Dispersionless integrable systems in 3D and Einstein-Weyl geometry

Eugene Ferapontov

Department of Mathematical Sciences, Loughborough University, UK

E.V.Ferapontov@lboro.ac.uk

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Plan:

- Formal linearisation: applications
- Einstein-Weyl geometry
- Integrability in 3D and Einstein-Weyl geometry
- Integrability in 4D and self-duality

Based on:

E.V. Ferapontov and B. Kruglikov, Dispersionless integrable systems in 3D and Einstein-Weyl geometry, arXiv:1208.2728v3.

Formal linearisation

Given a PDE

$$F(x^i, u, u_{x^i}, u_{x^i x^j}, \dots) = 0,$$

its formal linearisation results upon setting $u \to u + \epsilon v$, and keeping terms of the order ϵ. This leads to a linear PDE for v,

$$\ell_F(v) = 0,$$

where ℓ_F is the operator of formal linearisation,

$$\ell_F = F_u + F_{u_{x^i}} \mathcal{D}_{x^i} + F_{u_{x^i x^j}} \mathcal{D}_{x^i} \mathcal{D}_{x^j} + \dots .$$

Example: linearisation of the dKP equation, $u_{xt} - (uu_x)_x - u_{yy} = 0$, reads as $v_{xt} - (uv)_{xx} - v_{yy} = 0$.

Applications of formal linearisation

- *Stability analysis*
- *Symmetries*
- *Contact invariants of ODEs, generalised Laplace invariants of Darboux integrable Monge-Ampère equations*
- *Integrability of ODEs can be seen from the monodromy group of linearised equations*

Can one read the integrability of a given PDE off the geometry of its formal linearisation?

Yes, for broad classes of 3D dispersionless second order PDEs.

Types of PDEs studied:

Quasilinear wave equations:

$$f_{11}u_{xx} + f_{22}u_{yy} + f_{33}u_{tt} + 2f_{12}u_{xy} + 2f_{13}u_{xt} + 2f_{23}u_{yt} = 0.$$

Hirota-type equations:

$$F(u_{xx}, u_{xy}, u_{yy}, u_{xt}, u_{yt}, u_{tt}) = 0.$$

Equations possessing the 'central quadric ansatz':

$$(a(u))_{xx} + (b(u))_{yy} + (c(u))_{tt} + 2(p(u))_{xy} + 2(q(u))_{xt} + 2(r(u))_{yt} = 0.$$

The corresponding formal linearisations are second order linear PDEs. On every solution, their symbols define conformal structures. Which conformal geometries correspond to integrable PDEs? Which conformal geometries should be regarded as 'integrable'?

Einstein-Weyl geometry

This is a triple $(\mathbb{D},\ g,\ \omega)$ where $\mathbb{D}$ is a symmetric connection, g is a conformal structure and ω is a covector such that

$$\mathbb{D}_k g_{ij} = \omega_k g_{ij}, \quad R_{(ij)} = \Lambda g_{ij}.$$

Here $R_{(ij)}$ is the symmetrised Ricci tensor of $\mathbb{D}$, and Λ is some function (the first set of equations defines $\mathbb{D}$ uniquely, so it is sufficient to specify g and ω only).

Theorem (E. Cartan): The triple $(\mathbb{D},\ g,\ \omega)$ satisfies the Einstein-Weyl equations if and only if there exists a two-parameter family of surfaces which are totally geodesic with respect to $\mathbb{D}$, and null with respect to g.

Einstein-Weyl equations are integrable (Hitchin).

Main results

Theorem 1. A second order PDE is linearisable (by a transformation from the natural equivalence group) if and only if the conformal structure g is conformally flat on every solution (has vanishing Cotton tensor).

Theorem 2. A second order PDE is integrable by the method of hydrodynamic reductions if and only if, on every solution, the conformal structure g satisfies the Einstein-Weyl equations, with the covector $\omega = \omega_s dx^s$ given by the formula

$$\omega_s = 2g_{sj}\mathcal{D}_{x^k}(g^{jk}) + \mathcal{D}_{x^s}(\ln\det g_{ij}).$$

The corresponding two-parameter family of totally geodesic null surfaces is provided by the corresponding dispersionless Lax pair.

Example of dKP

As an illustration let us consider the dKP equation,

$$u_{xt} - (uu_x)_x - u_{yy} = 0.$$

The corresponding Einstein-Weyl structure is provided by the conformal metric $g = 4dxdt - dy^2 + 4udt^2$ and the covector $\omega = -4u_x dt$ (Dunajski, Mason, Tod). One can verify that they satisfy the Einstein-Weyl conditions. The dispersionless Lax pair is given by vector fields

$$X = \partial_y - \lambda\partial_x + u_x\partial_\lambda, \quad Y = \partial_t - (\lambda^2 + u)\partial_x + (u_x\lambda + u_y)\partial_\lambda,$$

such that the commutativity condition, $[X, Y] = 0$, is equivalent to dKP. Projecting integral surfaces of the distribution spanned by X, Y in the extended space x, y, t, λ to the space of independent variables x, y, t, one obtains a two-parameter family of surfaces which are null with respect to g, and totally geodesic in the Weyl connection $\mathbb{D}$ specified by g and ω.

Integrability in 4D and self-duality

Integrable equations of Monge-Ampére type in 4D were classified in

B. Doubrov and E.V. Ferapontov, On the integrability of symplectic Monge-Ampère equations, Journal of Geometry and Physics **60** *(2010) 1604-1616.*

- $u_{11} - u_{22} - u_{33} - u_{44} = 0$ (linear wave equation)
- $u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$ (second heavenly equation)
- $u_{13} = u_{12}u_{44} - u_{14}u_{24}$ (modified heavenly equation)
- $u_{13}u_{24} - u_{14}u_{23} = 1$ (first heavenly equation)
- $u_{11} + u_{22} + u_{13}u_{24} - u_{14}u_{23} = 0$ (Husain equation)
- $\alpha u_{12}u_{34} + \beta u_{13}u_{24} + \gamma u_{14}u_{23} = 0$ (general heavenly equation), $\alpha + \beta + \gamma = 0$.

Conjecture: A 4D second order dispersionless PDE is integrable if and only if the corresponding conformal structure is self-dual on every solution.

Questions:

- Contact-invariant approach to dispersionless integrability?
- Higher order PDEs and higher Einstein-Weyl geometry?
- Integrability in dimensions higher than four?