Flag structures on smooth manifolds: equivalence and applications

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(joint work with Igor Zelenko)

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Outline

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   - Motivating examples

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   - Conformal and symplectic structures
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4. Behind the scene
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Consider a differential equation of finite type (= finite-dimensional space of solutions).
Geometric structures on the solution space of differential equations

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- For example, systems of ordinary differential equations.
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Are there any natural geometric structures on this manifold?
Geometric structures on the solution space of differential equations

- Consider a differential equation of finite type (= finite-dimensional space of solutions).
- For example, systems of ordinary differential equations.
- It reduces (maybe after a series of prolongations) to a completely integrable distribution with solutions as fibers of this distribution.
- Locally we can define the structure of a smooth manifold on the solution space.
- Are there any natural geometric structures on this manifold?
- Questions: which geometric structures? what does “natural” mean?
Motivating examples

- Wunschmann (1905), S.-S. Chern (1939): conformal structures on 3rd order ODEs with vanishing Wunschmann invariant.
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- M. Dunajski, M. Godlinski (2012): $G_2$-structures on the solution space of 7-th order
- S. Casey, M. Dunajski, P. Tod (2012): ASD conformal on the solution space of a pair of 2nd order ODEs
Admissible curves in flag varieties

- Let $F_{\alpha}(V)$, $\alpha = (\alpha_1, \ldots, \alpha_r)$, be a flag variety:
  $$V_0 = 0 \subset V_1 \subset \cdots \subset V_r \subset V, \quad \dim V_i = \alpha_i.$$
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Let $W(t)$ be any curve of subspaces (of fixed dimension $k$). Then we define $W'(t)$ as:

$$W'(t) = \langle e_1(t), \ldots, e_k(t), e'_1(t), \ldots, e'_k(t) \rangle,$$

where $\{e_1(t), \ldots, e_k(t)\}$ is any frame in $W(t)$. This definition does not depend on the choice of the frame and is invariant under reparametrizations.
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- An (unparametrized) curve $\gamma \subset F_\alpha$ is called admissible, if for any (local) parametrization $\gamma(t) = \{V_i(t)\}$ we have:
  \[ V'_i(t) \subset V_{i+1}(t). \]
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- Examples: osculating flags of projective curves (or curves in Grassmanians).

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**Definition**

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Each non-linear system of ODEs (of order $\geq 2$) defines a flag structure on the solution manifold!
Flag structures via jet spaces

- An ordinary differential equation of order $n$ is a hypersurface $\mathcal{E} \subset J^n(\mathbb{R}, \mathbb{R})$ foliated by solutions. Contact geometry of $J^n$ defines a natural double fibration (Lie theorem):

$$\pi_1: \mathcal{E} \to S, \quad \pi_2: \mathcal{E} \to J^{(n-2)}(\mathbb{R}, \mathbb{R}).$$
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- For any solution $y(x) \in \mathcal{S}$ we have a family of curves $\mathcal{L}(x_0) \subset \mathcal{S}$ through $y(x)$:
  \[ \mathcal{L}(x_0) = \pi_1\pi_2^{-1}(j_{x_0}^{n-2}y). \]

That is: for any given solution $y(x)$ and any fixed $x_0 \in \mathbb{R}$ the curve $\mathcal{L}(x_0) \subset \mathcal{S}$ consists of all solutions having contact of order $(n - 2)$ with $y(x)$ at $x = x_0$. 
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- Tangent spaces to curves $\mathcal{L}(x_0)$ define a curve $\gamma$ of 1-dimensional subspaces in $T_{y(x)} S$. Osculating flags of this curve provide a natural flag structure on $S$. 

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Flag structures via linearization

- Tangent space to the solution space $S$ at $y(x)$ is defined via its linearization:

$$z^{(n)}(x) + p_1(x)z^{(n-1)}(x) + \cdots + p_n(x)z(x) = 0,$$

where $y(x) + \varepsilon z(x)$ is a first order variation (satisfies the given ODE up to $o(\varepsilon)$).
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2. Due to Wilczynski, each linear ODE of order $n$ defines a projective curve in $P^{n-1}$:

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- We get projective curve in $P(T_y(x)S)$ for any solution $y(x)$. The union of osculating flags defines a flag structure on $S$. 
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- **These two definitions are equivalent and produce the same flag structure on the solution space $S$ for any non-linear ODE of order $\geq 3$.**
Projective invariants of curves

- Local differential (relative) invariants of unparametrized curves in $P^{n-1}$:
  \[ \theta_3, \theta_4, \ldots, \theta_n. \]

E.Wilczynski, *Projective differential geometry of curves and ruled surfaces*, 1905.
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- For any non-linear $n$-th order ODE $\mathcal{E}$ the Wilczynski invariants $\theta_i$ of all linearizations of $\mathcal{E}$ define (relative) contact invariants of $\mathcal{E}$:

  $\Theta_3, \Theta_4, \ldots, \Theta_n$. 
Wilczynski invariant $\theta_3$ for $y''' + p_2y'' + p_1y' + p_0y = 0$:

$$\theta_3 = \frac{1}{6} p_2'' + \frac{1}{3} p_2 p_2' + \frac{2}{27} p_2^3 - \frac{1}{2} p_1' - \frac{1}{3} p_1 p_2 + p_0.$$
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Wilczynski invariant $\theta_3$ for a curve $[y_1 : y_2 : y_3] \subset P^2$ is the above $\theta_3$ for the linear equation with the solution space $\langle y_1, y_2, y_3 \rangle$. For a curve $[1 : x : y(x)]$ we get:

$$\theta_3 = 9 y''^2 y^{(5)} - 45 y'' y^{(3)} y^{(4)} + 40 (y^{(3)})^3.$$
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- Generalized Wilczynski invariant for $y''' = F(x, y, y', y'')$ coincides with Wunschmann invariant:
  \[ \Theta_3 = -\frac{1}{6} F_{2xx} + \frac{1}{3} F_2 F_{2x} - \frac{2}{27} F_2^3 + \frac{1}{2} F_{1x} - \frac{1}{3} F_1 F_2 - F_0. \]
We say that a curve $\gamma(t) = \{V_i(t)\}$ in the flag variety $F_\alpha(V)$ is compatible with a non-degenerate bilinear form $B$, if $B(V_i(t), V_{r+1-i}(t)) = 0$ for all $i = 1, \ldots, r - 1$. 
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An osculating flag of a projective curve $\gamma \subset P(V)$ is compatible with a non-degenerate bilinear form if and only if $\theta_{2i+1} = 0$ for all $i$. In this case the form is unique up to a scalar and is symmetric (dim $V$ is odd) or skew-symmetric (dim $V$ is even).
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**Theorem**

*The space of solutions of a single non-linear ODE of order $n \geq 3$ admits a natural conformal ($n$ is odd) or conformal symplectic ($n$ is even) structure if and only if $\Theta_{2i+1} = 0$ for all $i$.***
**Other $G$-structures**

- $G_2$-structure on a 7-dim manifold $M^7$ is defined as a $G$-structure with $G$ being the split real form $G_2(2)$ of the complex simple Lie group of type $G_2$ embedded into $GL(7)$. It can be defined by a 3-form on $M^7$. 

**Theorem**
The space of solutions of a single non-linear ODE of order 7 admits a natural $G_2$-structure if and only if $\Theta_3 = \Theta_4 = \Theta_5 = \Theta_7 = 0$. 

**Example:** curves of a constant projective curvature on $\mathbb{P}^2$:

\[ y^2 \left( S x x - S x^2 \right) + y^2 y^3 S x - 1/2 \left( 9 y^2 y^4 - 7 y^3 y^3 \right) S - c S^8/5 = 0, \]

where $S = 9 y^2 y^5 - 45 y^2 y^3 y^4 + 40 y^3 y^3$. 

There no other natural $G$-structures associated with single ODE for pure algebraic reason: there are no other non-trivial inclusions $GL(2) \subset G \subset GL(n)$. 

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Not all flag structures come from systems of ODEs (of mixed order). Those which come from ODEs satisfy extra semi-integrability conditions expressed in terms of local differential invariants. Other examples are known.
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Special properties of geometric structures that come from ODEs.
Generalization to other flag types

The same construction works for systems of ODEs: we get flag structures of type $\alpha = (m, 2m, \ldots, (n-1)m)$, where $m$ is a number of equations, $n$ the order of the system of ODEs.
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Other geometric structures
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  \[
y''(x) = F(x, y, z, y', z', z''), \quad z''' = G(x, y, z, y', z', z'').
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\]

- Caution: we get different types of flags if we consider different differential systems associated with the above equation!
- Point transformations preserve \( J^{2,3} \rightarrow J^{0,0} \) and lead to flag structure of type \((1, 3, 5)\). Contact transformations preserve \( J^{2,3} \rightarrow J^{1,2} \) and lead to flag structures of type \((2, 4, 5)\).
Generalization to other flag types

- The same construction works for systems of ODEs: we get flag structures of type $\alpha = (m, 2m, \ldots, (n - 1)m)$, where $m$ is a number of equations, $n$ the order of the system of ODEs.
- We get other types of flag varieties starting from systems of ODEs of mixed order such as:

$$y''(x) = F(x, y, z, y', z', z''), \quad z''' = G(x, y, z, y', z', z'').$$

- Caution: we get different types of flags if we consider different differential systems associated with the above equation!
- Point transformations preserve $J^{2,3} \rightarrow J^{0,0}$ and lead to flag structure of type $(1, 3, 5)$. Contact transformations preserve $J^{2,3} \rightarrow J^{1,2}$ and lead to flag structures of type $(2, 4, 5)$.
- In the trivial case $y'' = z''' = 0$ symmetry algebras of these two kinds of differential systems have the same dimension 15, but are not isomorphic!