Qualitative analysis of weakly coupled Landau- Lifshitz equations with spin-transfer torque terms

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A handful of references

Coupled Landau-Lifshitz-Slonczewski model (CLLS) of magnetization dynamics

Reduced version of CLLS: four-dimensional system

Analytic stability criterion for equilibrium states of the 4D system – two proofs

Another reduction of CLLS: slow-fast 3D system

Circular limit cycles - critical points of the 3D system

Noncircular limit cycles – averaging technique

Finding Lyapunov characteristic exponents: algorithm and computations

Summary
**Motivation**

**Spintronics**: Neologism for Spin Transport Electronics

**Spin valve**: a nanometer scale device with at least two magnetic layers. Its electrical resistance depends on relative alignments of the magnetization of the two (or more) layers.

**Spin-transfer torque**: effect in which the orientation of a magnetic layer in a spin valve can be modified using a spin-polarized current. Has potential applications in RAM (STT-RAM).

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**Practical implementation issues**: stability and efficient reading and writing at acceptably small currents. Better understanding of the magnetization dynamics and optimization of device geometry is needed.
CLLS model (a glimpse of physics)

\[
\frac{ds_j}{d\tau} = s_j \times [\gamma(H_j + \alpha_j s_j \times H_j) - \beta_j s_2 \times s_1] \quad (*)
\]

\( j = 1, 2, s_j = [x_j, y_j, z_j] \)

ASSUMPTION:
Uniaxial anisotropy, \( Z \) is the anisotropy axis
Notation:
\( H_j, j=1, 2: \) effective magnetic fields
\[
H_j = H_a + d_j (s_j \cdot e_3) e_3, \quad H_a = H e_3
\]
\( H, \gamma, \alpha_j, d_j: \) physical parameters, \( a_j > 0 \)
\( \beta_j: \) control parameters (involve applied DC)
\( a_j, \beta_j \) are small, \( \sim O(10^{-2}) \)

Conventional model: \( s_1 = \text{const} \) (fixed magnetization of layer 1)

Physics lingo: reference (fixed, pinned) layer, active (free) layer.

Eqs (*) imply that the magnitudes of \( s_j \) are conserved

\[\rightarrow \text{Only four scalar equations out of the six in (*) are independent}\]

Phase space: \( S^2 \times S^2 \)
**CLLS model (mathematics)**

Change \( t = -\gamma |d_2| \tau \) and division by \(-\gamma |d_2|\) give a dimensionless version of (*):

\[
\frac{d}{dt} s_j = -s_j \times \left[ h_j + a_j s_j \times h_j - b_j s_2 \times s_1 \right] \quad \text{(CLLS)}
\]

\( h_1 = h_1(z_1) = h + d z_1, \ h_2 = h_2(z_2) = h + s z_2, \ d = \frac{d_1}{|d_2|}, \ s = \text{sign} (d_2) \)

Four equilibrium states: \((0, 0, \pm 1; 0, 0, \pm 1)\)

Scalar form of the conventional model with \(s_1=[0,0,1]\), and \(s_2=[x,y,z]\):

\[
\begin{align*}
\frac{dx}{dt} &= -(h + sz)(y - a_2 xz) + b_2 xz \\
\frac{dy}{dt} &= (h + sz)(x - a_2 yz) + b_2 yz \\
\frac{dz}{dt} &= (1 - z^2)(a_2 (h + sz) - b_2)
\end{align*}
\]

**Observation:** matrix \(A\) of the linearization of \(x\)- and \(y\)-equations at any of the fixed points \((0, 0, \pm 1)\) shows complex structure. For \((0, 0, 1)\), \(A=Z\), where

\[
Z = \begin{bmatrix}
-a_2 h_2 + b_2 & -h_2 \\
h_2 & -a_2 h_2 + b_2
\end{bmatrix}, \quad h_2 = h_2(l) \quad \Rightarrow \quad \text{Asymptotic stability criterion:} \quad \text{Re}(z) < 0, \ z = -a_2 h_2 + b_2 + i h_2
4D system: choose equations for $x_j$ and $y_j$ in CLLS

Linearization of the 4D system at any of the four equilibrium points also reveals a complex structure

Example: Linearized 4D system at $(0, 0, 1; 0, 0, 1)$

\[
\frac{dU}{dt} = A_{pp} U, \quad U = [u_1, \ldots, u_4]^T,
\]

\[
A_{pp} = \begin{bmatrix}
Z_1 & b_1 E_2 \\
-b_2 E_2 & Z_2
\end{bmatrix}, \quad \text{where} \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
Z_1 = \begin{bmatrix}
-(a_1 h_1 + b_1) & -h_1 \\ h_1 & -(a_1 h_1 + b_1)
\end{bmatrix}, \quad Z_2 = \begin{bmatrix} - (a_2 h_2 - b_2) & -h_2 \\ h_2 & - (a_2 h_2 - b_2) \end{bmatrix},
\]

$h_j = h_j(1)$

$A_{pp}$ is a block matrix and the realification of a complex 2x2 matrix!
The Stability Criterion (includes ALL parameters of CLLS!)

Let \( M(0,0,z_{10};0,0,z_{20}) \), \( z_{10}=\pm 1, z_{20}=\pm 1 \), be a fixed point;
\[ h_1=h_1(z_{10}), \ h_2=h_2(z_{20}). \]

CLLS system (1) is asymptotically stable at \( M \) iff

\[ \text{Re}(z_1+z_2+w)<0, \text{ where } \ w = \sqrt{(z_1-z_2)^2 - 4b_1b_2}, \]

\[ z_1 = -z_{10}(a_1h_1+z_{20}b_1)+ih_1, \quad z_2 = -z_{20}(a_2h_2-z_{10}b_2)+ih_2 \quad (\ast) \]

**Proof 1** relies on the following

**Theorem.** The spectrum \( \sigma(A^R) \) of the realification \( A^R \) of a complex matrix \( A \) is the union

\[ \sigma(A) \cup \overline{\sigma}(A). \]

Without loss of generality, consider matrix \( A_{pp} \). It is the realification of the 2x2 complex matrix

\[ A = \begin{bmatrix} z_1 & b_1 \\ -b_2 & z_2 \end{bmatrix} \text{ with } \sigma(A) = \frac{1}{2}(z_1+z_2 \pm w) \text{ and } z_{10} = z_{20} = 1. \]

If the argument of complex numbers is chosen between \(-\pi\) and \(\pi\), then
\[ \text{Re}(w) \geq 0, \text{ and the criterion follows.} \]
Proof 2 [4]. Idea: The linearized system can be transformed into complex Riccati equation in 4 steps, solved explicitly, and the criterion can be derived from the explicit solution

**Tr1:**  \( \xi_1 = u_1 + iu_2, \quad \xi_2 = u_3 + iu_4 \)

\[
\frac{d\xi_1}{dt} = z_1 \xi_1 + b_1 \xi_2, \quad \frac{d\xi_2}{dt} = -b_2 \xi_1 + z_2 \xi_2.
\]

**Tr2:**  \( \xi_j = r_j(t) \exp(i\varphi_j(t)) \)  ( \( r = r_2 / r_1, \delta = \varphi_2 - \varphi_1 \) )

\[
\frac{d\ln(r_1)}{dt} = -a_1 h_1 - b_1 (1 - r \cos(\delta)), \quad \frac{d\varphi_1}{dt} = \text{Im}(z_1) + b_1 r \sin(\delta),
\]

\[
\frac{d\ln(r_2)}{dt} = a_2 h_2 - b_2 (\cos(\delta) / r + 1), \quad \frac{d\varphi_2}{dt} = \text{Im}(z_2) + b_2 \sin(\delta) / r.
\]

\[
\frac{d\delta}{dt} = \sin(\delta)(b_2 / r - b_1 r) + \Delta h, \quad \Delta h = h_2 - h_1
\]

\[
\Rightarrow \quad \frac{dr}{dt} = -\cos(\delta)(b_2 + b_1 r^2) + r \text{Re}(z_2 - z_1)
\]
Proof 2 cont’d

Riccati equation:

\[
\frac{dZ}{dt} = -b_1 Z^2 + (z_2 - z_1)Z - b_2, \quad \text{where} \quad Z = r \cos(\delta) + i \, r \sin(\delta)
\]

Solution:

\[
Z = \frac{1}{2b_1} \left( z_2 - z_1 + w \tanh(wt/2 + c) \right)
\]

\[
X = \Re(Z) = \frac{1}{2b_1} \left( \Re(z_2 - z_1) + w_1 \frac{\sinh(x) \cosh(x)}{\sinh^2(x) + \cos^2(x)} - w_2 \frac{\sin(y) \cosh(y)}{\sinh^2(x) + \cos^2(x)} \right)
\]

\[
Y = \frac{1}{2b_1} \Im(Z) = \ldots, \quad w_1 = \Re(w), \quad w_2 = \Im(w), \quad x = tw_1/2 + c_1, \quad y = tw_2/2 + c_2.
\]

The stability condition can be obtained from Tr2 as

\[
\Re(z_1) + b_1 \lim_{t \to \infty} X(t) < 0
\]

Finding the limit from (*), \( X_\infty = \Re(z_2 - z_1 + w)/(2b_1) \), completes the proof.

Explicit solution for the linearized system can be used for approximation of important parameters, relaxation time and frequency of oscillations.
Equilibrium point \((0,0,1,0,0,1)\)

Stability condition:

\[
b_2 - b_1 + |b_2 - b_1| - 2ah < 0
\]

\[
\Rightarrow b_2 < \frac{ahA}{A-1}, \quad A > 0; \quad b_2 > \frac{ahA}{A-1}, \quad 0 < A < 1
\]

where \(A = \frac{b_2}{b_1}\)
**Superimposed stability diagrams**

Parameters: 
\[ a_1 = a_2, \]
\[ s = 1, d = 1.8, \]
\[ h = 1.5 \]

- NP: Stability region for the state \((0,0,-1,0,0,1)\) (dashed)
- PP: Stability region for the state \((0,0,1,0,0,1)\) (solid)

Bistability region is bounded by two solid curves and lower dashed curve
Another reduction of CLLS: 3D system

CLLS system in cylindrical coordinates \((r_j, \phi_j, z_j)\), \(r_j = \sqrt{1 - z_j^2}\), \(j = 1, 2\):

\[
\frac{d\phi_1}{dt} = h + d z_1 + b_1 \sin(\delta) \frac{r_2}{r_1}, \quad \frac{d\phi_2}{dt} = h + s z_2 + b_2 \sin(\delta) \frac{r_1}{r_2}
\]

\[
dz_1 / dt = \ldots, \quad dz_2 / dt = \ldots
\]

simplifies to a 3D slow-fast system

\[
\frac{d\delta}{dt} = s z_2 - dz_1 + b_1 \sin(\delta) Ar_1/r_2 - r_2/r_1
\]

\[
\frac{dz_1}{dt} = b_1 \left( r_1^2 \left( k_1 h_1 z_1 + z_2 \right) - z_1 r_1 r_2 \cos(\delta) \right)
\]

\[
\frac{dz_2}{dt} = b_1 \left( r_2^2 \left( k_2 h_2 z_2 - Az_1 \right) + Az_2 r_1 r_2 \cos(\delta) \right)
\]

\[
r_j = \sqrt{1 - z_j^2}, \quad b_1 \ll 1, \quad k_j = a_j / b_1 = O(1)
\]

Circular limit cycles of CLLS are fixed points of the 3D system. They can be found numerically with high accuracy using reduction of the system (RHS(3D system)\(=0\) to a system of polynomial equations.
**Circular limit cycles**

**Example.** For $k_1=k_2=1$, $b_1=0.03$, $A=0.2$, $s=-1$, $d=-0.9$, $h=0.4$, a stable fixed point of the 3D system is $z_1 \approx 0.422$, $z_2 \approx 0.388$, $\delta \approx -0.320$.

Visualization of solutions $z_1$, $z_2$ to a polynomial system obtained from equations $\text{RHS(3D system)}=0$.

Numerical solution to the 3D system with parameters specified in the example.
**Noncircular limit cycles – averaging method**

3D system is a multiscale system with slow variables $z_1$, $z_2$ and fast variable $\delta$. RHS of the system is periodic in $\delta$. Classic averaging scheme (Bogolubov-Mitropolski, [5]) applies successfully.

**Algorithm**

1. Let $b_1=\varepsilon$. Assume $z_1=u+\varepsilon f(u,v,\omega)$, $z_2=v+\varepsilon g(u,v,\omega)$, $\delta=\omega+\varepsilon q(u,v,\omega)$, (*) where $f$, $g$, and $q$ are $2\pi$-periodic in $\omega$, and $u$, $v$, and $\omega$ solve the system

$$\frac{du}{dt} = \varepsilon M_1(u,v), \quad \frac{dv}{dt} = \varepsilon M_2(u,v), \quad \frac{d\omega}{dt} = s v - d u + \varepsilon \Omega(u,v) \quad (***)$$

2. Equate the time derivatives of $z_1$, $z_2$, and $\delta$ found in two ways:
   i. by linearization of RHS in (2) at $(u,v, \omega)$ and
   ii. by differentiation of RHS in (*)
   to obtain a system (***) (not shown).

3. Average: integrate (***) w/r to $\omega$ over $[0,2\pi]$. Then find $M_j$ and $\Omega$.

4. Solve (**) for $u$, $v$, and $\omega$. Then solve (***) for $f$, $g$, and $q$.

Exact solution of equations for $u$ and $v$ in (***) with $M_j$ found on step 3 is not available. On step 4, a stable fixed point $(u^*, v^*)$ is used instead.
Example. Parameters: $k_1=0.5$, $k_2=0.55$, $d=-0.8$, $h=0.3$, $b_1=0.04$, $h=0.3$, $A=0.2$. Fixed point of (**) system: $u^*=0.1114864865$, $v^*=-0.1054054054$.

Left: Superimposed plots of numerical solution $z_1$ for 3D system (red), $u$ for system (***) (blue), and a slow-fast approximation of $z_1$ in (*) (green);
Right: same for the variable $z_2$. Clearly, the slow-fast approximations of $z_1$, $z_2$ are in good agreement with numerical solutions. (Some 'rule of thumb' for adjustments of phases was used.)
Definition of chaoticity (one of many) [6]. A dynamical system is chaotic if it has a positive Lyapunov exponent.

Lyapunov exponents are defined by the long-term evolution of the axes of an infinitesimal sphere of states centered at a point in the phase space.

Computational Algorithm [7] (just one step)

Given a system of n nonlinear ODEs, a point $X_0$ in the phase space, and an orthonormal frame $F_0$ anchored to the point,
1. Find the solution $X=X(t,X_0)$ of the system.

2. Integrate the linearized system with $n$ sets of initial conditions defined by the vectors of the frame $F_0$ to obtain $n$ evolved vectors $v_j$.

3. Use Gram-Schmidt renorthonormalization procedure on $v_j$ to obtain a new frame $F_1$.

4. Update estimates of LCEs as projections of the evolved vectors onto the new orthonormal frame $F_1$. 

Finding Lyapunov characteristic exponents
**Example.** Parameters: $s=-1$, $k_1=-0.2$, $k_2=-0.25$, $d=0.8$, $h=0.3$, $b_1=-0.1$, $h=0.3$, $A=1.3$. Initial conditions: $z_1=-0.2, z_2=-0.38$, $\delta=0$.

Left: $z_1-z_2$ dynamics, 3D system.

Right: LCEs estimates. One of them is positive (blue curve), signature of chaos.
Poincare sections, plane $\cos(\delta) = \cos(\delta_0)$

**Left:** $\delta_0 = 0.3$  
**Right:** Superposition: $\delta_0 = 0.3$ and $\delta_0 = 0.301$

Sensitivity to initial conditions, another illustration of chaos
Summary

- A CLLS system of ODEs for modeling dynamics of a magnetic bilayer has been introduced.
- ANALYTIC asymptotic stability criterion for four equilibrium magnetic states in terms of all parameters of the model has been derived.
- Circular and noncircular limit cycles have been found using 3D reduction of CLLS and the averaging technique.
- Simulations suggest that the system is chaotic. Positive Lyapunov exponents have been found for certain sets of the model parameters.
Thank you!