is *compatible with a non-degenerate bilinear form* B, if $B(V_i(t), V_{r+1-i}(t)) = 0$ for all $i = 1, \ldots, r-1$.

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- An osculating flag of a projective curve $\gamma \subset P(V)$ is compatible with a non-degenerate bilinear form if and only if $\theta_{2i+1} = 0$ for all i. In this case the form is unique up to a scalar and is symmetric (dim V is odd) or skew-symmetric (dim V is even).

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Theorem

The space of solutions of a single non-linear ODE of order $n \geq 3$ admits a natural conformal (n is odd) or conformal symplectic (n is even) structure if and only if $\Theta_{2i+1} = 0$ for all i.

Other G-structures

- G_2-structure on a 7-dim manifold M^7 is defined as a G-structure with G being the split real form $G_{2(2)}$ of the complex simple Lie group of type G_2 embedded into $GL(7)$. It can be defined by a 3-form on M^7.

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- Example: curves of a constant projective curvature on P^2:

$$y_2(SS_{xx} - \tfrac{7}{6}S_x^2) + y_2 y_3 SS_x - \tfrac{1}{2}(9y_2y_4 - 7y_3^3)S - cS^{8/5} = 0,$$

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- **There no other natural G-structures associated with single ODE for pure algebraic reason: there are no other non-trivial inclusions $GL(2) \subset G \subset GL(n)$.**

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- Flag structures on contact manifolds (isotropic flags inside the contact distribution) and applications to the geometry of generic vector distributions.
- Relationship between flag structures and Tanaka theory.
- Special properties of geometric structures that come from ODEs.

Generalization to other flag types

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- Point transformations preserve $J^{2,3} \to J^{0,0}$ and lead to flag structure of type $(1, 3, 5)$. Contact transformations preserve $J^{2,3} \to J^{1,2}$ and lead to flag structures of type $(2, 4, 5)$.

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- In the trivial case $y'' = z''' = 0$ symmetry algebras of these two kinds of differential systems have the same dimension 15, but are not isomorphic!