Qualitative analysis of weakly coupled Landau- Lifshitz equations with spin-transfer torque terms

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Conference on Nonlinear Mathematical Physics:
Twenty Years of JNMP,
June 4 – June 14, 2013
Nordfjordeid, Norway

A handful of references

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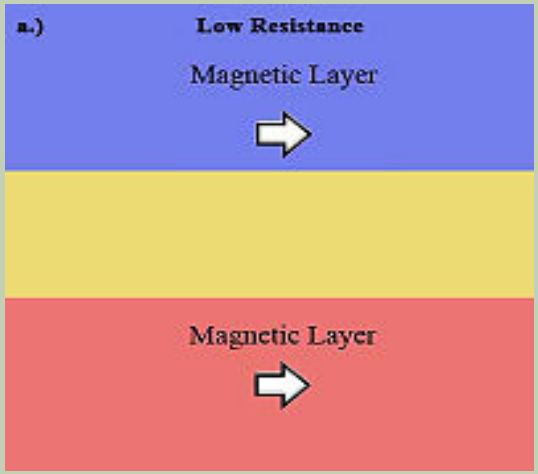
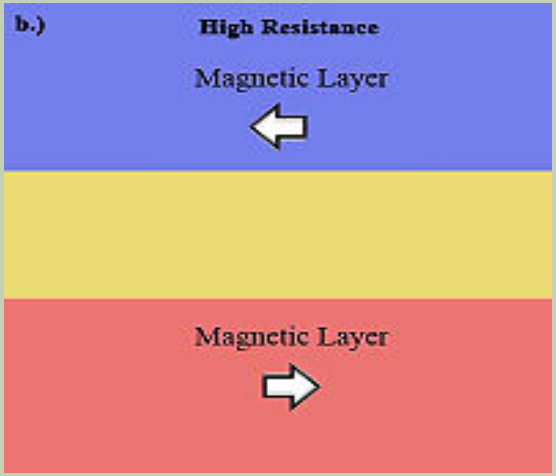
Outline

- Coupled Landau-Lifshitz- Slonczewski model (CLLS) of magnetization dynamics
- Reduced version of CLLS: four-dimensional system
- **Analytic** stability criterion for equilibrium states of the 4D system – two proofs
- Another reduction of CLLS: slow-fast 3D system
- Circular limit cycles - critical points of the 3D system
- Noncircular limit cycles – averaging technique
- Finding Lyapunov characteristic exponents: algorithm and computations
- Summary

Motivation

Spintronics: Neologism for Spin Transport Electronics
Spin valve: a nanometer scale device with at least two magnetic layers. Its electrical resistance depends on relative alignments of the ***magnetization*** of the two (or more) layers
Spin-transfer torque: effect in which the orientation of a magnetic layer in a spin valve can be modified using a spin-polarized current. Has potential applications in RAM (STT-RAM)





Practical implementation issues: stability and efficient reading and writing at acceptably small currents ⟶ better understanding of the magnetization dynamics and optimization of device geometry is needed

CLLS model (a glimpse of physics)

$$\frac{d\mathbf{s}_j}{d\tau} = \mathbf{s}_j \times [\gamma(\mathbf{H}_j + \alpha_j \mathbf{s}_j \times \mathbf{H}_j) - \beta_j \mathbf{s}_2 \times \mathbf{s}_1] \quad (*)$$

$$j = 1, 2, \mathbf{s}_j = [x_j, y_j, z_j]$$

ASSUMPTION:

Uniaxial anisotropy, Z is the anisotropy axis

Notation:

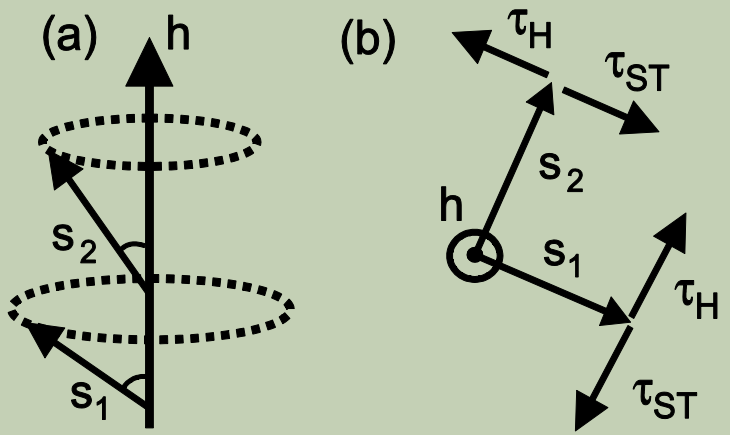
$\mathbf{H}_j$, j=1, 2: effective magnetic fields

$$\mathbf{H}_j = \mathbf{H}_a + d_j(\mathbf{s}_j \cdot \mathbf{e}_3)\mathbf{e}_3, \quad \mathbf{H}_a = H\mathbf{e}_3$$

H, γ, α_j,d_j: physical parameters, α_j>0

β_j: control parameters (involve applied DC)

α_j,β_j are small, ~$O(10^{-2})$



Schematics of precessional motion of two nanomagnets

Conventional model: $\mathbf{s}_1$=const (fixed magnetization of layer 1)

Physics lingo: reference (fixed, pinned) layer, active (free) layer.

Eqs (*) imply that the magnitudes of $\mathbf{s}_j$ are conserved

⟹ **Only four scalar equations out of the six in (*) are independent**

Phase space: $S^2 \times S^2$

CLLS model (mathematics)

Change $t = -\gamma |d_2| \tau$ and division by $-\gamma |d_2|$ give a dimensionless version of (*):

$$\frac{d}{dt}\mathbf{s}_j = -\mathbf{s}_j \times \left[\mathbf{h}_j + a_j \mathbf{s}_j \times \mathbf{h}_j - b_j \mathbf{s}_2 \times \mathbf{s}_1 \right] \quad \text{(CLLS)}$$

$$h_1 = h_1(z_1) = h + d\, z_1,\ h_2 = h_2(z_2) = h + s\, z_2,\ d = \frac{d_1}{|d_2|},\ s = \text{sign}\ (d_2)$$

Four equilibrium states: (0, 0, ±1; 0, 0, ±1)

Scalar form of the conventional model with $\mathbf{s}_1$=[0,0,1], and $\mathbf{s}_2$=[x,y,z]:

$$\frac{dx}{dt} = -(h + sz)(y - a_2 xz) + b_2 xz$$

$$\frac{dy}{dt} = (h + sz)(x - a_2 yz) + b_2 yz$$

$$\frac{dz}{dt} = (1 - z^2)(a_2(h + sz) - b_2)$$

Observation: matrix *A* of the linearization of x- and y-equations at any of the fixed points (0, 0, ±1) shows complex structure. For (0,0,1), ***A*=*Z***, where

$$Z = \begin{bmatrix} -a_2 h_2 + b_2 & -h_2 \\ h_2 & -a_2 h_2 + b_2 \end{bmatrix},\ h_2 = h_2(1) \quad \Rightarrow$$

Asymptotic stability criterion:

$$\mathbf{Re}(z) < \mathbf{0},\ z = -a_2 h_2 + b_2 + i\, h_2$$

4D reduced system and its linear stability analysis

4D system: choose equations for x_j and y_j in CLLS
Linearization of the 4D system at any of the four equilibrium points also reveals a complex structure

Example: Linearized 4D system at (0, 0, 1; 0, 0, 1)

$$\frac{d\mathbf{U}}{dt} = \mathbf{A}_{pp}\mathbf{U}, \quad \mathbf{U} = [u_1, \ldots, u_4]^T,$$

$$\mathbf{A}_{pp} = \begin{bmatrix} \mathbf{Z}_1 & b_1\mathbf{E}_2 \\ -b_2\mathbf{E}_2 & \mathbf{Z}_2 \end{bmatrix}, \quad \text{where} \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{Z}_1 = \begin{bmatrix} -(a_1h_1 + b_1) & -h_1 \\ h_1 & -(a_1h_1 + b_1) \end{bmatrix}, \mathbf{Z}_2 = \begin{bmatrix} -(a_2h_2 - b_2) & -h_2 \\ h_2 & -(a_2h_2 - b_2) \end{bmatrix},$$

$$h_j = h_j(1)$$

$\mathbf{A}_{pp}$ is a block matrix and the realification of a complex 2x2 matrix!

Stability criterion

The Stability Criterion (includes ALL parameters of CLLS!)

Let $M(0,0,z_{10};0,0,z_{20})$, $z_{10}=\pm 1, z_{20}=\pm 1$, be a fixed point;

$h_1=h_1(z_{10})$, $h_2=h_2(z_{20})$.

CLLS system (1) is asymptotically stable at M iff

$$\mathrm{Re}(z_1+z_2+w)<0, \text{ where } \quad w=\sqrt{(z_1-z_2)^2-4b_1b_2},$$

$$z_1=-z_{10}(a_1h_1+z_{20}b_1)+ih_1, \quad z_2=-z_{20}(a_2h_2-z_{10}b_2)+ih_2 \qquad (*)$$

Proof 1 relies on the following

Theorem. The spectrum $\sigma(A^R)$ of the *realification* A^R of a complex matrix A is the union $\sigma(A)\bigcup\bar{\sigma}(A)$.

Without loss of generality, consider matrix A_{pp}. It is the realification of the 2x2 complex matrix

$$A=\begin{bmatrix} z_1 & b_1 \\ -b_2 & z_2 \end{bmatrix} \text{ with } \sigma(A)=\frac{1}{2}(z_1+z_2\pm w) \text{ and } z_{10}=z_{20}=1.$$

If the argument of complex numbers is chosen between $-\pi$ and π, then $\mathrm{Re}(w)\geq 0$, and the criterion follows. ■

Analytic stability criterion – Proof 2

Proof 2 [4]. Idea: The linearized system can be transformed into complex **Riccati equation** in 4 steps, solved explicitly, and the criterion can be derived from the explicit solution

$$\mathrm{Tr1}: \quad \xi_1 = u_1 + iu_2, \quad \xi_2 = u_3 + iu_4$$

$$\frac{d\xi_1}{dt} = z_1\xi_1 + b_1\xi_2, \quad \frac{d\xi_2}{dt} = -b_2\xi_1 + z_2\xi_2.$$

$$\mathrm{Tr2}: \quad \xi_j = r_j(t)\exp(i\varphi_j(t)) \quad (r = r_2/r_1, \delta = \varphi_2 - \varphi_1)$$

$$\frac{d(\ln(r_1))}{dt} = -a_1h_1 - b_1(1 - r\cos(\delta)), \quad \frac{d\varphi_1}{dt} = \mathrm{Im}(z_1) + b_1 r\sin(\delta),$$

$$\frac{d(\ln(r_2))}{dt} = a_2h_2 - b_2(\cos(\delta)/r + 1), \quad \frac{d\varphi_2}{dt} = \mathrm{Im}(z_2) + b_2\sin(\delta)/r.$$

$$\Rightarrow \quad \begin{aligned} \frac{d\delta}{dt} &= \sin(\delta)(b_2/r - b_1 r) + \Delta h, \quad \Delta h = h_2 - h_1 \\ \frac{dr}{dt} &= -\cos(\delta)(b_2 + b_1 r^2) + r\,\mathrm{Re}(z_2 - z_1) \end{aligned}$$

Proof 2 cont'd

Tr3 … Tr4 …
Riccati equation:

$$\frac{dZ}{dt} = -b_1 Z^2 + (z_2 - z_1)Z - b_2, \text{ where } \quad Z = r\cos(\delta) + i\, r \sin(\delta)$$

$$\text{Solution:} \quad Z = \frac{1}{2b_1}\left(z_2 - z_1 + w \tanh(wt/2 + c)\right)$$

$$X = \mathrm{Re}(Z) = \frac{1}{2b_1}\left(\mathrm{Re}(z_2 - z_1) + w_1 \frac{\sinh(x)\cosh(x)}{\sinh^2(x) + \cos^2(x)} - w_2 \frac{\sin(y)\cosh(y)}{\sinh^2(x) + \cos^2(x)}\right) (*)$$

$$Y = \frac{1}{2b_1}\mathrm{Im}(Z) = \ldots, \quad w_1 = \mathrm{Re}(w),\ w_2 = \mathrm{Im}(w),\ x = t\, w_1/2 + c_1,\ y = t\, w_2/2 + c_2.$$

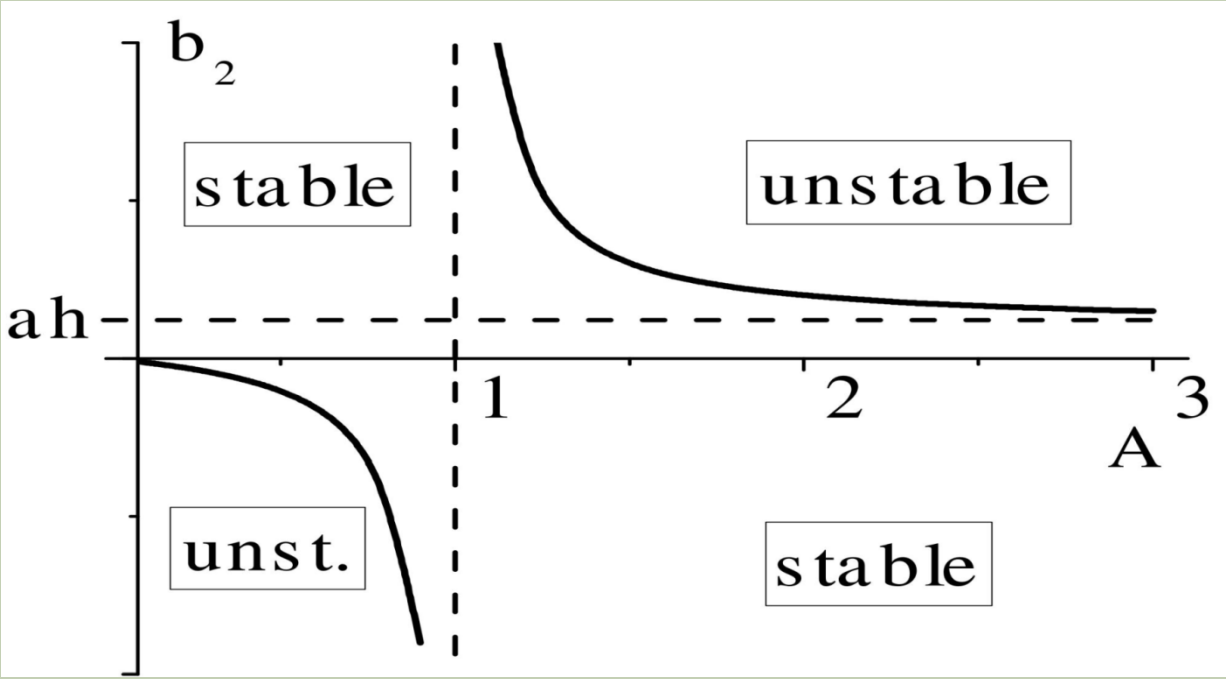
The stability condition can be obtained from Tr2 as

$$\mathrm{Re}(z_1) + b_1 \lim_{t\to\infty} X(t) < 0$$

Finding the limit from (*), $X_\infty = \mathrm{Re}(z_2 - z_1 + w)/(2b_1)$, completes the proof. ■

Explicit solution for the linearized system can be used for approximation of important parameters, relaxation time and frequency of oscillations.

Stability diagram: Identical layers [3]



Equilibrium point (0,0,1,0,0,1)

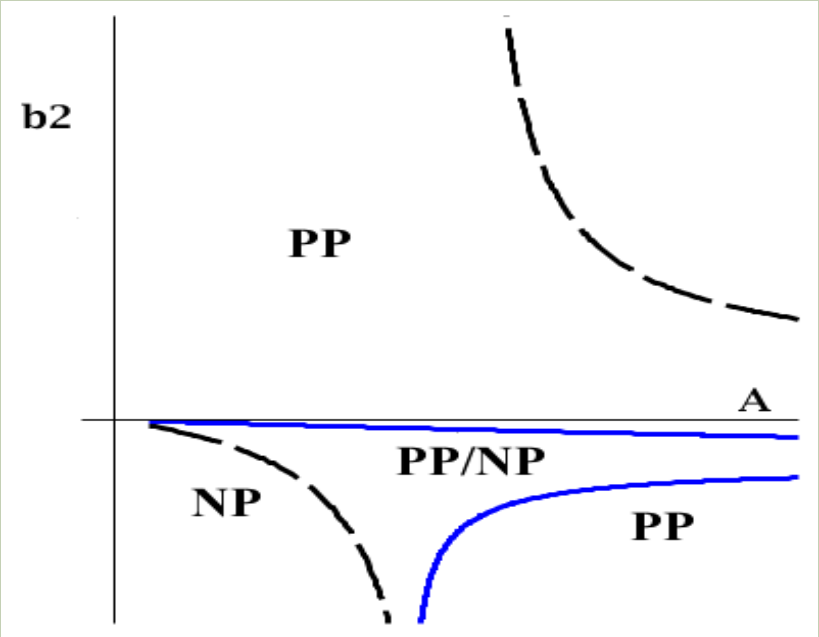
Stability condition: $b_2 - b_1 + |b_2 - b_1| - 2ah < 0$

$$\Rightarrow b_2 < \frac{ahA}{A-1}, A > 0; b_2 > \frac{ahA}{A-1}, 0 < A < 1$$

$$where\ A = b_2 / b_1$$

Superimposed stability diagrams

Parameters:
a1=a2,
s=1,d=1.8,
h=1.5



NP: Stability region for the state (0,0,-1,0,0,1) (dashed)
PP: Stability region for the state (0,0,1,0,0,1) (solid)
Bistability region is bounded by two solid curves and lower dashed curve

Another reduction of CLLS: 3D system

CLLS system in cylindrical coordinates (r_j, φ_j, z_j), $r_j = \sqrt{(1-z_j^2)}$, $j=1,2$:

$$\frac{d\varphi_1}{dt} = h + d\, z_1 + b_1 \sin(\delta)\, r_2 / r_1), \quad \frac{d\varphi_2}{dt} = h + s\, z_2 + b_2 \sin(\delta)\, r_1 / r_2)$$

$$dz_1 / dt = \ldots, \quad dz_2 / dt = \ldots$$

simplifies to a 3D slow-fast system

$$\frac{d\delta}{dt} = s\, z_2 - d\, z_1 + b_1 \sin(\delta) A r_1 / r_2 - r_2 / r_1)$$

$$\frac{dz_1}{dt} = b_1 \left(r_1^2 \left(k_1 h(z_1) + z_2 \right) - z_1 r_1 r_2 \cos(\delta) \right)$$

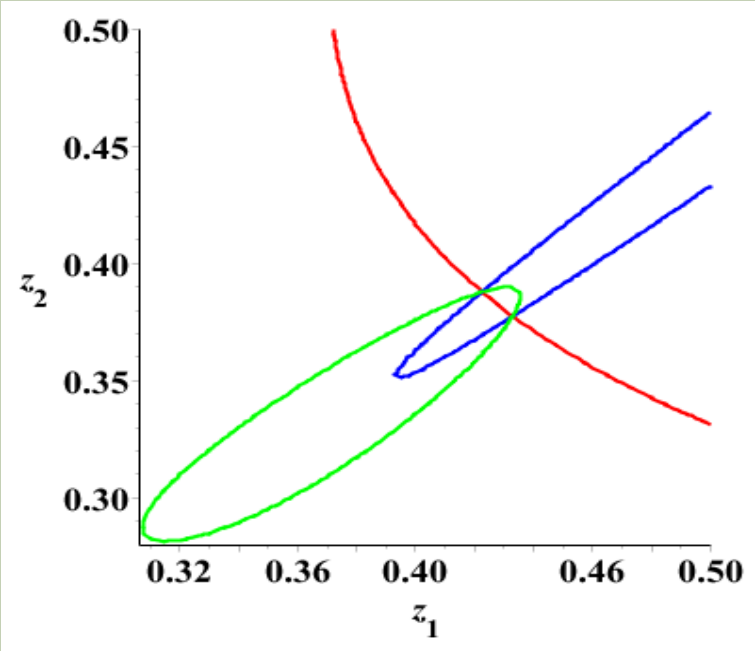
$$\frac{dz_2}{dt} = b_1 \left(r_2^2 \left(k_2 h_2(z_2) - A z_1 \right) + A z_2 r_1 r_2 \cos(\delta) \right)$$

$$r_j = \sqrt{1 - z_j^2}, \quad b_1 << 1, \quad k_j = a_j / b_1 = O(1)$$

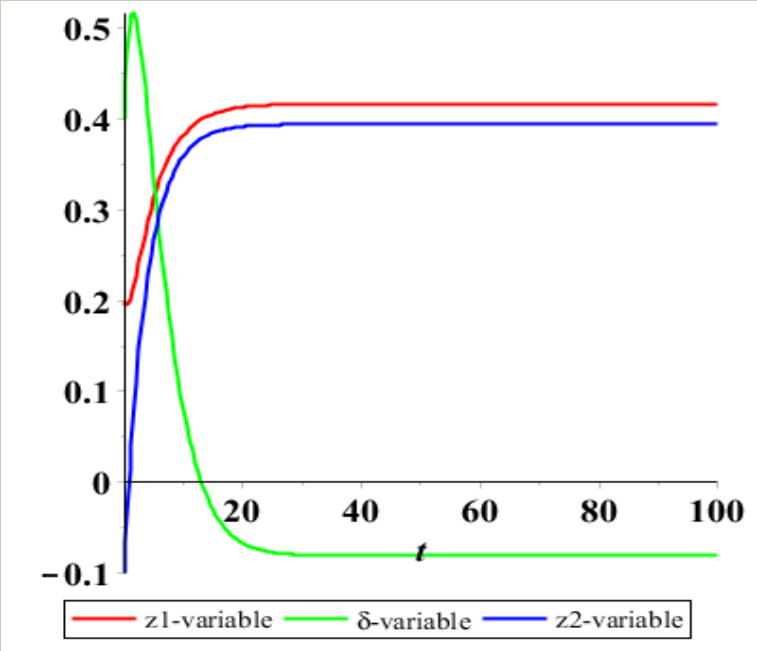
Circular limit cycles of CLLS are fixed points of the 3D system. They can be found numerically with high accuracy using reduction of the system (RHS(3D system)=0 to a system of polynomial equations.

Circular limit cycles

Example. For $k_1=k_2=1$, $b_1=0.03$, $A=0.2$, $s=-1$, $d=-0.9$, $h=0.4$, a stable fixed point of the 3D system is $z_1 \approx 0.422$, $z_2 \approx 0.388$, $\delta \approx -0.320$.



Visualization of solutions z_1, z_2 to a polynomial system obtained from equations RHS(3D system)=0.



Numerical solution to the 3D system with parameters specified in the example.

Noncircular limit cycles – averaging method

3D system is a multiscale system with slow variables z_1, z_2 and fast variable δ. RHS of the system is periodic in δ. Classic averaging scheme (Bogolubov-Mitropolski, [5]) applies successfully.

Algorithm

1. Let $b_1=\varepsilon$. Assume $z_1=u+\varepsilon f(u,v,\omega)$, $z_2=v+\varepsilon g(u,v,\omega)$, $\delta=\omega+\varepsilon q(u,v,\omega)$,(*) where f, g, and q are 2π-periodic in ω, and u, v, and ω solve the system

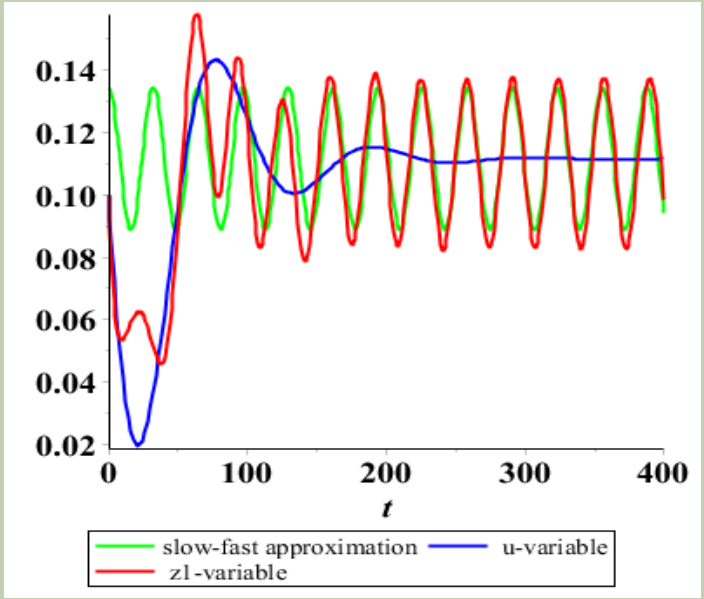
$$\frac{du}{dt}=\varepsilon\, M_1(u,v),\ \frac{dv}{dt}=\varepsilon\, M_2(u,v),\ \frac{d\omega}{dt}=s\,v-d\,u+\varepsilon\Omega(u,v) \quad (\ **)$$

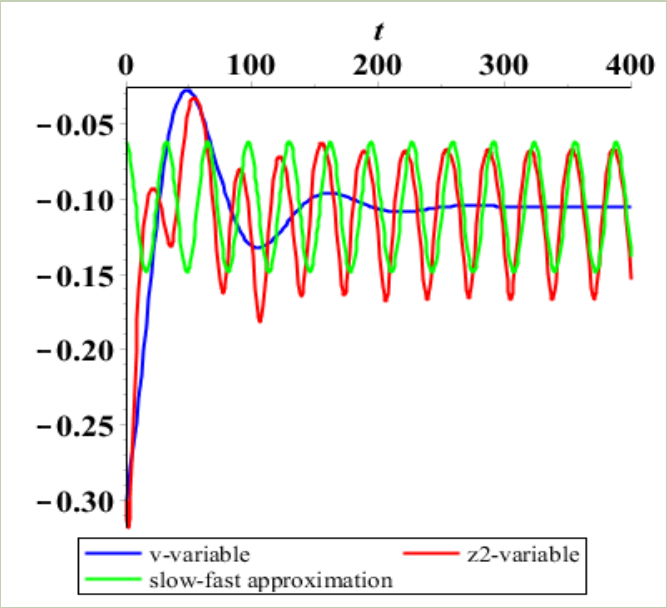
2. Equate the time derivatives of z_1, z_2, and δ found in two ways:
   i. by linearization of RHS in (2) at (u,v,ω) and
   ii. by differentiation of RHS in (*)
   to obtain a system (***) (not shown).
3. Average: integrate (***) w/r to ω over $[0,2\pi]$. Then find M_j and Ω.
4. Solve (**) for u,v, and ω. Then solve (***) for f, g, and q.

Exact solution of equations for u and v in (**) with M_j found on step 3 is not available. On step 4, a stable fixed point (u^*,v^*) is used instead.

Averaging for 3D system: Example

Example. Parameters: k_1=0.5, k_2=0.55, d=-0.8, h=0.3, b_1=0.04, h=0.3, A=0.2.
Fixed point of (**) system: u*= 0.1114864865, v*=-0.1054054054.





Left: Superimposed plots of numerical solution z_1 for 3D system (red), u for system (**) (blue), and a slow-fast approximation of z_1 in (*) (green); **Right:** same for the variable z_2. Clearly, the slow-fast approximations of z_1, z_2 are in good agreement with numerical solutions. (Some 'rule of thumb' for adjustments of phases was used.)

Finding Lyapunov characteristic exponents

Definition of chaoticity (one of many) [6]. A dynamical system is chaotic if it has a positive Lyapunov exponent.
Lyapunov exponents are defined by the long-term evolution of the axes of an infinitesimal sphere of states centered at a point in the phase space.

Computational Algorithm [7] (just one step)
Given a system of n nonlinear ODEs, a point X_0 in the phase space, and an orthonormal frame F_0 anchored to the point,
1. Find the solution $X=X(t,X_0)$ of the system.

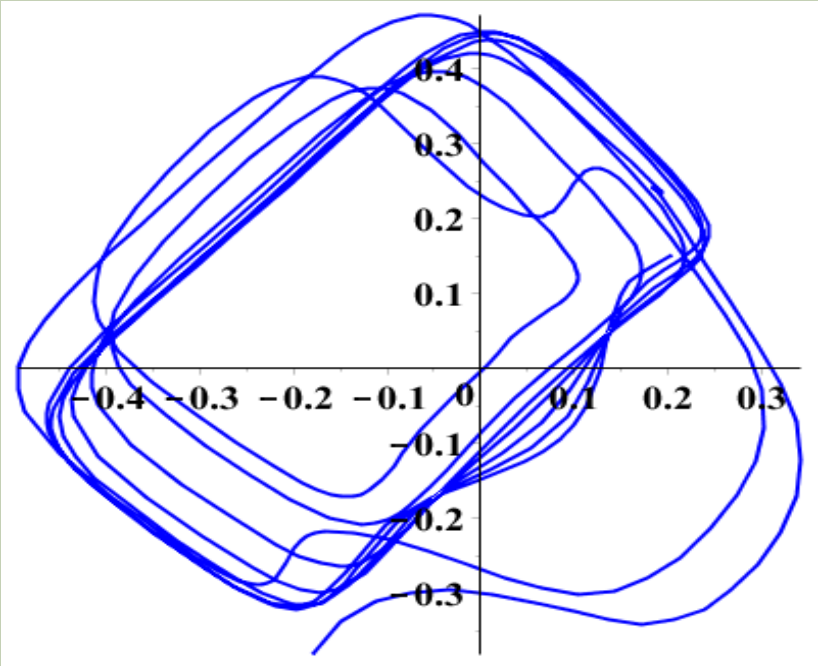
2. Integrate the linearized system with n sets of initial conditions defined by the vectors of the frame F_0 to obtain n evolved vectors $\mathbf{v}_j$.

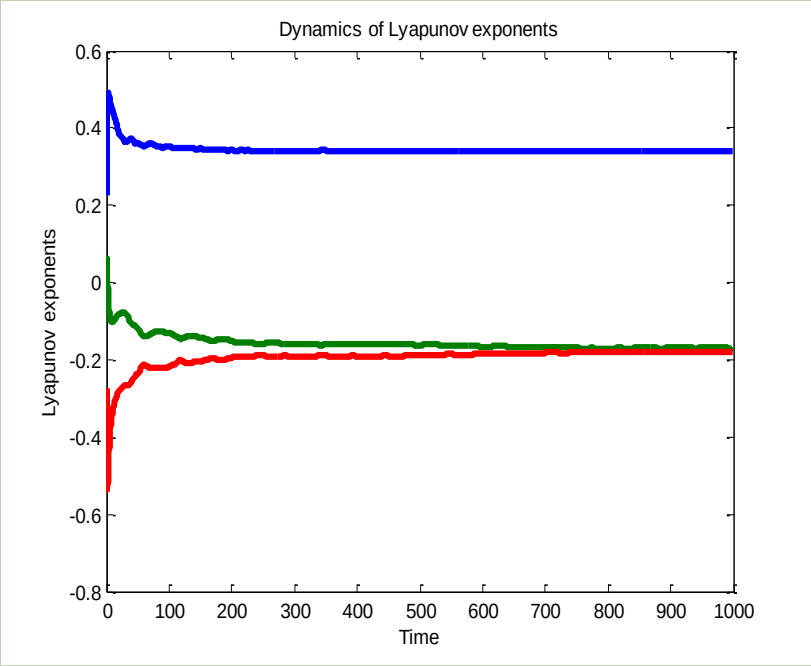
3. Use Gram-Schmidt renorthonormalization procedure on $\mathbf{v}_j$ to obtain a new frame F_1.

4. Update estimates of LCEs as projections of the evolved vectors onto the new orthonormal frame F_1.

LCSs - Example

Example. Parameters: $s=-1$, $k_1=-0.2$, $k_2=-0.25$, $d=0.8$, $h=0.3$, $b_1=-0.1$, $h=0.3$, $A=1.3$. Initial conditions: $z_1=-0.2$, $z_2=-0.38$, $\delta=0$.





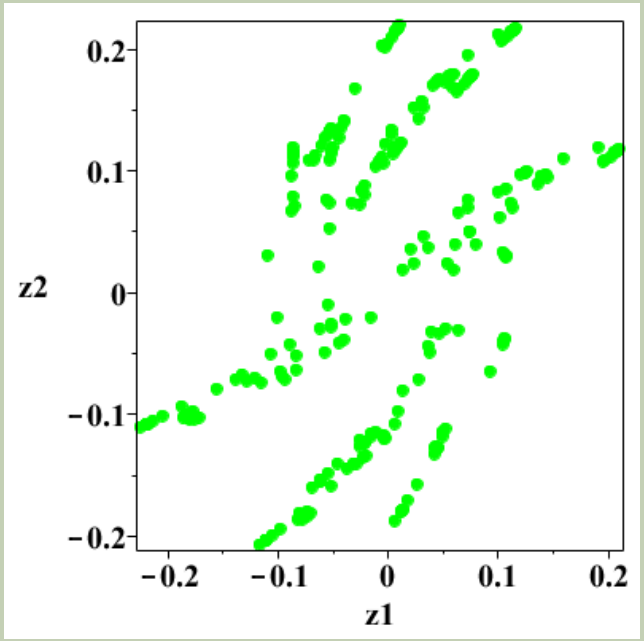
Left: z_1-z_2 dynamics, 3D system.

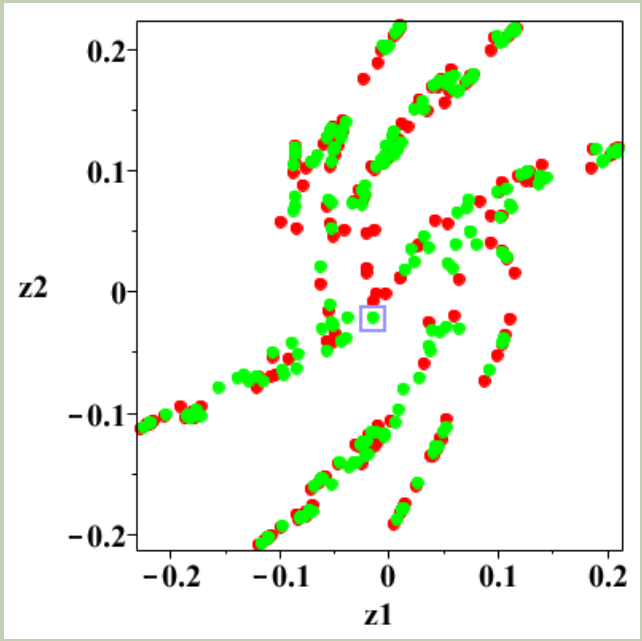
Right: LCEs estimates. One of them is positive (blue curve), signature of chaos

Poincare sections

Poincare sections, plane cos(δ)=cos(δ0)

Left:δ0=0.3 Right: Superposition: δ0=0.3 and δ0=0.301





Sensitivity to initial conditions, another illustration of chaos

Summary

- A CLLS system of ODEs for modeling dynamics of a magnetic bilayer has been introduced
- ANALYTIC asymptotic stability criterion for four equilibrium magnetic states in terms of all parameters of the model has been derived
- Circular and noncircular limit cycles have been found using 3D reduction of CLLS and the averaging technique
- Simulations suggest that the system is chaotic. Positive Lyapunov exponents have been found for certain sets of the model parameters.

Thank you!