New variational principle for integrable systems

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(partly joint work with Sarah Lobb, Pavlos Xenitidis and Sikarin Yoo-Kong)

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The problem

Multidimensional consistency: We know that many "integrable" equations, discrete and continuous possess the property of multidimensional consistency.

- continuous: commuting flows, higher symmetries & master symmetries, hierarchies;
- discrete: consistency-around-the-cube, Bäcklund transforms, higher continuous symmetries, commuting discrete flows

In all these cases we can think of the *dependent* variable a (possibly vector-valued) function of many (discrete and continuous) variables

$$u = u(n, m, h, \dots; x, t_1, t_2, \dots)$$

on which we can impose many equations simultaneously, and it is the *compatibility* of those equations that makes the integrability manifest.

Key question: How to capture the property of multidimensional consistency within a Lagrange formalism?

The main problem is that the variational principle, through the EL equations, only produces one equation per component of the dependent variables, but not an entire system of compatible equations on one and the same dependent variable!

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- S. Lobb & FWN: *Lagrangian multiforms and multidimensional consistency*, J. Phys. A:Math Theor. **42** (2009) 454013
- S. Lobb, FWN & R. Quispel, *Lagrangian multiform structure for the lattice KP system*, J. Phys. A:Math Theor. **42** (2009) 472002
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- P. Xenitidis, FWN & S. Lobb, *On the Lagrangian formulation of multidimensionally consistent systems*, Proc. Roy. Soc. **A467** # 2135 (2011) 3295-3317
- S. Yoo-Kong, S. Lobb and F.W. Nijhoff, *Discrete-time Calogero-Moser system and Lagrangian 1-form structure*, J. Phys. A: **44** (2011) 365203
- J. Atkinson, S.B. Lobb and F.W. Nijhoff, *An integrable multicomponent quad equation and its Lagrangian formalism*, Theor. Math. Phys 173 (2012) # 3 pp. 1644-1653
- S. Yoo-Kong and F.W. Nijhoff, *Discrete-time Ruijsenaars-Schneider system and Lagrangian 1-form structure*, arXiv:1112.4576 (Dec. 2011)
- Yu. Suris, *Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms*, arXiv: 1212.3314 (Dec. 2012)
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I.o.w. the Lagrangians become differential (or difference) forms that are closed *subject to the equations of the motion*.

- These are forms in a multidimensional space of independent variables, where the action functional, defined as the integral of such forms over geometric objects (curves, surfaces, etc), depends on those geometries;
- The closedness of the form under the equations of motion, is crucial to allow the geometrical object to be deformed (fixing boundaries), and in order for the variational equations to be compatible with the deformation;
- Explicit examples were considered for 2-forms and 3-forms, and 1-forms (the easiest scenario to study variational formalisms involving Lagrangian forms).
- For the 1-form case (case of ODEs and commuting flows) we established the system of EL equations on arbitrary curves in the space of independent variables (work with S. Yoo-Kong and S. Lobb, 2011).
- Recently, for the 1-form case the connection with the Hamiltonian structure of (Liouville-integrable) finite-dimensional systems was established (Suris, Boll et al.)
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Discrete Lagrangian 1-form structure

We consider a system described by N dependent variables $\mathbf{x}(\mathbf{n}) = (x_(\mathbf{n}), \ldots, x_N(\mathbf{n}))$ (particle positions) depending on an arbitrary set of discrete multi-time variables $\mathbf{n} = (\ldots, n_i, \ldots, n_j, \ldots)$.

The action functional is defined as the sum

$$S(\mathbf{x}(\mathbf{n}); \Gamma) = \sum_{\gamma(\mathbf{n}) \in \Gamma} L_i(\mathbf{x}(\mathbf{n}), \mathbf{x}(\mathbf{n} + \mathbf{e}_i)).$$

over an arbitrary curve Γ (with or without boundary) in the space of independent discrete variables:

Such a discrete curve Γ is built from connected oriented links $\gamma_i(\mathbf{n}) = (\mathbf{n}, \mathbf{n} + \mathbf{e}_i)$ on the edges of the lattice.
The action on the curve by summing up the contributions L_i from each of the those oriented links over the entire curve.
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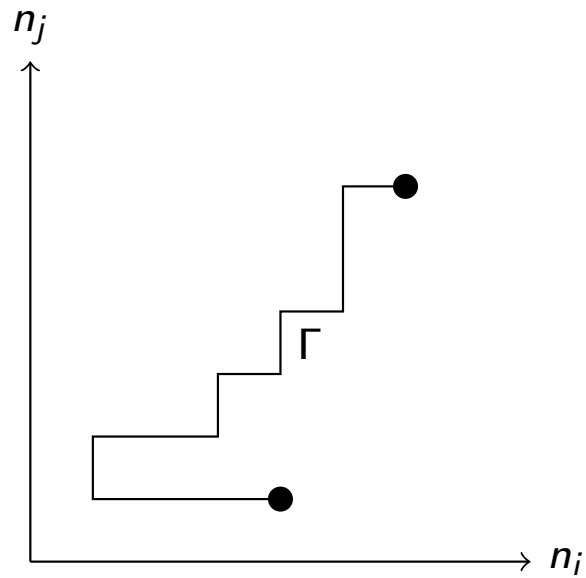

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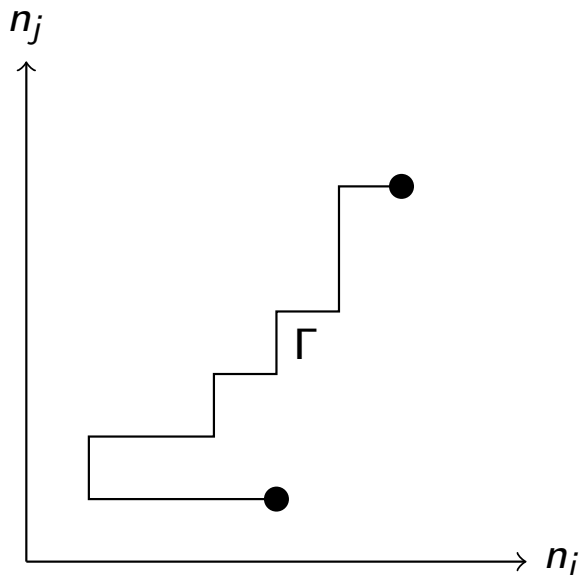

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The system of Euler-Lagrange equations for discrete 1-forms

The basic relations constituting the discrete multi-time Euler Lagrange (EL) equations were first given in 2011 in the thesis of S. Yoo-Kong[1], and reproduced in a recent preprint by Yu. Suris[2].

They read:

case (a): From the action for the discrete curve in this case

$$S[\mathbf{x};\Gamma_{(a)}] = L_i(\mathbf{x}, T_i\mathbf{x}) + (T_iL_j)(T_i\mathbf{x}, T_iT_j\mathbf{x}) \quad \Rightarrow \quad \frac{\partial L_i(\mathbf{x}, T_i\mathbf{x})}{\partial T_i\mathbf{x}} + \frac{\partial (T_iL_j)(T_i\mathbf{x}, T_iT_j\mathbf{x})}{\partial T_i\mathbf{x}} = 0\,.$$

Case (b): Here the action reads

$$S[\mathbf{x};\Gamma_{(b)}] = L_j(\mathbf{x}, T_j\mathbf{x}) + (T_jL_i)(T_j\mathbf{x}, T_iT_j\mathbf{x}) \quad \Rightarrow \quad \frac{\partial L_j(\mathbf{x}, T_j\mathbf{x})}{\partial T_j\mathbf{x}} + \frac{\partial (T_jL_i)(T_j\mathbf{x}, T_iT_j\mathbf{x})}{\partial T_j\mathbf{x}} = 0\,.$$

Case (c): The action for a discrete curve in case (c) is

$$S[\mathbf{x};\Gamma_{(c)}] = L_i(\mathbf{x}, T_i\mathbf{x}) + (T_iL_i)(T_i\mathbf{x}, T_iT_j\mathbf{x}) \quad \Rightarrow \quad \frac{\partial L_i(\mathbf{x}, T_i\mathbf{x})}{\partial T_i\mathbf{x}} + \frac{\partial (T_iL_i)(T_i\mathbf{x}, T_iT_j\mathbf{x})}{\partial T_i\mathbf{x}} = 0\,.$$

Case (d): The action in this case is:

$$S[\mathbf{x};\Gamma_{(d)}] = L_j(\mathbf{x}, T_j\mathbf{x}) + (T_jL_j)(T_j\mathbf{x}, T_iT_j\mathbf{x})\,. \quad \Rightarrow \quad \frac{\partial L_j(\mathbf{x}, T_j\mathbf{x})}{\partial T_j\mathbf{x}} + \frac{\partial (T_jL_j)(T_j\mathbf{x}, T_iT_j\mathbf{x})}{\partial T_j\mathbf{x}} = 0\,.$$

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[1] S Yoo-Kong, *Calogero-Moser type systems, associated KP systems, and Lagrangian structure*, Thesis, University of Leeds (2011).

[2] Y.B. Suris, *Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms*, arXiv:1212.3314.

The system of Euler-Lagrange equations for discrete 1-forms

The basic relations constituting the discrete multi-time Euler Lagrange (EL) equations were first given in 2011 in the thesis of S. Yoo-Kong[1], and reproduced in a recent preprint by Yu. Suris[2].
They read:
case (a): From the action for the discrete curve in this case

$$S[\mathbf{x};\Gamma_{(a)}] = L_i(\mathbf{x}, T_i\mathbf{x}) + (T_iL_j)(T_i\mathbf{x}, T_iT_j\mathbf{x}) \quad \Rightarrow \quad \frac{\partial L_i(\mathbf{x}, T_i\mathbf{x})}{\partial T_i\mathbf{x}} + \frac{\partial (T_iL_j)(T_i\mathbf{x}, T_iT_j\mathbf{x})}{\partial T_i\mathbf{x}} = 0\,.$$

Case (b): Here the action reads

$$S[\mathbf{x};\Gamma_{(b)}] = L_j(\mathbf{x}, T_j\mathbf{x}) + (T_jL_i)(T_j\mathbf{x}, T_iT_j\mathbf{x}) \quad \Rightarrow \quad \frac{\partial L_j(\mathbf{x}, T_j\mathbf{x})}{\partial T_j\mathbf{x}} + \frac{\partial (T_jL_i)(T_j\mathbf{x}, T_iT_j\mathbf{x})}{\partial T_j\mathbf{x}} = 0\,.$$

Case (c): The action for a discrete curve in case (c) is

$$S[\mathbf{x};\Gamma_{(c)}] = L_i(\mathbf{x}, T_i\mathbf{x}) + (T_iL_i)(T_i\mathbf{x}, T_iT_j\mathbf{x}) \quad \Rightarrow \quad \frac{\partial L_i(\mathbf{x}, T_i\mathbf{x})}{\partial T_i\mathbf{x}} + \frac{\partial (T_iL_i)(T_i\mathbf{x}, T_iT_j\mathbf{x})}{\partial T_i\mathbf{x}} = 0\,.$$

Case (d): The action in this case is:

$$S[\mathbf{x};\Gamma_{(d)}] = L_j(\mathbf{x}, T_j\mathbf{x}) + (T_jL_j)(T_j\mathbf{x}, T_iT_j\mathbf{x})\,. \quad \Rightarrow \quad \frac{\partial L_j(\mathbf{x}, T_j\mathbf{x})}{\partial T_j\mathbf{x}} + \frac{\partial (T_jL_j)(T_j\mathbf{x}, T_iT_j\mathbf{x})}{\partial T_j\mathbf{x}} = 0\,.$$

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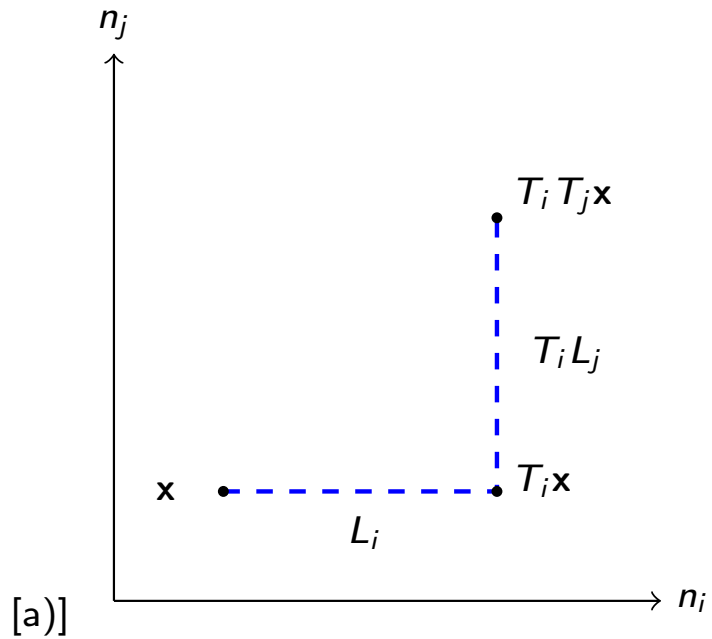
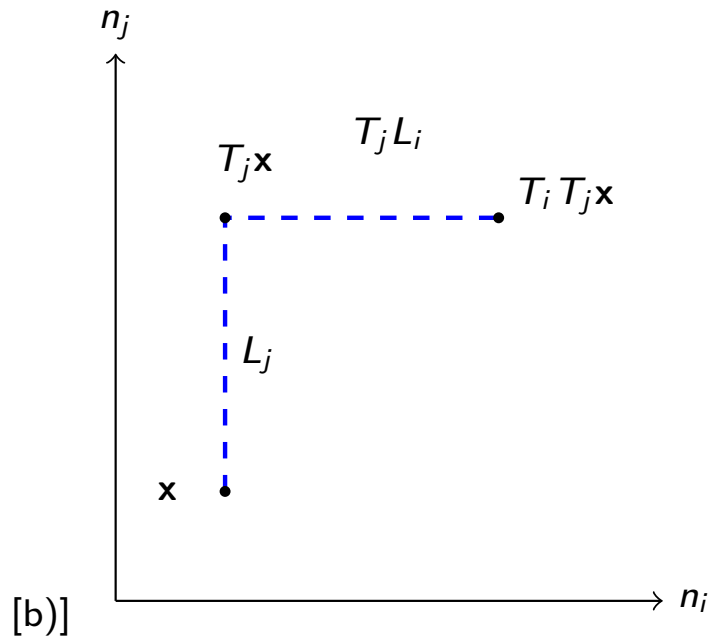
Elementary curves in space of discrete independent variables

They are given as the set of simultaneous equations for the following set of elementary curves in the space of two discrete time variables:



[a)]



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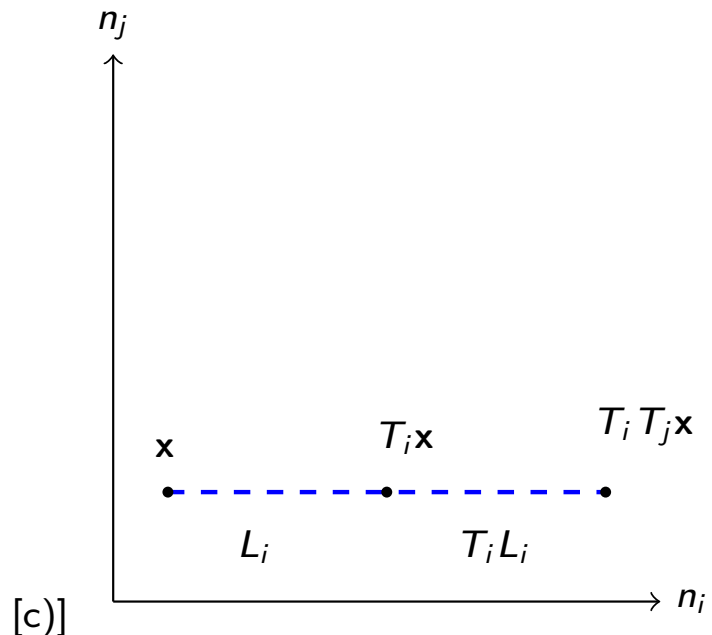

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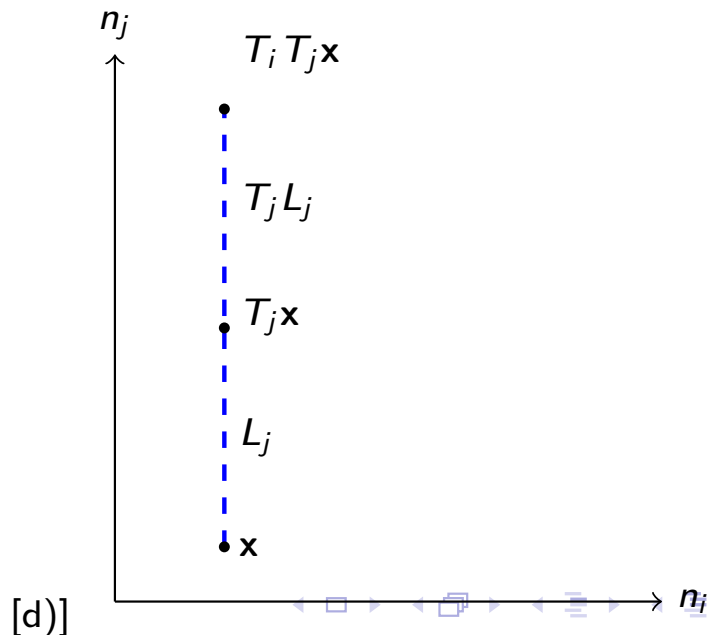

[d)]

Closure relation

The consistency of the four EL relations is guaranteed by the *closure relation*

$$\Delta_i L_j = \Delta_j L_i \quad \text{where} \quad \Delta_i = T_i - \mathrm{id} \ ,$$

which holds on solutions of the EL equations!

This also guarantees that on solutions of the equations of motion the action is stationary under changes of the curve (keeping the end points fixed):

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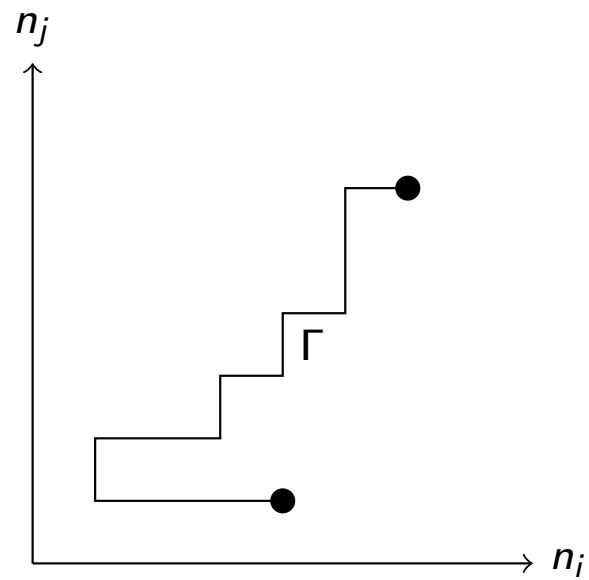

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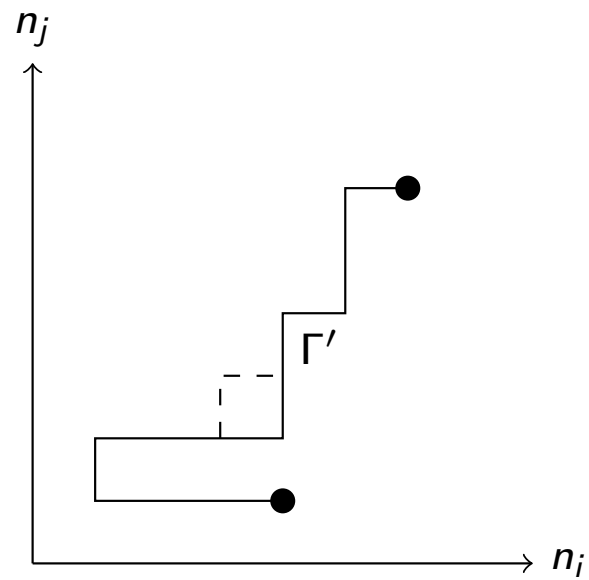

Examples

The example we investigated of discrete 1-form Lagrangians took on the form:

- Discrete-time Calogero-Moser system:

$$L_i(\mathbf{x}, T_i\mathbf{x}; p_i) = \sum_{k,l=1}^{N} f(x_k - T_i x_l) - \frac{1}{2}\sum_{\substack{k,l=1\\ l\neq k}}^{N} [f(x_k - x_l) + f(T_i x_k - T_i x_l)] - p_i(\Xi - T_i\Xi) \ ,$$

where $\Xi = \sum_{k=1}^{N} x_k$, and where the function f can adopt the forms:

$$f(x) = \log(x) \quad , \quad f(x) = \log \sinh(x) \quad , \quad f(x) = \log \sigma(x) \ .$$

- Disrete-time Ruijsenaars-Schneider system:

$$\begin{aligned} L_i(\mathbf{x}, T_i\mathbf{x}; p_i) \quad = \quad & \sum_{k,l=1}^{N} [F(x_k - T_i x_l) - F(x_k - T_i x_l - \lambda)] - \frac{1}{2}\sum_{\substack{k,l=1\\ l\neq k}}^{N} [F(x_k - x_l + \lambda) \\ & + F(T_i x_k - T_i x_l + \lambda)] - \ln|p_i|\,(\Xi - T_i\Xi) \ , \end{aligned}$$

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The variational principle for Lagrangian 1-forms

Assume that we have a system with dependent variables $\mathbf{x} = \mathbf{x}(t_1, t_2, \dots)$ in multiple times $t_1, t_2, \dots$ and described by Lagrangians of the type:

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(The formulae are motivated by the example of the 3-particle CM system).
Consider a parametrized curve Γ in the space of independent variables:

$$\Gamma : \ (t_1(s), t_2(s)) \ , \quad s \in [s_0, s_1] \ .$$

Thus, the action becomes:

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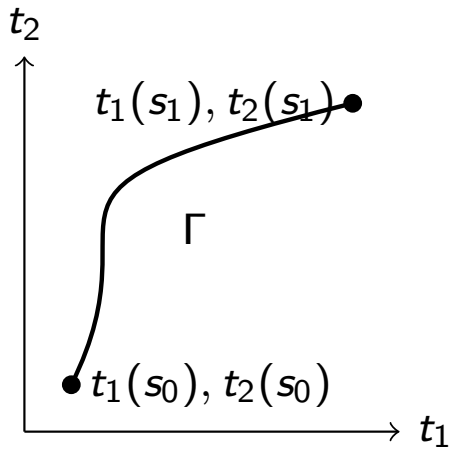

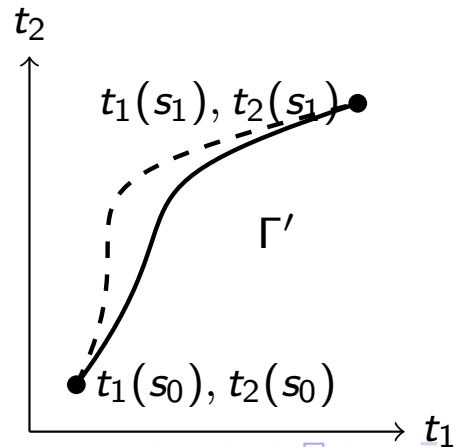

The variational calculus in terms of 1-forms involves two types of variations:

- variations $\mathbf{t}(s) \to \mathbf{t}(s) + \delta\mathbf{t}(s)$ of the curve, i.e. $\Gamma \to \Gamma'$
- variations $\mathbf{x}(\mathbf{t}) \to \mathbf{x} + \delta\mathbf{x}(\mathbf{t})$ of the dependent variables on the curve Γ (while keeping the curve fixed);

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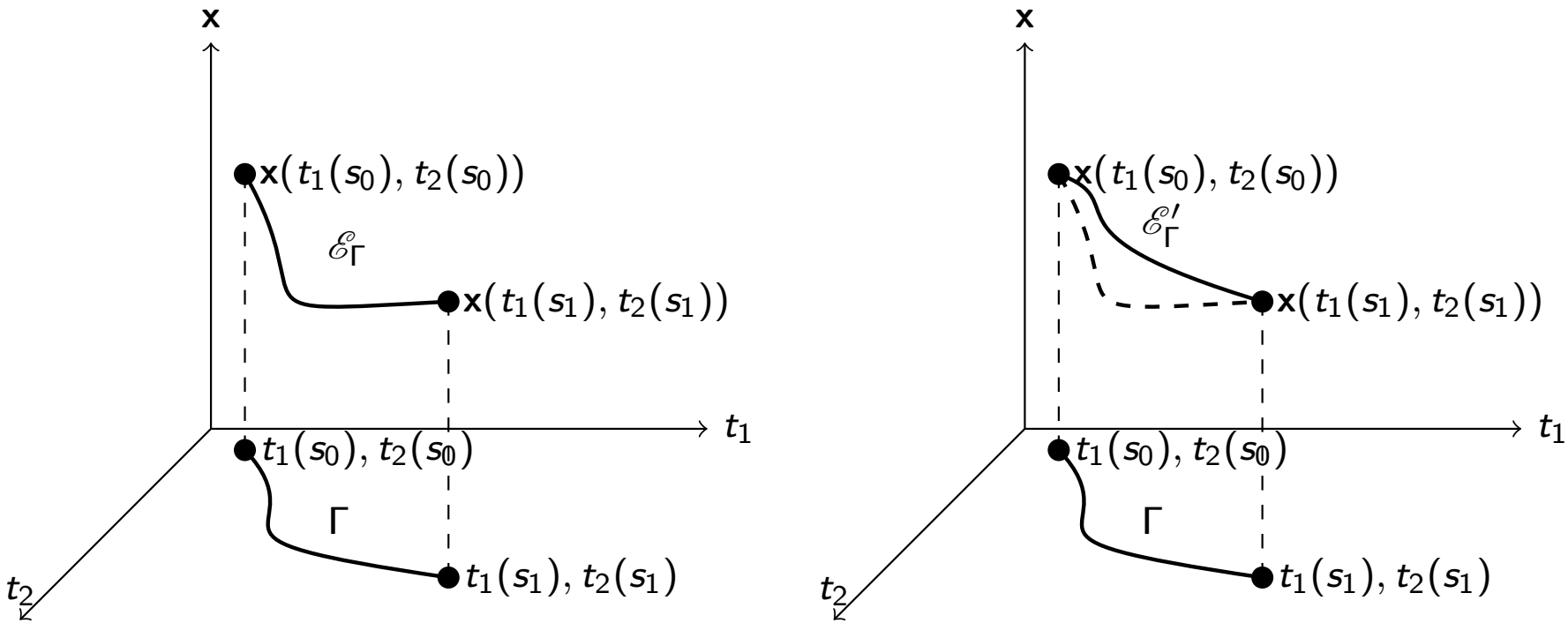

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Lagrangian 1-form structure implies integrability

We will now consider a class of Lagrangians with given kinetic terms, but arbitrary potentials.
The form of the Lagrangian 1-form components of $\mathbf{L} = \mathscr{L}_1\, dt_1 + \mathscr{L}_2\, dt_2$, are assumed to be:

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$$\frac{\partial \mathscr{L}_2}{\partial (\frac{\partial X_i}{\partial t_1})} = 3\alpha \left(\frac{\partial X_i}{\partial t_1} \right)^2 + \frac{\partial X_i}{\partial t_2} + \sum_{j \neq i = 1}^{N} W(X_i - X_j) = 0 \ .$$

Lagrangian 1-form structure implies integrability

We will now consider a class of Lagrangians with given kinetic terms, but arbitrary potentials.
The form of the Lagrangian 1-form components of $\mathbf{L} = \mathscr{L}_1\, dt_1 + \mathscr{L}_2\, dt_2$, are assumed to be:

$$\begin{aligned}
\mathscr{L}_1 &= \sum_{i=1}^{N} \frac{1}{2} \left(\frac{\partial X_i}{\partial t_1} \right)^2 + \sum_{i \neq j}^{N} V(X_i - X_j) \ , \\
\mathscr{L}_2 &= \sum_{i=1}^{N} \left[\left(\frac{\partial X_i}{\partial t_1} \right) \frac{\partial X_i}{\partial t_2} + \alpha \left(\frac{\partial X_i}{\partial t_1} \right)^3 \right] + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{\partial X_i}{\partial t_1} W(X_i - X_j) \ .
\end{aligned}$$

The Euler-Lagrange equations for the t_1- resp. t_2 components yields

$$\begin{aligned}
\frac{\partial^2 X_i}{\partial t_1^2} &= 2 \sum_{j \neq i} V'(X_i - X_j) \ , \\
\frac{\partial^2 X_i}{\partial t_1 \, \partial t_2} &= \sum_{j \neq i} \left[W'(X_i - X_j) \frac{\partial X_i}{\partial t_1} - W'(X_j - X_i) \frac{\partial X_j}{\partial t_1} \right] \ .
\end{aligned}$$

Furthermore, we have the the constraint:

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The consistency between the constraint and the EL equations lead to the folowing relation between the potentials V and W:

$$W(x) = -6\alpha V(x) + \text{constant} \ .$$

We now consider the closure relation:

$$\frac{\partial \mathscr{L}_1}{\partial t_2} = \frac{\partial \mathscr{L}_2}{\partial t_1} \ .$$

which leads next to the following condition on W:

$$\sum_{\substack{i,j,l=1 \\ i\neq j, i\neq l, j\neq l}}^{N} W'(X_i - X_j) W(X_j - X_l) = 0 \ .$$

In particular this should hold for the 3-particle case, and hence setting $N = 3$ we get a functional relation for W of the form:

$$W'(x)\,[W(y) - W(x+y)] - W'(y)\,[W(x) - W(x+y)] = W'(x+y)\,[W(x) - W(y)]$$

which we easily recognise as the addition formula for the Weierstrass $\wp$-function:

$$\begin{vmatrix} 1 & \wp(x) & \wp'(x) \\ 1 & \wp(y) & \wp'(y) \\ 1 & \wp(x+y) & -\wp'(x+y) \end{vmatrix} = 0 \ .$$

thus leading to the general solution of the functional equation: $W(x) = \beta\wp(x) + \gamma$ (up to a (trivial) scaling freedom in the argument x). Thus, we recover precisely the integrable cases of Calogero-Moser potentials from the Lagrangian 1-form structure!

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Relativistic case

Here the Lagrangian components L_1 and L_2 of the 1-form action are:

$$
\begin{aligned}
L_1 &= \sum_{i=1}^{N} \frac{\partial X_i}{\partial t_1} \ln\left|\frac{\partial X_i}{\partial t_1}\right| + \sum_{i\neq j}^{N} \frac{\partial X_j}{\partial t_1}\, V(X_i - X_j)\ , \\
L_2 &= \sum_{i=1}^{N} \left(\frac{\partial X_i}{\partial t_2} \ln\left|\frac{\partial X_i}{\partial t_1}\right| + \alpha \left(\frac{\partial X_i}{\partial t_1}\right)^2 + \beta \frac{\partial X_i}{\partial t_2} \right) \\
&\quad + \sum_{i\neq j}^{N} \left[\frac{\partial X_j}{\partial t_2}\, W(X_i - X_j) + \frac{\partial X_i}{\partial t_1}\frac{\partial X_j}{\partial t_1}\, U(X_i - X_j) \right]\ .
\end{aligned}
$$

The EL equations for the t_1-component, $\frac{\partial L_1}{\partial X_i} - \frac{\partial}{\partial t_1}\left(\frac{\partial L_1}{\partial(\partial X_i/\partial t_1)}\right) = 0$, gives:

$$
\frac{\partial^2 X_i}{\partial t_1^2} = \sum_{j\neq i} \partial X_i/\partial t_1 \, \frac{\partial X_j}{\partial t_1}\, \left[V'(X_i - X_j) - V'(X_j - X_i)\right]\ ,
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\frac{\partial^2 X_i}{\partial t_1\, \partial t_2} &= \sum_{j\neq i} \frac{\partial X_i}{\partial t_1}\, \frac{\partial X_j}{\partial t_2}\, \left[W'(X_i - X_j) - W'(X_j - X_i)\right] \\
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$$\frac{\partial L_2}{\partial(\frac{\partial X_i}{\partial t_1})} = 2\alpha\frac{\partial X_i}{\partial t_1} + \frac{\partial X_i}{\partial t_2}\Big/\frac{\partial X_i}{\partial t_1} + \sum_{j\neq i=1}^{N}\frac{\partial X_j}{\partial t_1}\left[U(X_i - X_j) + U(X_j - X_i)\right] = 0\,.$$

Taking its derivative with respect to t_1 and eliminating the second order derivatives using the EL eqs. we get a condition of the form:

$$\sum_{j\neq i}\left[A(X_i - X_j)\left(\frac{\partial X_i}{\partial t_1}\right)^2\frac{\partial X_j}{\partial t_1} + B(X_i - X_j)\frac{\partial X_i}{\partial t_1}\left(\frac{\partial X_j}{\partial t_1}\right)^2\right] + \sum_{\substack{j,l\neq i\\ j\neq l}} C(X_i, X_j, X_l)\frac{\partial X_i}{\partial t_1}\frac{\partial X_j}{\partial t_1}\frac{\partial X_l}{\partial t_1}\,,$$

where the coefficients A, B and C have to vanish identically. The conditions $A = 0$, $B = 0$ lead to the relations:

$$\begin{aligned} &2[U'(x) - U'(-x)] + 4\alpha[V'(x) - V'(-x)] \\ &\qquad = [U(x) + U(-x)]\,[W'(x) - W'(-x) + V'(x) - V'(-x)] \\ &[U'(x) - U'(-x)] + 2\alpha[W'(x) - W'(-x)] \\ &\qquad = [U(x) + U(-x)]\,[V'(x) - V'(-x)] \end{aligned}$$

respectively, from which by eliminating the U' terms we obtain:

$$[4\alpha + U(x) + U(-x)]\,\left[V'(x) - V'(-x) - W'(x) + W'(-x)\right] = 0\,.$$

Furthermore, we have the constraint:

$$\frac{\partial L_2}{\partial(\frac{\partial X_i}{\partial t_1})} = 2\alpha\frac{\partial X_i}{\partial t_1} + \frac{\partial X_i}{\partial t_2}\Big/\frac{\partial X_i}{\partial t_1} + \sum_{j\neq i=1}^{N}\frac{\partial X_j}{\partial t_1}\left[U(X_i - X_j) + U(X_j - X_i)\right] = 0\,.$$

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$$\sum_{j\neq i}\left[A(X_i - X_j)\left(\frac{\partial X_i}{\partial t_1}\right)^2\frac{\partial X_j}{\partial t_1} + B(X_i - X_j)\frac{\partial X_i}{\partial t_1}\left(\frac{\partial X_j}{\partial t_1}\right)^2\right] + \sum_{\substack{j,l\neq i\\ j\neq l}} C(X_i, X_j, X_l)\frac{\partial X_i}{\partial t_1}\frac{\partial X_j}{\partial t_1}\frac{\partial X_l}{\partial t_1}\,,$$

where the coefficients A, B and C have to vanish identically. The conditions $A = 0$, $B = 0$ lead to the relations:

$$2[U'(x) - U'(-x)] + 4\alpha[V'(x) - V'(-x)] = [U(x) + U(-x)]\,[W'(x) - W'(-x) + V'(x) - V'(-x)]$$

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respectively, from which by eliminating the U' terms we obtain:

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The condition $C = 0$ leads to the following functional relation for the potentials (where $z = x + y$)

$$\begin{aligned} &[U(x)+U(-x)]\left[V'(y)-V'(-y)+V'(z)-V'(-z)\right] \\ &+[U(z)+U(-z)]\left[V'(x)-V'(-x)-V'(y)+V'(-y)\right] \\ &=[U(y)+U(-y)]\left[W'(x)-W'(-x)+W'(z)-W'(-z)\right] \ , \end{aligned}$$

For the first option, case **i)**, leads to the condition

$$W(x)+W(-x) = -2(V(x)+V(-x))+\text{const.} \ ,$$

and inserting this into the above relation yields:
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In the second option case **ii)**, the system reduces to the equation

$$[U(x)+U(-x)-2\alpha]\left[V'(x)-V'(-x)\right] = U'(x)-U'(-x) \ ,$$

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Introducing the function $\Phi(x) := V(x) + V(-x)$ we obtain the functional relation

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with $z = x + y$. A solution of this functional equation is given by

$$\Phi(x) = -\ln\left(a + b\wp(x)\right) \ , \quad a, b \ \text{ constants} \ .$$

which corresponds to the full elliptic case of the Ruijsenaars flows. It can be shown by explicit computation that the closure relation

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Conclusion: We thus see, that by posing general forms for the potentials in the higher-order Lagrangians, but specific forms for the kinetic terms, the Lagrangian 1-form structure selects the integrable cases.

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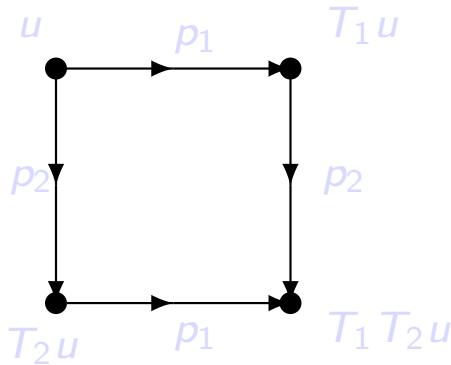
Multidimensional consistency on the 2D lattice

quadrilateral PΔEs on the 2D lattice:

$$\mathbf{Q}(u, T_1u, T_2u, T_1T_2u; p_1, p_2) = 0$$



notation of shifts on the elementary quadrilateral on a rectangular lattice:

$u := u(n_1, n_2)$, $T_1u = u(n_1+1, n_2)$

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Consistency-around-the cube:

Verifying consistency: Values at the black disks are initial values, values at open circles are uniquely determined from them, but there are three different ways to compute $T_1T_2T_3u$.

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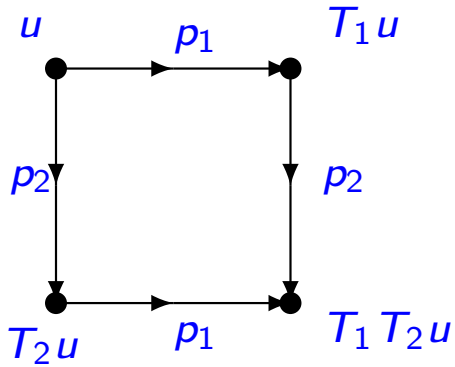

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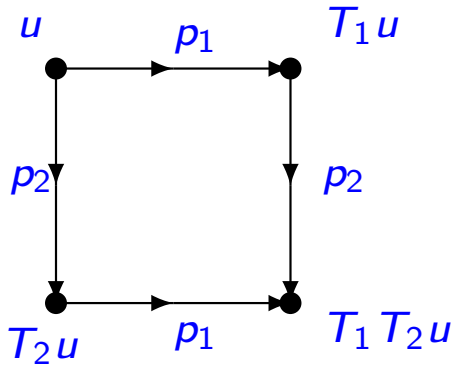

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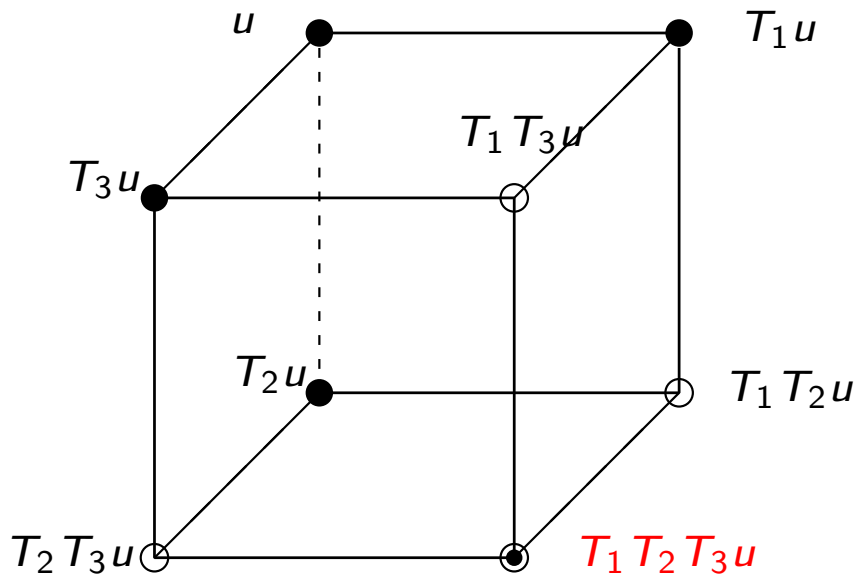

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Conventional variational formalism: discrete Euler-Lagrange equations

Define an action functional:

$$S[u(n_1, n_2)] = \sum_{n_1, n_2 \in \mathbb{Z}} \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) \ .$$

Following the usual *least-action principle*, the lattice equations for u are determined by the demand that S attains a minimum under local variations $u(n_1, n_2) \to u(n_1, n_2) + \delta u(n_1, n_2)$. Thus,

$$\begin{aligned} \delta S \quad = \quad & \sum_{n_1, n_2 \in \mathbb{Z}} \left\{ \frac{\partial}{\partial u} \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) \delta u + \frac{\partial}{\partial T_1 u} \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) \delta(T_1 u) \right. \\ & \left. + \frac{\partial}{\partial T_2 u} \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) \delta(T_2 u) \right\} = 0 \end{aligned}$$

Setting $\delta(T_i u) = T_i \delta u$, and resumming each of the terms we get:

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(ignoring boundary terms) and since δu is arbitrary the discrete Euler-Lagrange (EL) equation follow:

$$\begin{aligned} & \frac{\partial}{\partial u} \Big(\mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) + \mathscr{L}(T_1^{-1} u, u, T_1^{-1} T_2 u; p_1, p_2) \\ & \qquad + \mathscr{L}(T_2^{-1} u, T_1 T_2^{-1} u, u; p_1, p_2) \Big) = 0 \end{aligned}$$

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Conventional variational formalism: discrete Euler-Lagrange equations

Define an action functional:

$$S[u(n_1, n_2)] = \sum_{n_1, n_2 \in \mathbb{Z}} \mathscr{L}(u, T_1 u, T_2 u; p_1, p_2) \ .$$

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Example: H1

The equation is

$$Q(u, T_1u, T_2u, T_1T_2u; p_1, p_2) = (u - T_1T_2u)(T_1u - T_2u) + p_1^2 - p_2^2 = 0$$

The equation in the "3-leg form" is

$$(u + T_1u) - (u + T_2u) + \frac{p_1^2 - p_2^2}{u - T_1T_2u} = 0$$

The corresponding 3-point Lagrangian is given as[3]

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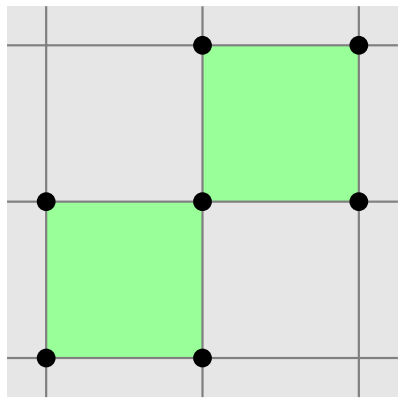

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Closure relation and Lagrangian multiform structure

- MDC implies that on *the same dependent variable* $u = u(n_1, n_2, n_3, \dots, p_1, p_2, p_3, \dots)$ one can simultaneously impose a multitude of compatible PΔEs, $Q_{i,j} = 0$ in any pair of independent variables n_i, n_j.
- **The aim:** To give a complete Lagrangian description of multidimensional consistent (MDC) integrable systems.
- **problem:** Conventional least-action principle yields only one single equation (per component of the dependent variable u) as EL equation, giving the "dynamical" equation of the system;
- Thus, we need a new variational approach enabling the derivation of the multitude of equations of the MDC system.
- **Answer:** The Lagrangians will become extended objects, i.e. components of a *Lagrangian multiform*.

The existence and consistency of this approach resides on the following key observation:

Proposition: MDC systems of lattice equations, possess Lagrangians which obey the following closure relation:

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Here $\Delta_i = T_i - \mathrm{id}$ denotes the difference operator, i.e.. on functions f of $u = u(n_1, n_2, n_3)$ we have: $\Delta_i f(u) = f(T_i u) - f(u)$.

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- MDC implies that on *the same dependent variable* $u = u(n_1, n_2, n_3, \dots, p_1, p_2, p_3, \dots)$ one can simultaneously impose a multitude of compatible PΔEs, $Q_{i,j} = 0$ in any pair of independent variables n_i, n_j.
- **The aim:** To give a complete Lagrangian description of multidimensional consistent (MDC) integrable systems.
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- Thus, we need a new variational approach enabling the derivation of the multitude of equations of the MDC system.
- **Answer:** The Lagrangians will become extended objects, i.e. components of a *Lagrangian multiform*.

The existence and consistency of this approach resides on the following key observation:

Proposition: MDC systems of lattice equations, possess Lagrangians which obey the following closure relation:

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Closure relation for other cases

- In a similar way the cases of H2, Q1 and A1 can be verified[4].
- The cases H1 and $(Q1)_{\delta=0}$ involve $F(u) = \ln u$, the cases H2 and $(Q1)_{\delta\neq 0}$ and A1 involve $F(u) = u \ln u$.
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$$\begin{aligned}\mathrm{Li}_2\left(\frac{x}{1-y}\frac{y}{1-x}\right) \quad = \quad & \mathrm{Li}_2\left(\frac{x}{1-y}\right) + \mathrm{Li}_2\left(\frac{y}{1-x}\right) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) \\ & - \ln(1-x)\ln(1-y)\end{aligned}$$

- For Q2, Q3, Q4 it is more difficult to verify closure by direct computation due to the implicit nature of those cases. Also for Q4 it requires new functional identities for an elliptic dilogarithm function:

$$F(u) = \int^u \ln(\sigma(x))\, dx$$

- Bobenko and Suris[5] showed subsequently that the closure relation holds for all quadrilateral equations in the ABS list.

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[4] S B Lobb and F.W. Nijhoff. Lagrangian multiforms and multidimensional consistency. *Journal of Physics A: Mathematical and Theoretical*. **42** (2009) 454013.

[5] A I Bobenko and Yu B Suris, Lett. Math. Phys. **92** (2010) 1731.

Universal Lagrangian

For quadrilateral *affine-linear* equations for scalar dependent variable $u = u(\mathbf{n})$

$$Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u, u_i, u_j, u_{ij}) = 0 \quad , \quad u_i := u(\mathbf{n} + \mathbf{e}_i) \ , \quad u_{ij} := u(\mathbf{n} + \mathbf{e}_i + \mathbf{e}_j) \ ,$$

where the function possesses the symmetries of the square:

$$Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u, u_i, u_j, u_{ij}) = Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u_i, u, u_{ij}, u_j) = Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u_j, u_{ij}, u, u_i) = -Q_{\mathfrak{p}_j,\mathfrak{p}_i}(u, u_j, u_i, u_{ij})$$

let us introduce the biquadratic functions:

$$\begin{aligned} Q_{u_j} Q_{u_{ij}} - Q\, Q_{u_j u_{ij}} &=: K_{\mathfrak{p}_i,\mathfrak{p}_j} h_{\mathfrak{p}_i}(u, u_i) \\ Q_{u_i} Q_{u_{ij}} - Q\, Q_{u_i u_{ij}} &=: K_{\mathfrak{p}_j,\mathfrak{p}_i} h_{\mathfrak{p}_j}(u, u_j) \\ Q_{u} Q_{u_{ij}} - Q\, Q_{u u_{ij}} &=: -K_{\mathfrak{p}_i,\mathfrak{p}_j} h_{\mathfrak{p}_{ij}}(u_i, u_j) \end{aligned}$$

where $K_{\mathfrak{p},\mathfrak{q}} = -K_{\mathfrak{q},\mathfrak{p}}$ is a function of the lattice parameters $\mathfrak{p}$, $\mathfrak{q}$ only. We now introduce the Lagrangian[6]

$$\begin{aligned} \mathscr{L}(u, u_i, u_j) \quad = \quad & \int_{u^0}^{u} \int_{u_i^0}^{u_i} \frac{dx\, dy}{h_{\mathfrak{p}_i}(x, y)} - \int_{u^0}^{u} \int_{u_j^0}^{u_j} \frac{dx\, dy}{h_{\mathfrak{p}_j}(x, y)} - \int_{u_i^0}^{u_i} \int_{u_j^0}^{u_j} \frac{dx\, dy}{h_{\mathfrak{p}_{ij}}(x, y)} \\ & + \int_{u_i^0}^{u_i} dx \int_{u_j^0}^{Y(u^0, x, u_{ij}^0)} \frac{dy}{h_{\mathfrak{p}_{ij}}(x, y)} + \int_{u_j^0}^{u_j} dy \int_{u_i^0}^{X(u^0, y, u_{ij}^0)} \frac{dx}{h_{\mathfrak{p}_{ij}}(x, y)} \end{aligned}$$

where the functions X and Y are solutions of the equations

$$Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u^0, x, Y, u_{ij}^0) = 0 \quad \text{respectively} \quad Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u^0, X, y, u_{ij}^0) = 0 \ .$$

[6] **P. Xenitidis, F.W. Nijhoff and S. Lobb**, *On the Lagrangian formulation of multidimensionally consistent systems*, (Proc. Roy. Soc. in press).

Interplay Discrete $\longleftrightarrow$ Continuous

The lattice systems we consider here admit a role reversal:
lattice *parameters* $p, q, \ldots$ $\leftrightarrow$ lattice *variables* $n, m, \ldots$
For all quadrilateral PΔEs we have *fully consistent system of equations* comprising three types of equations, all compatible discrete as well as continuous.

PΔE $\leftrightarrow$ DΔE $\leftrightarrow$ PDE

These D(Δ)Es can be simultaneously imposed on the same dependent variables:

$$u = u(n_1, n_2, n_3, \ldots ; p_1, p_2, p_3, \ldots)$$

and possess the property of multidimensional consistency.
The consistency can be expressed as the condition that on solutions of these equations the operators $(T_i u)(\ldots, n_i, \ldots) = u(\ldots, n_i + 1, \ldots)$, mutually commute in all lattice directions and with the differential operators $\partial_{p_i}, \partial_{p_j}, \ldots$, i.e. among these equations hold on the solutions

$$T_i T_j u = T_j T_i u \quad , \quad \frac{\partial}{\partial p_i}\left(\frac{\partial u}{\partial p_j}\right) = \frac{\partial}{\partial p_j}\left(\frac{\partial u}{\partial p_i}\right) \quad , \quad T_i\left(\frac{\partial u}{\partial p_j}\right) = \frac{\partial}{\partial p_j} T_i u \ .$$

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Differential-difference equations: master-symmetries

DΔE The DΔE analogue of H1 is given by the relations:

$$\frac{\partial u}{\partial p_i} = \frac{2n_i p_i}{T_i u - T_i^{-1} u}$$

It can be shown that this equation is compatible with H1.

A Lagrangian for the DΔE is given by

$$\mathscr{L}_i = u \frac{\partial}{\partial p_i} T_i u + 2n_i p_i \log\left(\frac{\partial}{\partial p_i} T_i u\right)$$

PDE The following coupled system of PDEs follows from the DΔE and the PΔE:

$$\frac{\partial^2 u}{\partial p_i \, \partial p_j} = \frac{1}{p_i^2 - p_j^2}\left[2n_i p_i \frac{\partial u}{\partial p_j} - 2n_j p_j \frac{\partial u}{\partial p_i} - 2v_{ij} \frac{\partial u}{\partial p_i}\frac{\partial u}{\partial p_j}\right],$$

$$\frac{\partial^2 v_{ij}}{\partial p_i \, \partial p_j} = \frac{\partial}{\partial p_i}\left[\frac{(\partial u/\partial p_j) v_{ij}^2 + 2n_j p_j v_{ij}}{p_i^2 - p_j^2}\right] + \frac{\partial}{\partial p_j}\left[\frac{(\partial u/\partial p_i) v_{ij}^2 - 2n_i p_i v_{ij}}{p_i^2 - p_j^2}\right],$$

where $v_{ij} = (T_i - T_j)u$.

It derives as EL equations from the following Lagrangian:

$$\mathscr{L}_{ij} = v_{ij}\left(\frac{\partial^2 u}{\partial p_i \, \partial p_j} - \frac{2n_i p_i (\partial u/\partial p_j) - 2n_j p_j (\partial u/\partial p_i)}{p_i^2 - p_j^2}\right) + \frac{1}{p_i^2 - p_j^2} v_{ij}^2 \frac{\partial u}{\partial p_i}\frac{\partial u}{\partial p_j} .$$

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Scalar Lagrangian: "hierarchy generating PDE"

Here the variational equations are given by

$$\frac{\delta \mathscr{L}_{ij}}{\delta u} = 0 \quad , \quad \frac{\delta \mathscr{L}_{ij}}{\delta v_{ij}} = 0 \ ,$$

The latter Lagrangian obeys the following *closure relation*

$$\frac{\mathscr{L}_{ij}}{\partial p_k} + \frac{\mathscr{L}_{jk}}{\partial p_i} + \frac{\mathscr{L}_{ki}}{\partial p_j} = 0 \ ,$$

where the "form field" v_{ij} is subject to:

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Alternatively, we can have a higher-degree Lagrangian of the form

$$\mathscr{L}_{ij} = \frac{1}{4}(p_i^2 - p_j^2)\frac{(\partial_{p_i}\partial_{p_j} u)^2}{(\partial_{p_i} u)\partial_{p_j} u} + \frac{1}{p_i^2 - p_j^2}\left(n_i^2 p_i^2 \frac{\partial_{p_j} u}{\partial_{p_i} u} + n_j^2 p_j^2 \frac{\partial_{p_i} u}{\partial_{p_j} u}\right) \ ,$$

in terms of only u and its derivatives. The Euler-Lagrange equation:

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The closure relation suggests that the $_{ij}$ should be considered as components of a *Lagrangian 2-form*, which is closed but only *on the solutionsof the equations of the motion!*

Thus, we can define an action functional

$$S[u(p_1, p_2, p_3, \dots); \sigma] = \int_\sigma \sum_{i,j} \mathscr{L}_{ij} \mathrm{d}p_i \wedge \mathrm{d}p_j$$

to be evaluated for any field configuration u associated with any (smooth) surface σ in the space of 3 (or more) independent variables.
The closure relation, due to Stokes' theorem, expresses the fact that S is independent of the surface σ (with given boundaries) on solutions of the equations of the motion.
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The closure relation, due to Stokes' theorem, expresses the fact that S is independent of the surface σ (with given boundaries) on solutions of the equations of the motion.
The derivation of the relevant set of EL equations is in progress[7]

[7] S.Lobb & F.W.Nijhoff, in preparation.

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Discrete 2-forms

We now need the notion of a discrete Lagrangian 2-form.
These are oriented expressions of the form:

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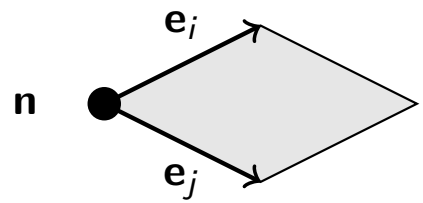

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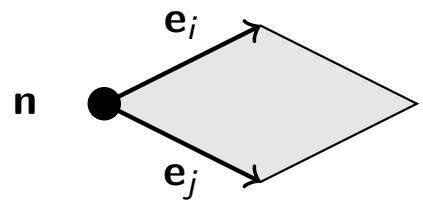

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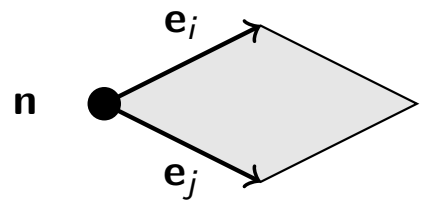

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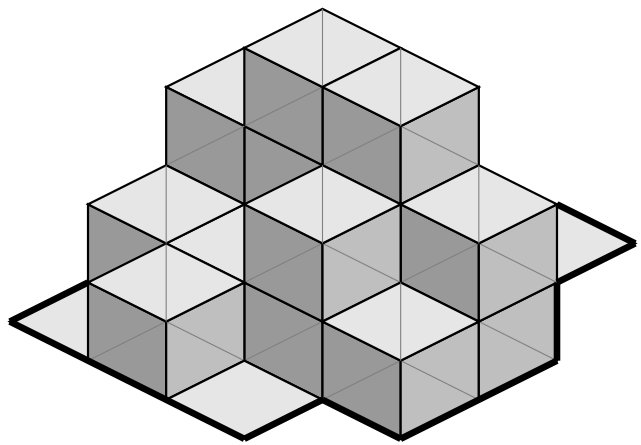

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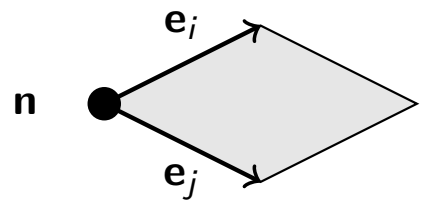

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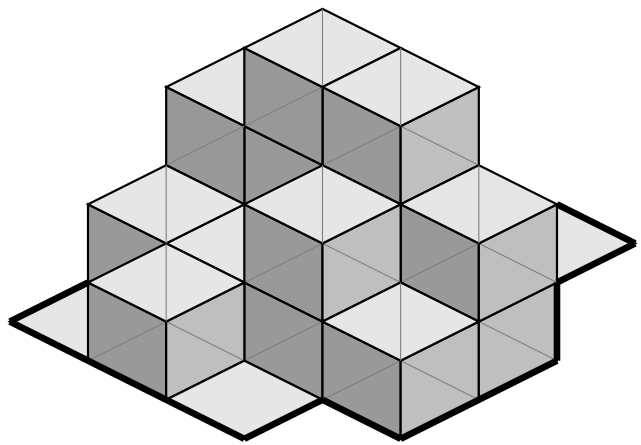

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Basic configurations for 2D lattice EL system

The following configurations constitute the elementary surface actions from which the system of 2D discrete EL equations for 2-forms are derived.

Figure: Flat 2d space.

This represents the "planar" EL equations.
The following represent EL over curved surfaces:

Figure: Configurations when embedded in 3d.

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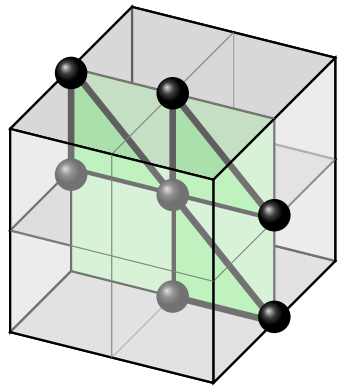

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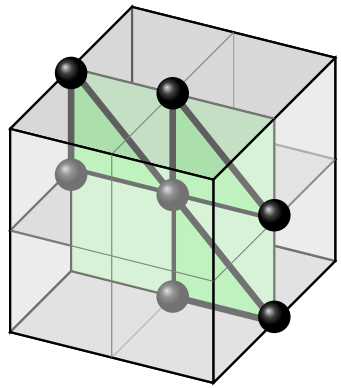

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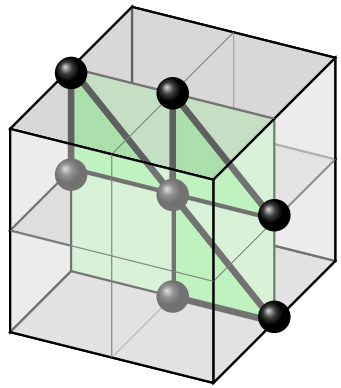

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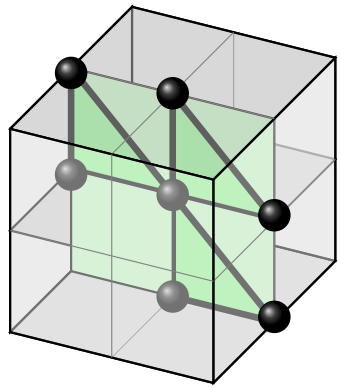

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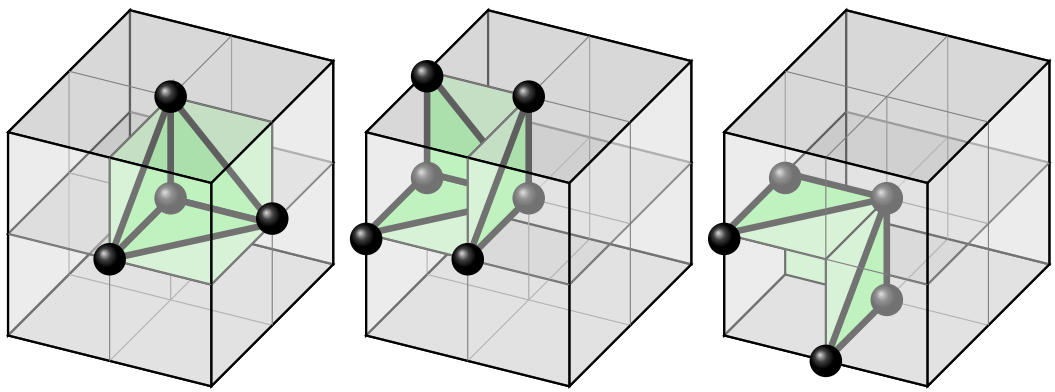

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System of EL eqs for 2-forms

Proposition: For any action which is the sum of 3-point Lagrangians $L(u, T_i(u), T_j(u); \alpha_i, \alpha_j)$ in the multidimensional regular lattice, the configurations given above, involving the arbitrary point u, represent an exhaustive list of elementary actions, the EL for which are given by the system of equations:

$$\frac{\partial}{\partial u}\Big(L(T_i^{-1}u, u, T_i^{-1}T_j u; p_i, p_j) + L(u, T_i u, T_j u; p_i, p_j) + L(T_j^{-1}u, T_i T_j^{-1}u, u; p_i, p_j)\Big) = 0,$$

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Equivalent configurations

The following diagrams represent elementary surfaces different from the earlier, but which give equivalent sets of EL equations:

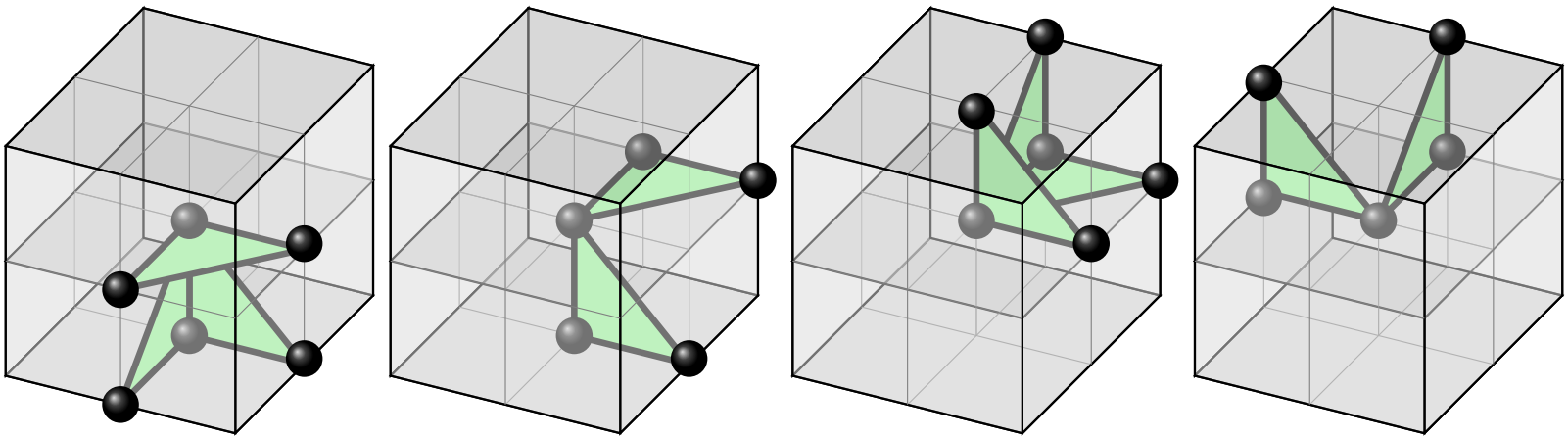

Figure: Equivalent configurations.

Discussion

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- The main motivation is to formulate a least-action principle that produces the whole system of multidimensionally consistent equations, rather than a single equation of the motion.
- This new variational principle brings in an essential way *the geometry of the independent variables*. In fact, the closure arises from variations w.r.t. independent variables as functions of the parameters.
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- There is an important interplay between compatible *continuous* and *discrete* structures, i.e. compatibility between discrete systems of OΔEs and of ODEs.
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- The explicit examples we have studied so far, seem to indicate that the relevant variational principle for integrable (multidimensionally consistent) systems is that of *Lagrangian multiforms* rather than of scalar Lagrangians.
- The main motivation is to formulate a least-action principle that produces the whole system of multidimensionally consistent equations, rather than a single equation of the motion.
- This new variational principle brings in an essential way *the geometry of the independent variables*. In fact, the closure arises from variations w.r.t. independent variables as functions of the parameters.
- The variational principle determines not only the equations for the classical trajectories of the system, but also the *admissable Lagrangians*. In fact, the integrable cases are selected by the structure. In fact, the Lagrangians are solutions of a system of eqautions, comprising the EL equations, the constraints and the closure.
- There is an important interplay between compatible *continuous* and *discrete* structures, i.e. compatibility between discrete systems of OΔEs and of ODEs.
- Implications of the new variational principle for corresponding quantum problems, and the role that Lagrangian multi-form structures play there (in the spirit of Dirac & Feynman) is under development.