Infinite Symmetries and Infinite Conservation Laws

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1. Arbitrary functions in the generators of symmetry groups
2. Infinite symmetries and conservation laws
3. Infinite symmetries and Linear PDE.
4. Symmetries of hydrodynamic-type system
   - Open problems
Equations that admit symmetries with arbitrary functions:

- Presence of arbitrary functions in the generators of symmetry groups (pseudo-groups), history, different cases of arbitrary functions.

- Infinite-dimensional symmetries and conservation laws for Lagrangian differential systems. Essential conservation laws.

- Linear PDE and infinite-dimensional symmetries.

- Hydrodynamic-type system and infinite symmetries.

- Open problems.
Potential Kadomtsev-Petviashvili equation

\[ u_{xxxx} + 6u_x u_{xx} + 3s^2 u_{yy} + 4u_{xt} = 0, \quad s^2 = \mp 1 \]  \hspace{1cm} (1)

(KP-I and KP-II). Lagrangian

\[ L = \frac{u_{xx}^2}{2} - u_x^3 - 2u_x u_t - \frac{3s^2 u_y^2}{2}, \]  \hspace{1cm} (2)
Arbitrary functions

Lie point symmetry group [David, Kamran, Levi, Winternitz, 1986]:

\[ X_m = m \frac{\partial}{\partial x} + \left[ \frac{2xm'}{3} - 4s^2y^2m''/9 \right] \frac{\partial}{\partial u}, \]
\[ X_g = g \frac{\partial}{\partial y} - \left( \frac{2s^2yg'}{3} \right) \frac{\partial}{\partial x} + \left[ -4s^2xyg''/9 + 8y^3f'''/81 \right] \frac{\partial}{\partial u}, \]
\[ X_f = f \frac{\partial}{\partial t} + \left( \frac{2yf'}{3} \right) \frac{\partial}{\partial y} + \left[ \frac{xf'}{3} - 2s^2y^2f''/9 \right] \frac{\partial}{\partial x} \]
\[ + \left[ -uf'/3 + x^2f''/9 - 4s^2xy^2f'''/27 + 4y^4f^{(IV)}/243 \right] \frac{\partial}{\partial u}, \]
\[ X_h = yh \frac{\partial}{\partial u}, \quad X_l = l \frac{\partial}{\partial u}, \]

where \( m(t), g(t), f(t), h(t), l(t) \) are arbitrary functions.
Infinite symmetry algebras with arbitrary functions and relations to conservation laws have not been studied very extensively.

- $f(x^1, x^2, ..., x^n)$ do not lead to conservation laws. Noether II: Infinite variational symmetries with arbitrary functions of all independent variables lead to a certain relation between equations of original system (the original system is underdetermined).

- $f(x^1, x^2, ..., x^k), k < n$ lead to a finite number of essential conservation laws. $f(t)$.

Arbitrary functions of not all independent variables lead to a finite number of essential conservation laws, and each essential conservation law is determined by a specific form of boundary conditions. [Rosenhaus, JMP, 2002]
Infinite symmetries and corresponding essential conservation laws

- Potential Zabolotskaya–Khokhlov equation, [JNMP, 2006]
Infinite symmetry algebras with arbitrary functions of dependent variables - radically different situation leading to an infinite set of essential conservation laws (local conserved densities) [V.R. TMP 2005, 2007].


Another situation - hydrodynamic type equations.
Theorem

Equations with infinite variational symmetry algebras parametrized by arbitrary functions of dependent variables possess infinite sets of essential conservation laws.
By a conservation law for a differential system

\[ \omega^a(x, u, u_1, u_2, \ldots) = 0, \quad a = 1, \ldots, n, \]

is meant a divergence expression

\[ D_i K_i(x, u, u_1, u_2, \ldots) = 0, \]

vanishing for all solutions of the original system (\(=\)).

\(x = (x^1, x^2, \ldots, x^{m+1}),\) \(x^{m+1} = t,\)
\(u = (u^1, u^2, \ldots, u^n),\)
\(u^{(r)} - \text{rth-order derivatives of } u,\) \(\omega^a\) and \(K_i\) are differential functions: smooth functions of \(x, u,\) and \(u^{(r)}\).
Essential Conservation Laws

**Definition**
Two conservation laws \( K \) and \( \tilde{K} \) are *equivalent* if they differ by a trivial conservation law [Olver].

**Definition**
A conservation law \( D_i P_i = 0 \) is *trivial* if a linear combination of two kinds of triviality is taking place:
1. \((m+1)\)-tuple \( P \) vanishes on the solutions of the original system: \( P_i = 0 \).
2. The divergence identity is satisfied in the whole space: \( D_i P_i = 0 \).
By an **essential** conservation law, we mean such non-trivial conservation law $D_i K_i = 0$, which gives rise to a non-vanishing conserved quantity

$$D_t \int_D K_t \, dx^1 \, dx^2 \cdots dx^m = 0, \quad x \in D \subset \mathbb{R}^{m+1}, \quad K_t \neq 0. \quad (4)$$
The approach

We consider functions \( u^a = u^a(x) \) defined on a region \( D \) of \((m+1)\)-dimensional space-time. Let

\[
S = \int_D L(x, u, u^{(1)}, \ldots) \, d^{m+1}x
\]

be the action functional, where \( L \) is the Lagrangian density. Then the equations of motion are

\[
E^a(L) \equiv \omega^a(x, u, u^{(1)}, u^{(2)}, \ldots) = 0, \tag{5}
\]

where

\[
E^a = \frac{\partial}{\partial u^a} - \sum_i D_i \frac{\partial}{\partial u^a_i} + \sum_{i \leq j} D_i D_j \frac{\partial}{\partial u^a_{ij}} + \cdots, \tag{6}
\]

is the Euler–Lagrange operator.
The approach

Consider an infinitesimal (one-parameter) transformation with the canonical infinitesimal operator

\[ X_\alpha = \alpha^a \frac{\partial}{\partial u^a} + \sum_i (D_i \alpha^a) \frac{\partial}{\partial u_i^a} + \sum_{i<j} (D_i D_j \alpha^a) \frac{\partial}{\partial u_{ij}^a} + \cdots, \] (7)

\[ \alpha^a = \alpha^a(x, u, u^{(1)}, \ldots). \]

Variation of the functional \( S \) under the transformation with operator \( X_\alpha \) is

\[ \delta S = \int_D X_\alpha L \, d^{m+1}x. \] (8)

\( X_\alpha \) is a variational (Noether) symmetry if

\[ X_\alpha L = D_i M_i, \] (9)

where \( M_i = M_i(x, u, u^{(1)}, \ldots) \) are smooth functions.
The approach

The Noether identity. Relates the operator $X_\alpha$ to $E^a$,

$$X_\alpha = \alpha^a E^a + D_i R_{\alpha i},$$  \hspace{1cm} (10)

$$R_{\alpha i} = \alpha^a \frac{\partial}{\partial u^a_i} + \left\{ \sum_{k \geq i} (D_k \alpha^a) - \alpha^a \sum_{k \leq i} D_k \right\} \frac{\partial}{\partial u^a_{ik}} + \cdots.$$  \hspace{1cm} (11)

Applying the Noether identity to $L$ we obtain

$$D_i (M_i - R_{\alpha i} L) = \alpha^a \omega^a,$$  \hspace{1cm} (12)

On the solution manifold ($\omega = 0$, $D_i \omega = 0$, $\ldots$) is

$$D_i (M_i - R_{\alpha i} L) \equiv 0,$$  \hspace{1cm} (13)

Thus, any variational one-parameter symmetry transformation $X_\alpha$ gives rise to a conservation law (13) - First Noether Theorem.
Consider an infinite variational symmetry with a characteristic $\alpha$ of the form

$$\alpha^a = \alpha^{a0} p(x) + \alpha^{ai} D_i p(x) + \sum_{i \leq j} \alpha^{aij} D_i D_j p(x) + \cdots ,$$  \hspace{1cm} (14)

where $p(x)$ are smooth functions, and $\alpha^{a0}$, $\alpha^{ai}$, $\alpha^{aij}$, $\ldots$ are differential functions.

$p(x)$ is an arbitrary function of all base variables According to Noether II a consequence of an infinite symmetry (11) of functional $S$ is not a conservation law but a certain relation between the original differential equations [Noether 1918].

$p(x)$ is an arbitrary function of not all base variables [Rosenhaus JMP 2002].
Boundary conditions

For a Noether symmetry operator \( X_\alpha \) we have

\[
\delta S = \int_D \delta L \, d^{m+1}x = \int_D X_\alpha L \, d^{m+1}x = \int_D D_i M_i \, d^{m+1}x = 0. \tag{15}
\]

Therefore, the following conditions for \( M_i \) (Noether boundary conditions) should be satisfied

\[
M_i(x, u, \ldots) \bigg|_{x \to \partial D} = 0, \quad i = 1, \ldots, m + 1, \tag{16}
\]

where \( \to \) denotes the limit along the \( i \)th axis. Equations (16) are usually satisfied for a regular asymptotic behavior: \( u^a \to 0 \) and \( u_i^a \to 0 \) at infinity, or for periodic solutions.

Integrating (13) over the space \( \Omega (x^1, x^2, \ldots, x^m) \) we get

\[
\int_\Omega dx^1 \ldots dx^m D_t (M_t - R_{\alpha t}L) = \int_\Omega dx^1 \ldots dx^m \sum_{i=1}^m D_i (R_{\alpha i}L - M_i). \tag{17}
\]
Boundary conditions

Applying the Noether boundary condition and requiring the LHS to vanish on the solution manifold leads to the “strict” boundary conditions

\[ R_{\alpha 1} L \bigg|_{x \to \partial \Omega} = R_{\alpha 2} L \bigg|_{x \to \partial \Omega} = \cdots = R_{\alpha m} L \bigg|_{x \to \partial \Omega} = 0. \]  \hspace{1cm} (18)

In the case \( L = L(x, u, u^{(1)}) \), the strict boundary conditions take a simple form

\[ \alpha^a \frac{\partial L}{\partial u^a_l} \bigg|_{x \to \partial \Omega} = 0, \quad l = 1, \ldots, m. \]  \hspace{1cm} (19)

Thus, Noether and strict boundary conditions are necessary for the existence of essential conservation laws. In case of arbitrary functions of independent variables these conditions allow only a finite number of essential conservation laws.
Arbitrary functions of dependent variables

Consider infinite symmetries whose characteristics contain an arbitrary smooth function \( f \) of \textbf{dependent} variables and their derivatives, i.e.

\[
\alpha^a = \alpha^{a0} f + \alpha^{as} \partial_s f + \cdots ,
\]

where \( s \) numerates arguments of \( f = f(u, u(1), \ldots) \), and \( \alpha^{a0}, \alpha^{as}, \ldots \) are some differential functions. The conservation law (17) then has the form

\[
D_l(M_l - R_{\alpha l} L) + D_t(M_t - R_{\alpha t} L) = 0, \quad l = 1, \ldots, m,
\]

(20)

where

\[
M_i = M^{0}_i f + M^s_i \partial_s f + \cdots , \quad R_{\alpha i} L = P^{0}_i f + P^s_i \partial_s f + \cdots
\]

(21)

for some differential functions \( M^{0}_i, M^s_i, \ldots, \) and \( P^{0}_i, P^s_i, \ldots \).
Arbitrary functions of dependent variables

In order for the system to possess (Noether) local conserved quantities, both Noether (16) and strict boundary conditions (18) have to be satisfied. Let

\[ f(u, u_{(1)}, \ldots) = g(\xi(u, u_{(1)}, \ldots)) \]  

(22)

where \( g \) is some smooth function. Assuming regular boundary conditions \( u^a \to 0, u^a_i \to 0, \ldots \) at infinity, the Noether boundary conditions take the form

\[
M_i \left( g(\xi(0, \ldots, 0), g'(\xi(0, \ldots, 0), \ldots) \right) - \\
M_i \left( g(\xi(0, \ldots, 0), g'(\xi(0, \ldots, 0), \ldots) \right) = 0.
\]  

(23)
Arbitrary functions of dependent variables

These conditions are satisfied for any smooth function $g$ and

$$\left| \xi(0, \ldots, 0) \right| < \infty.$$  \hspace{1cm} (24)

The strict boundary are also satisfied if conditions (24) are met. Thus, in the case of infinite symmetries with arbitrary functions of dependent variables $\alpha^a = f^a(u, u(1), \ldots)$ unlike the case with independent variables, there are no serious restrictions for functions $f^a$ to lead to local conservation laws, and the continuity equation provides an infinite number of essential conservation laws [Rosenhaus, TMP, 2007]. The corresponding Noether conserved quantities

$$D_t \int_\Omega dx^1 dx^2 \ldots dx^m (M_t - R_{\alpha t} L) \div 0.$$  \hspace{1cm} (25)
1. Which equations would admit infinite variational symmetries with an **arbitrary function of** \( u \): \( f(u) \)?

\[
\alpha = pf(u) + qf'(u) + rf''(u),
\]  
(26)

where \( p, q, r = g_j(x, t, u, u_x, u_t), j = 1, 2, 3 \). We look for equations with Lagrangians of the first order

\[
L = L(u, u_x, u_t).
\]  
(27)

Requiring \( X_\alpha \) to be a variational symmetry

\[
X_\alpha L = D_x M_x + D_t M_t,
\]

\[
M_x = Af(u) + Bf'(u) + Cf''(u),
\]  
(28)

\[
M_t = Ef(u) + Ff'(u) + Gf''(u),
\]

\((A, B, C, E, F, G = h_j(x, t, u, u_x, u_t))\) we obtain:
\[ L(u, u_x, u_t) = R \left( \frac{u_x}{u_t}, u \right) + u_t S \left( \frac{u_x}{u_t}, u \right), \]  

(29)

where \( R \) and \( S \) are arbitrary functions. [V.R.TMP 2007]

Any equations with Lagrangians of the form (29) have an infinite number of essential conservation laws given by expression (13).
2. Look for equations admitting infinite variational symmetries with an arbitrary function of the first derivatives $f(u_x, u_t)$

$$\alpha = pf(\xi) + qf'(\xi) + rf''(\xi).$$

(30)

Requiring $X_\alpha$ to be a variational symmetry

$$X_\alpha L = D_x M_x + D_t M_t,$$

$$M_x = Af(\xi) + Bf'(\xi) + Cf''(\xi),$$

(31)

$$M_t = Ef(\xi) + Ff'(\xi) + Gf''(\xi),$$

$(A, B, C, E, F, G, p, q, r = h_j(x, t, u, u_x, u_t))$. We obtain

$$L = u_t P \left( \frac{u_x}{u_t}, u \right)$$

(32)

$$L = Q(u_x, u_t).$$

(33)

with arbitrary functions $P, Q$
The function $\xi(z, w)$ should satisfy

$$\xi_w^2 L_{zz} - 2r \xi_w \xi_z L_{zw} + \xi_z^2 L_{ww} = 0. \quad (34)$$

To ensure the existence of real solutions for $\xi$ the following condition has to hold.

$$L_{zw}^2 - L_{zz} L_{ww} \geq 0. \quad u_x \equiv z, u_t \equiv w). \quad (35)$$

Any equations with Lagrangians of the form (32) or (33) have an infinite number of essential conservation laws given by expression (13).
An interesting example of the class (33) is Born-Infeld equation

\[
(1 - u_t^2)u_{xx} + 2u_t u_x u_{xt} - (1 + u_x^2)u_{tt} = 0,
\] (36)

with the Lagrangian density

\[
L = (1 + u_x^2 - u_t^2)^{1/2}.
\] (37)

Born–Infeld Lagrangian is related to classical relativistic string Lagrangian in 4-d space. Infinite set of conservation laws for Born–Infeld equation and its infinite group of contact transformations [Koiv, Rosenhaus, Algebras Groups Geom., 1986].
Another interesting example is hyperbolic Fermi–Pasta–Ulam equation [Juras, J. Diff. Eq., 2000]

\[ u_{tt} - ku_x^2 u_{xx} = 0, \quad (38) \]

with the Lagrangian density

\[ L = \frac{u_t^2}{2} - \frac{ku_x^3}{3}. \quad (39) \]
3. Look for equations possessing infinite variational symmetries with an arbitrary function of the dependent variables and its first and second derivatives. We look for equations with first-order Lagrangians, \( L = L(u, u_x, u_t) \), possessing \( X_\alpha \) as a variational symmetry for each characteristic \( \alpha \) of the form

\[
\alpha = Pf(\xi) + Qf'(\xi) + Rf''(\xi),
\]

where \( f \) is an arbitrary smooth functions of a single argument, \( \xi \) is a smooth function of \( u \) and its first and second derivatives: \( \xi = \xi(u, u^{(1)}, u^{(2)}) \), and \( P, Q \) and \( R \) are some differential functions. Then we have [V.R. 2013]

\[
D_t \xi \equiv 0.
\]

(or \( D_x \xi \equiv 0 \)). Equations of Liouville type.
The most common equation of this class is the Liouville equation

\[ u_{xt} = e^u \quad \text{with} \quad L = \frac{u_x u_t}{2} + e^u, \quad (42) \]

for which we have

\[ \alpha = u_x f'(\xi) + D_x f'(\xi) = u_x f'(\xi) + \xi x f''(\xi), \]

where

\[ \xi = u_{xx} - \frac{u_x^2}{2}. \]

Indeed,

\[ D_t \xi = u_{xxt} - u_x u_{xt} = -D_x \omega + u_x \omega = 0 \]

where \( \omega = e^u - u_{xt}. \)
In the field theory the presence of arbitrary functions of independent variables $g(x)$ in the generators of a symmetry group is gauge invariance. However, the total symmetry group includes an arbitrary function of dependent variables as well [Kiiranen, Rosenhaus, 1989]. For example, for Maxwell equations with the action functional

$$S = \int \int F_{\mu\nu} F^{\mu\nu} \, d^4 x$$

(43)

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \quad A_{\mu,\nu} = \frac{\partial A_\mu}{\partial x^\nu}, \quad \mu, \nu = 1, \ldots, 4,$$
a variational symmetry transformation

\[ X_{\alpha} = \alpha^\mu \frac{\partial}{\partial A^\mu} + \sum_{\mu,\nu} (D_\nu \alpha^\mu) \frac{\partial}{\partial A_{\mu,\nu}} + \cdots \]

takes a form

\[ \alpha^\mu = D_\mu g (x, A, A, \nu, \ldots) \] (44)

\[ D_\mu = \frac{\partial}{\partial x^\mu} + A^\nu_{,\mu} \frac{\partial}{\partial A^\nu} + \cdots \]

g is an arbitrary function of all \( x^\nu \) as well as 4-potentials \( A^\mu \) and their derivatives. This is an extended form of the gauge (gradient) transformation for Maxwell equation.
Similarly, for the Yang-Mills equations (non-abelian case) a local gauge transformation can also be extended to include arbitrary functions $g$ of potentials $A$ and their derivatives:

$$g = g(x, A, A, \nu, \ldots).$$
The presence of arbitrary functions in symmetry generators and components of a continuity equation is not unusual for linear equations. The wave equation

\[ u_{xx} - u_{tt} = 0, \quad (45) \]

has infinite conservation laws:

\[ D_t (f(u_t + u_x) + g(u_t - u_x)) + D_x (-f(u_t + u_x) + g(u_t - u_x)) = 0. \quad (46) \]

with arbitrary functions \( f \) and \( g \).
Similarly, for a parabolic equation

\[ R^2 u_{xx} + 2Ru_{xt} + u_{tt} = 0, \quad R = \text{const}. \]  \hspace{1cm} (47)

infinite conservation laws are given by

\[ D_t (f(u_t + Ru_x)) + D_x (f(u_t + Ru_x)) = 0. \]  \hspace{1cm} (48)

with an arbitrary function \( f \).
Let us look at a connection between equations with infinite symmetries and linear equations. Consider the class (33) of equations with symmetries containing an arbitrary function of the first derivatives $f(u_x, u_t)$. The Lagrangian function (33)

$$L = Q(u_x, u_t), \quad (49)$$

leads to the equation

$$D_x (L_{u_x}) + D_t (L_{u_t}) = 0. \quad (50)$$

In the notations $u_x \equiv z, u_t \equiv w$ we will get the following system

$$u_{xx} L_{zz} + 2u_{xt} L_{zw} + u_{tt} L_{ww} = 0, \quad (51)$$

$$z_t = w_x.$$
Using hodograph transformation:

\[ z(x, t), \ w(x, t) \rightarrow x(z, w), \ t(z, w) \]

the system (51) will take a form

\[
\begin{align*}
& t_w L_{zz} - (x_w + t_z) L_{zw} + x_z L_{ww} = 0, \\
& x_w = t_z,
\end{align*}
\]

(52)

which is a linear system.
Consider the class (29) of equations possessing symmetries with an arbitrary function of $u$

$$L = R \left( \frac{u_x}{u_t}, u \right),$$  \hspace{1cm} (53)

where function $R$ is arbitrary.
Corresponding Euler-Lagrange equation will have a form

\[-u_t^2 u_{xx} + 2u_x u_t u_{xt} - u_x^2 u_{tt} = u_t^4 f \left( \frac{u_x}{u_t}, u \right), \tag{54}\]

where

\[f \left( \frac{u_x}{u_t}, u \right) = -\frac{R_u \left( \frac{u_x}{u_t}, u \right)}{R_{\xi \xi} \left( \frac{u_x}{u_t}, u \right)}, \tag{55}\]

and

\[\xi = \frac{u_x}{u_t}.\]
Applying a hodograph transformation again $u(x, t) \rightarrow t(x, u)$ the equation (54) can be given a following form

$$t_{xx}(x, u) = f(u, -t_x(x, u)).$$  \hspace{1cm} (56)

The equation (56) in general, is not linear although it is linear with respect to second derivatives.
Joint work with Jonathan Roy.
Systems of hydrodynamic type were introduced by Dubrovin and Novikov [Sov. Math. Dokl. 1983] as **quasi-linear systems of first order PDE** which possessed Hamiltonian structure. Hydrodynamic-type systems describe various physics phenomena; hydrodynamics, gas dynamics, MHD magnetohydrodynamics, nonlinear elasticity, nonlinear plasticity models, and other.
We will not require a Hamiltonian structure and consider a general n-component homogeneous system of first order PDE in the form

\[ u_t^i = \sum_{j=1}^{n} v_j(u^1, u^2, \ldots, u^n, x, t) u_x^j, \]  

(57)

where \( u^i = u^i(x, t), \ i = 1, 2, \ldots, n \): with \( n \) functions \( u^i \) that depend on time \( t \) and one space variable \( x \). We will further assume that the system is diagonalizable (in terms of its Riemann invariants) and the coefficients \( v^i \) do not depend on \( x \) or \( t \) explicitly:

\[ u_t^i = v^i(u^1, u^2, \ldots, u^n) u_x^i, \quad i = 1, 2, \ldots, n. \]  

(58)
Symmetries of hydrodynamic-type system

Many papers have been written on systems of hydrodynamic-type, incl. work related to invariance properties of this system: Tsarev, Ferapontov, Rogers, Sheftel, others.


a) Strictly hyperbolic systems: \( v^i \neq v^j \) for \( i \neq j \). We will not assume the condition of non-degeneracy of the spectrum.
b) Coefficients \( v^i \) had explicit dependence on \( t \).
c) Looking for hydrodynamic flows commuting with the flows determined by the original system:

\[
\begin{align*}
\frac{u^i}{u^*_\tau} &= \sum_{j=1}^{n} A^i_j(u^1, u^2, \ldots, u^n, x, t) u^j_x, \\
i &= 1, 2, \ldots, n.
\end{align*}
\] (59)

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Infinite Symmetries
(Sheftel, Grundland, Winternitz) called these commuting flows hydrodynamic symmetries. These symmetries are not classical Lie point or tangent symmetries.

We will look for point and tangent transformation group for the system (58), and our main interest is the presence of infinite symmetries with arbitrary functions. Start with the simplest case \( n = 1 \).

We have

\[
\omega = u_t - v(u)u_x = 0.
\]  
(60)
We are looking for a symmetry operator

\[ X_\alpha = \alpha(u, u_x) \frac{\partial}{\partial u} + (D_x \alpha(u, u_x)) \frac{\partial}{\partial u_x} + (D_t \alpha(u, u_x)) \frac{\partial}{\partial u_t}, \] (61)

such that

\[ X_\alpha \omega = 0. \] (62)

We will get the following result:
Symmetries of hydrodynamic-type system

\[ v'(u) \neq 0 \quad \alpha(u, u_x) = C(u)u_x, \quad C(u) \text{ is arbitrary.} \]

\[ v'(u) = 0 \quad \alpha(u, u_x) \text{ is arbitrary.} \]

Thus, \( n = 1 \) hydrodynamic systems admit **infinite-dimensional symmetry with arbitrary functions** either of \( u \) or both \( u \) and \( u_x \).
The symmetry operator will take a form

\[ X_\alpha = \alpha^1(u^1, u^2, u_x^1, u_x^2) \frac{\partial}{\partial u^1} + \alpha^2(u^1, u^2, u_x^1, u_x^2) \frac{\partial}{\partial u^2} + \ldots \]  

(64)

The specific form of symmetry admitted by the system will depend on the form of functions \( v^i(u^1, u^2) \).
1. \( v^1(u^1, u^2) \neq v^2(u^1, u^2) \). In this case

\[
\alpha^1 = \alpha^1_1(u^1, u^2) + \alpha^1_2(u^1, u^2)u_x^1,
\]

\[
\alpha^1 = \alpha^2_1(u^1, u^2) + \alpha^2_2(u^1, u^2)u_x^2.
\]

(65)

Many sub-cases leading to various symmetries; where coefficients \( \alpha^i_k(u^1, u^2) \) include arbitrary functions \( l(u^1), m(u^2) \) or \( S(u^1, u^2) \).
2. \( v^1(u^1, u^2) = v^2(u^1, u^2) = v(u^1, u^2) \).

2a. \( v_{u^1} = v_{u^2} = 0 \) \hspace{1cm} (66)

\( \alpha^1(u^1, u^2, u^1_x, u^2_x), \alpha^2(u^1, u^2, u^1_x, u^2_x) \) are completely arbitrary functions of four variables.

2b. \( v = v(u^1) \) \hspace{1cm} (67)

The coefficient \( v(u^1, u^2) \) is a function of a single variable.
Symmetries of hydrodynamic-type system

\[ \alpha^1(u^1, u^2, u^1_x, u^2_x) = C^1 \left( u^1, u^2, \frac{u^2_x}{u^1_x} \right) u^1_x, \]  

(68)

\[ \alpha^2(u^1, u^2, u^1_x, u^2_x) = C^1 \left( u^1, u^2, \frac{u^2_x}{u^1_x} \right) u^2_x + C^2 \left( u^1, u^2, \frac{u^2_x}{u^1_x} \right). \]  

(69)

with arbitrary functions \( C^1, C^2 \).

Symmetry vectors have arbitrary functions of three variables.
2c. \( v_{u^1}, v_{u^2} \neq 0. \) \hspace{1cm} (70)

The coefficient \( v(u^1, u^2) \) is a function of two variables. In this case we obtain

\[
\alpha^1(u^1, u^2, u^1_x, u^2_x) = f \left( u^1, u^2, \frac{u^2_x}{u^1_x} \right) u^1_x - v_{u^2} g \left( u^1, u^2, \frac{u^2_x}{u^1_x} \right), \hspace{1cm} (71)
\]

\[
\alpha^2(u^1, u^2, u^1_x, u^2_x) = f \left( u^1, u^2, \frac{u^2_x}{u^1_x} \right) u^2_x + v_{u^1} g \left( u^1, u^2, \frac{u^2_x}{u^1_x} \right). \hspace{1cm} (72)
\]

Again, the symmetry group (pseudo-group) has arbitrary functions of three variables.
Symmetries of hydrodynamic-type system

Corollary

In degenerate case $v^1 = v^2$ the symmetry of 2x2 system of hydrodynamic-type drastically changes to include arbitrary functions of the derivatives $u^1_x, u^2_x$. The same effect is taking place for cases of higher $n$. 
Symmetries of hydrodynamic-type system

\[ n = 3. \]

1. \( v^1 (u^1, u^2, u^3) \neq v^2 (u^1, u^2, u^3) \neq v^3 (u^1, u^2, u^3). \)

\[ \alpha^i = \alpha^i_1 (u^i) + \alpha^i_2 (u^1, u^2, u^3) u^i_x, \quad i = 1, 2, 3 \quad (73) \]

Many sub-cases leading to various symmetries; coefficients \( \alpha^i_k \) include arbitrary functions \( l(u^1), m(u^2), n(u^3), p(u^1, u^2), q(u^1, u^3), \) or \( r(u^2, u^3). \)
Symmetries of hydrodynamic-type system

2. \( v^2 = v^3 \). \hfill (74)

\[
\alpha^1 = \alpha_1^1(u^1) + \alpha_2^1(u^1, u^2, u^3)u_x^1, \\
\alpha^2 = \alpha_1^2(u^2, u^3) + \alpha_2^2(u^1, u^2, u^3)u_x^2 + \alpha_4^2 \left( u^2, u^3, \frac{u_x^3}{u_x^2} \right), \\
\alpha^3 = \alpha_1^3(u^2, u^3) + \alpha_2^3(u^1, u^2, u^3)u_x^2 + \alpha_4^3 \left( u^2, u^3, \frac{u_x^3}{u_x^2} \right) \hfill (75)
\]

In the degenerate case arbitrary functions of derivatives show up.
3. \( v^1 = v^2 = v^3. \) (76)

\( \alpha^i \) have arbitrary functions \( C^i \left( u^1, u^2, u^3, \frac{u_x^2}{u_x^1}, \frac{u_x^3}{u_x^1} \right). \)

More degeneracy - more degrees of freedom in the derivatives.

Relations between infinite symmetries and conservation laws for hydrodynamic-type (non-Lagrangian) systems.
Open problems

- Physical meaning of infinite number of conservation laws.
- Relations between systems with infinite-dimensional symmetries and known integrable systems