Scalar and vector nonlinear Schrödinger systems with non-zero boundary conditions

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in collaboration with:

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Introduction

- **Nonlinear Schrödinger equation:**
  \[ iq_t + q_{xx} - 2\nu(|q|^2 - q_0^2)q = 0, \]
  \( \nu = \pm 1 \): focusing/defocusing.

- **Boundary conditions (BCs):**
  \[ q(x, t) \rightarrow q_\pm \text{ as } x \rightarrow \pm \infty \]
  (constant w.r.t. time),
  with \( |q_\pm| = q_0 \).

- **\( q_0 = 0 \): zero BCs (ZBCs),**
- **\( q_0 \neq 0 \): non-zero BCs (NZBCs).**
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- Vector NLS system:
  \[ i\mathbf{q}_t + \mathbf{q}_{xx} - 2\nu(|\mathbf{q}|^2 - q_o^2)\mathbf{q} = 0 , \]

- Case \( N = 2: \) Manakov system.

- BCs: \( \mathbf{q}(x, t) \to \mathbf{q}_\pm \text{ as } x \to \pm\infty. \)

- All integrable, solvable by IST...
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• All integrable, solvable by IST...
But the story is not yet complete.

- Scalar case w/ ZBC:
Zakharov-Shabat, 1972
- Scalar defocusing case w/ NZBC:
Zakharov-Shabat, 1973
(also Faddeev-Takhtajan, 1987)
- Manakov system, ZBC:
Manakov, 1974
\(N > 2\): Ablowitz-Prinari-Trubatch 2004)

- Defocusing Manakov w/ NZBC:
Prinari-Ablowitz-GB, 2006
(also GB-Kraus, in preparation)
- \(N > 2\) defocusing with NZBC:
Prinari-GB-Trubatch, 2011 (partial!)
- Scalar focusing case w/ NZBC:
Ma, 1979; GB-Kovacic, submitted
(also GB-Fagerstrom, in preparation)
- Focusing Manakov with NZBC:
GB-Kraus, in progress
Nonlinear Schrödinger equation:

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1. IST for focusing NLS with NZBC
   Direct problem
   Inverse problem

2. Soliton solutions of focusing NLS with NZBC

3. The Benjamin-Feir instability revisited

4. Defocusing Manakov system with NZBC

5. Focusing Manakov system with NZBC

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Lax pair and Riemann surface

- **NLS Lax pair:** \( \phi_x = X \phi \quad & \quad \phi_t = T \phi \),

\[
X = ik\sigma_3 + Q, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(x, t) = \begin{pmatrix} 0 & q \\ \nu q^* & 0 \end{pmatrix},
\]

\[
T = -2ik^2\sigma_3 + i\sigma_3(Q_x - Q^2 - q_o^2) - 2kQ.
\]

We formulate the IST so as to allow the reduction \( q_o \to 0 \).
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We formulate the IST so as to allow the reduction \( q_o \to 0 \).

- Asymptotic scattering problem: \( \phi_x = X_\pm \phi, \quad X_\pm = i k \sigma_3 + Q_\pm \).

- The eigenvalues of \( X_\pm \) are \( \pm i \lambda \), with \( \lambda^2 = q_o^2 + k^2 \).
Lax pair and Riemann surface

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- **Branch points:** values of \( k \) s.t. \( \lambda(k) = q_o^2 + k^2 = 0 \), i.e., \( k = \pm iq_o \).

Branch cut: \( i[-q_o, q_o] \).

- **Introduce the two-sheeted Riemann surface** \( \mathbb{C}_I \cup \mathbb{C}_{II} \) defined by \( \lambda(k) \).

- **Uniformization variable:** \( z = k + \lambda \).

\[
k \in \mathbb{C}_I \Leftrightarrow |z| > q_o, \; k \in \mathbb{C}_{II} \Leftrightarrow |z| < q_o.
\]

- express all \( k \) dependence as \( z \) dependence:

\[
k = \frac{1}{2} (z - q_o^2/z), \; \lambda = \frac{1}{2} (z + q_o^2/z).
\]
Jost solutions, analyticity and scattering matrix

- Eigenvector matrices: \( Y_{\pm} = I + (i/z)\sigma_3Q_{\pm}. \) s.t. \( X_{\pm}Y_{\pm} = Y_{\pm}i\lambda\sigma_3. \)
- Continuous spectrum: \( k \) s.t. \( \lambda \in \mathbb{R}: \) 
  \[
  k \in \mathbb{R} \cup i[-q_0, q_0] \iff z \in \Sigma = \mathbb{R} \cup C_o \quad (C_o = \text{circle of radius } q_o)
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- Asymptotic time evolution: $\dot{\phi} = T_\pm \phi$, $T_\pm = -2kX_\pm \Rightarrow X_\pm$ & $T_\pm$ admit common eigenvectors (as they should).

- Jost solutions $\phi_{\pm}$: simultaneous solutions of both parts of Lax pair s.t.
  
  $\phi_{\pm}(x, t, z) = Y_{\pm} e^{i\theta \sigma_3} + o(1)$ as $x \to \pm \infty$,

  $\theta(x, t, z) = \lambda (x - 2kt)$. (Formally: Volterra integral equations.)

  This way scattering matrix and norming constants are independent of time.
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- Scattering matrix:
  \( \phi_+ = \phi_- S(z) \quad \forall z \in \Sigma, \)
  \( b/c \ det \phi_{\pm} = det \mu_{\pm} = det Y_{\pm} = 1 + q_o^2 / z^2 \neq 0 \quad \forall z \neq 0, \pm iq_o \)
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- Remove oscillations: \( \mu(x, t, z) = \phi e^{-i\theta \sigma_3} \) (s.t. \( \mu_{\pm} \to I \) as \( x \to \pm\infty \))

- \( \text{Im } \lambda > 0 \) in \( \mathbb{C}^+ \) (gray), \( \text{Im } \lambda < 0 \) in \( \mathbb{C}^- \) (yellow)

- Analyticity: \( \mu_{-,1}, \mu_{+,2} \& s_{2,2} : \mathbb{C}^- \), \( \mu_{+,1}, \mu_{-,2} \& s_{1,1} : \mathbb{C}^+ \).
Symmetries and discrete spectrum

- The scattering problem admits two independent symmetries:
  (unlike focusing NLS with ZBC, and like defocusing NLS with NZBC)

\[ z \mapsto z^* \text{ (UHP/LHP)} \iff (k, \lambda) \mapsto (k^*, \lambda^*) \text{ (same sheet)}, \]
\[ z \mapsto -q_0^2/z \text{ (outside/inside } C_0), \iff (k, \lambda) \mapsto (k, -\lambda) \text{ (opposite sheets)}, \]

- Symmetries of Jost solutions:
  \[ \phi_\pm(x, t, z) = \sigma_2 \phi_\pm(x, t, z^*) \sigma_2, \quad \phi_\pm(x, t, z) = (i/z) \phi_\pm(x, t, -q_0^2/z) \sigma_3 Q_\pm. \]

- Symmetries of scattering coeffs:
  \[ s_{2,2}(z) = s_{1,1}^*(z^*), \quad s_{1,2}(z) = -s_{2,1}^*(z^*). \]
  \[ s_{1,1}(z) = (q_+/q_-) s_{2,2}(-q_0^2/z), \quad s_{1,2}(z) = (q_+/q_-) s_{2,1}(-q_0^2/z). \]
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- Discrete spectrum: \( z_1, \ldots, z_N \in \mathbb{C}^+ \) s.t. \( s_{1,1}(z_n) = 0 \):
  \[ \phi_{+,1}(x, t, z_n) = b_n \phi_{-,2}(x, t, z_n), \]

- Symmetries \( \Rightarrow \) eigenvalues appear in quartets:
  \( \{z_n, z_n^*, -q_o^2/z, -q_o^2/z^*\}_{n=1}^N \).
Symmetries and discrete spectrum

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- Notation: \( \zeta_n = z_n, \zeta_{n+n} = -q_0^2 / z_n \quad n = 1, \ldots, N. \)

Norming constants: \( C_n = b_n / s_{1,1}^*(z_n) \quad \forall n = 1, \ldots, N, \)
+ symmetric counterparts.
Riemann-Hilbert problem

Starting point (as usual): scattering relation, \( \mu_+ = \mu_- e^{i\theta \sigma_3} S e^{-i\theta \sigma_3} \quad \forall z \in \Sigma. \)

- Sectionally meromorphic matrices:
  \[
  M^+(x, t, z) = (\mu_{+,1}, \mu_{-,2}/s_{1,1}), \quad M^-(x, t, z) = (\mu_{-,1}/s_{2,2}, \mu_{+,2}).
  \]
  (Subscripts \( \pm \): normalization as \( x \to \pm \infty \). Superscripts \( \pm \): analyticity.)

- Riemann-Hilbert problem (RHP):
  \[
  M^- = M^+(I - G) \quad \forall z \in \Sigma.
  \]
  Jump matrix:
  \[
  G(x, t, z) = \begin{pmatrix}
  0 & -e^{2i\theta} \tilde{b} \\
  e^{-2i\theta} b & b \tilde{b}
  \end{pmatrix}.
  \]
  Reflection coefficients:
  \[
  b(z) = s_{2,1}/s_{1,1}, \quad \tilde{b}(z) = s_{1,2}/s_{2,2} = -b^*(z^*).\]
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- Asymptotics:
  \[
  \mu_{\pm} = I + O(1/z) \quad \text{as} \quad z \to \infty, \quad \mu_{\pm} = (i/z)\sigma_3 Q_{\pm} + O(1) \quad \text{as} \quad z \to 0.
  \]

- Regularize the RHP: subtract the asymptotic behavior as \(z \to \infty\)
  and the pole contributions at \(z = 0\) and \(z = \zeta_n, \ z = \zeta_n^*, \ n = 1, \ldots, 2N.\)
  (Assuming zeros of scattering coefficients are simple and finite.)
Solution of RHP and reconstruction formula

- Apply Cauchy transform and use Plemelj’s formulae:
  \[ M(x, t, z) = I + \frac{i}{z} \sigma_3 Q_- + \sum_{n=1}^{2N} \left( \frac{\text{Res}_{\zeta_n} M^+}{z - \zeta_n} + \frac{\text{Res}_{\zeta_n^*} M^-}{z - \zeta_n^*} \right) \]
  \[ + \frac{1}{2\pi i} \int_{\Sigma} \frac{M^+(x, t, \zeta)}{\zeta - z} G(x, t, \zeta) \, d\zeta. \]

- Evaluate residue conditions as usual ⇒ closed integral-algebraic system.

- Evaluate the asymptotics of the solution of the RHP as \( z \to \infty \).
  Compare with asymptotics of eigenfunctions ⇒ reconstruction formula:
  \[ q(x, t) = q_- + i \sum_{n=1}^{2N} \tilde{C}_n e^{2i\theta(\zeta_n^*)} \mu_{-1,1}(\zeta_n^*) + \frac{1}{2\pi} \int_{\Sigma} (M^+ G)_{1,2}(\zeta) \, d\zeta. \]
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- Reflectionless potentials: \( q(x, t) = q_\text{-} - i \det G^\text{ext} / \det G \),

  [note \( G \) is \( 2N \times 2N \) and \( G^\text{ext} \) is \( (2N+1) \times (2N+1) \)]
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- Apply Cauchy transform and use Plemelj’s formulae:

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M(x, t, z) = I + \frac{i}{z} \sigma_3 Q - \sum_{n=1}^{2N} \left( \frac{\text{Res}_{\zeta_n} M^+}{z - \zeta_n} + \frac{\text{Res}_{\zeta_n^*} M^-}{z - \zeta_n^*} \right) + \frac{1}{2\pi i} \int_{\Sigma} \frac{M^+(x, t, \zeta)}{\zeta - z} G(x, t, \zeta) \, d\zeta.
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[note \( G \) is \( 2N \times 2N \) and \( G^\text{ext} \) is \((2N+1) \times (2N+1)\)]

- Trace formulae:

\[
s_{1,1}(z) = \prod_{n=1}^{2N} \frac{z - \zeta_n}{z - \zeta_n^*} \exp \left\{ - \frac{1}{2\pi i} \int_{\Sigma} \log \left[ 1 + b(\zeta) b^*(\zeta^*) \right] \frac{1}{z - \zeta} \, d\zeta \right\},
\]

- “Theta” condition:

\[
\arg(q_+/q_-) = -4 \sum_{n=1}^{N} \arg z_n.
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Stationary one-soliton solutions

- Stationary one-soliton solutions: Let $N = 1$, $z_1 = iZq_o$, with $Z > 1$:

$$q(x, t) = q_- \frac{\cosh \chi + \frac{1}{2} A (c_1 \sin s - ic_2 \cos s)}{\cosh \chi + A \sin s},$$

$$\chi = \frac{1}{2} q_o (Z - 1/Z)x + \xi,$$

$$s = q_o^2 (Z^2 - 1/Z^2)t + \varphi,$$

$$A = 2/(Z + 1/Z), \quad c_1 = Z^2 + 1/Z^2, \quad c_2 = Z^2 - 1/Z^2.$$
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$$\chi = \frac{1}{2} q_o(Z - 1/Z)x + \xi,$$

$$s = q_o^2(Z^2 - 1/Z^2)t + \varphi,$$

$$A = 2/(Z + 1/Z), \quad c_1 = Z^2 + 1/Z^2, \quad c_2 = Z^2 - 1/Z^2.$$

- This solution:
  - is $t$-periodic
  - tends to $q_-$ as $|x| \to \infty$ ($\Delta \theta = 0$ here)
  - (opposite behavior compared to the homoclinic solutions of NLS with periodic BCs...)
  - was found by Kuznetsov in 1977 and Ma in 1979
  - reduces to the bright-soliton solution of NLS as $q_o \to 0$,
  - in the limit $Z \to 1$ yields the Peregrine solution (Peregrine, 1983):

$$q_{\text{peregrine}}(x, t) = \frac{(16t^2 + 4(x^2 - 4it) - 3)(16t^2 + 4x^2 + 1)}{(16t^2 + 4x^2 + 1)}.$$
Traveling 1-soliton solutions

Four-parameter family of 1-soliton solutions
(Watanabe-Tajiri 1991; Zakharov-Gelash 2011)

Left: \( z_1 = 1 + i (\Delta \theta = \pi) \).
Right: \( z_1 = 2 e^{i \pi/6} (\Delta \theta = 2\pi/3) \).
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Unlike NLS with ZBC, these are not simply Galilean boosts of the stationary solutions!
(no asymptotic carrier phase, different spectrum)
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Unlike NLS with ZBC, these are not simply Galilean boosts of the stationary solutions!
(no asymptotic carrier phase, different spectrum)

(But as \( q_o \to 0 \) both the “traveling” solitons and the Galilean-boosted stationary solutions do reduce to the traveling solitons of NLS with ZBC.)
One-parameter family generalization of Peregrine soliton

Explicit form of the traveling 1-soliton solutions:

\[ q(x, t) = \frac{c_o \cosh(\chi + 2i\alpha) + c_1(Z^2 \sin(s + 2\alpha) - \sin s) - ic_2(Z^2 \cos(s + 2\alpha) - \cos s)}{c_o \cosh \chi + Z^2 \sin(s + 2\alpha) - \sin s}, \]

\[ z_1 = iZ_0 e^{-i\alpha}, \quad \chi = i(\theta(z_1) - \theta(z_1^*)) + \xi, \quad s = \theta(z) + \theta(z_1^*) + \varphi, \quad c_j = \ldots \]
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\]

The limit \(Z \to 1\) yields Akhmediev’s “breathers” (1988)

\[
q_\alpha(x, t) = \frac{\cosh(2 \sin(2\alpha)t - 2i\alpha) - \cos \alpha \cos(2 \sin(\alpha)x)}{\cosh(2 \sin(2\alpha)t) - \cos \alpha \cos(2 \sin(\alpha)x)}.
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This solution:
- is \( x \)-periodic, with period \( \pi / \sin \alpha \),
- has \( q_{\text{max}} = (\cos \alpha - \cos(2\alpha))/(1 - \cos \alpha) \),
- tends to \( e^{\pm 2i\alpha} \) as \( t \to \pm \infty \), reduces to Peregrine’s as \( \alpha \to 0 \).

\( \alpha = \pi / 4 \)

\( \alpha = \pi / 3 \)

\( \alpha = 2\pi / 5 \)

\( \alpha = 0 \)
Multi-soliton solutions

Recall \( q(x, t) = q_- - i \det G^{\text{ext}} / \det G \).

Of course one can also easily write down multi-soliton solutions.

Two-soliton solutions:

Left: \( z_1 = 2i, z_2 = 3i \).

Right: \( z_1 = -1 + 2i, z_2 = 2 + i \).

And so on...
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The Benjamin-Feir instability revisited

- Benjamin-Feir instability (1967):
  plane wave solutions of focusing NLS exhibit modulational instability (MI)
  (i.e., the uniform solution \( q(x, t) = q_0 e^{2i\omega_0 t} \) is linearly unstable.)

- Essence of phenomenon w/ periodic BCs is known (homoclinic solutions)
  [Ablowitz-Ma 1981; Forest-McLaughlin-Overman 1991].
  - But it’s useful to understand differences btw periodic case and whole line
    (e.g.: w/ periodic BCs, there is a threshold for the presence of unstable modes)
  - Also, IST for periodic case cannot solve the full IVP!
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- Benjamin-Feir instability (1967): plane wave solutions of focusing NLS exhibit modulational instability (MI) (i.e., the uniform solution \( q(x, t) = q_o e^{2iq_o^2 t} \) is linearly unstable.)

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  - Also, IST for periodic case cannot solve the full IVP!

- Special case: piecewise constant, box-like ICs.

\[
q(x, 0) = \begin{cases} 
q_- & x < -L, \\
A e^{i\alpha} & |x| < L, \\
q_+ & x > L.
\end{cases}
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  - But it’s useful to understand differences btw periodic case and whole line (e.g.: w/ periodic BCs, there is a threshold for the presence of unstable modes)
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  \]

- Theorem: when \( A > q_o \), no threshold in \( L_2 \) norm for discrete eigenvalues (this is unlike ZBC, and like KdV and defocusing NZBC case)

- This suggests that the nonlinear stage of MI manifests itself through the formation of solitons [cf. Zakharov-Gelash 2011, 2012]
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Defocusing Manakov with NZBC: analyticity defect

- $3 \times 3$ Lax pair for the defocusing Manakov system: $\phi_x = X \phi$ & $\phi_t = T \phi,$

\[ X = -ikJ + Q, \quad J = \begin{pmatrix} 1 & 0^T \\ 0 & -I_2 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q^T \\ r & O_2 \end{pmatrix}, \quad r = q^* . \]

- Jost solutions: $\phi_\pm(x, t, z) = Y_\pm(z) e^{i \Theta(x, t, z)} + o(1)$ as $x \to \pm \infty,$

\[ Y_\pm = \begin{pmatrix} 1 & 0 & -iq_o/z \\ ir_\pm/z & q_\pm^T/q_o & r_\pm/q_o \end{pmatrix}, \quad \Theta = \Lambda(z)x - \Omega(z)t, \]

$\Lambda = \text{diag}(-\lambda, k, \lambda), \quad \Omega = -\text{diag}(-2k\lambda, k^2 + \lambda^2, 2k\lambda), \quad \lambda^2 = k^2 - q_o^2.$

- Uniformization: $z = k + \lambda; \quad k \in \mathbb{C}_I \Rightarrow z \in \text{UHP}, \quad k \in \mathbb{C}_\Pi \Rightarrow z \in \text{LHP}.$
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- **Uniformization:** \( z = k + \lambda; \quad k \in \mathbb{C}_I \Rightarrow z \in \text{UHP}, \quad k \in \mathbb{C}_II \Rightarrow z \in \text{LHP}. \)

- **Continuous spectrum:** \( k \) s.t. \( \lambda \in \mathbb{R}, \) i.e. \( k \in (-\infty, -q_o] \cup [q_o, \infty) \Rightarrow z \in \mathbb{R}. \)

- **Scattering matrix:**

\[
\forall z \in \mathbb{R} \quad \phi_- = \phi_+ A, \quad \text{or} \quad \phi_+ = \phi_- B, \quad \text{with} \quad B = A^{-1}.
\]

- **Analyticity:**

\( \phi_{-,1}, \phi_{+,3}, a_{1,1}, b_{3,3} : \quad z \in \text{UHP}, \quad \phi_{+,1}, \phi_{-,3}, b_{1,1}, a_{3,3} : \quad z \in \text{UHP}. \)

- **Problem:** \( \phi_{-,2} \) & \( \phi_{+,2} \) are nowhere analytic!
Defocusing Manakov with NZBC: adjoint problem

- “Adjoint” problem: \( \tilde{\phi}_x = (ikJ + Q^*)\tilde{\phi} \) & \( \tilde{\phi}_t = T^*\tilde{\phi} \).
  [cf. Kaup 1976, IST for three-wave interaction equations]

- Lemma: if \( \tilde{u}(x, t, z) \) & \( \tilde{v}(x, t, z) \) solve the adjoint Lax pair,
  \[ w(x, t, z) = e^{i\theta} J(\tilde{u} \times \tilde{v}) \]
  solves the original Lax pair [with \( \theta(x, t, z) = kx + (k^2 + \lambda^2)t \)].
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  solves the original Lax pair [with \( \theta(x, t, z) = kx + (k^2 + \lambda^2)t \)].
- Auxiliary eigenfunctions:
  \[ \chi(x, t, z) = -e^{i\theta} J(\tilde{\phi}_{-,3} \times \tilde{\phi}_{+,1})/\gamma, \quad \bar{\chi}(x, t, z) = -e^{i\theta} J(\tilde{\phi}_{-,1} \times \tilde{\phi}_{+,3})/\gamma. \]
  ○ Lemma: \( \chi \) & \( \bar{\chi} \) are analytic for \( z \in \text{UHP} \) & \( z \in \text{LHP} \), respectively,
- Corollary: \((\phi_{-,1}, \chi, \phi_{+,3})\) & \((\phi_{+,1}, \bar{\chi}, \phi_{-,3})\) form fundamental analytic eigenfunctions in \( \text{UHP} \) and \( \text{LHP} \), respectively.
Defocusing Manakov system with NZBC: adjoint problem

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  - Corollary: \( (\phi_{-1}, \chi, \phi_{+3}) \) & \( (\phi_{+1}, \bar{\chi}, \phi_{-3}) \) form fundamental analytic eigenfunctions in UHP and LHP, respectively.

- Symmetries:
  \[ \phi_\pm(x, t, z) = \phi_\pm(x, t, q_o^2/z) \prod(z) = J(\phi_{\pm}^\dagger(x, t, z^*))^{-1} C(z), \]
  \[ \prod(z) = \begin{pmatrix} 0 & 0 & -iq_o/z \\ 0 & 1 & 0 \\ iq_o/z & 0 & 0 \end{pmatrix}, \quad C = \text{diag}(-\gamma, 1, \gamma), \quad \gamma = \det Y_\pm = 1 - q_o^2/z^2. \]
  - Note \( (M^\dagger)^{-1} = (M_2 \times M_3, M_3 \times M_1, M_1 \times M_2)/\det M. \)
Defocusing Manakov with NZBC: dark-bright solitons

- \( k_n \in [-q_o, q_o] \Rightarrow |z_n| = q_o \Rightarrow \) dark solitons (same as scalar NLS) (apart from constant unit polarization vector)

- \( k_n \notin \mathbb{R} \Rightarrow |z_n| < q_o \Rightarrow \) dark-bright solitons:
  
  \[
  q_1(x, t) = q_o (\cos \alpha + i \sin \alpha \tanh S), \quad q_2(x, t) = \nu \sin \alpha \sqrt{q_o^2 - |z_o|^2} \sech S e^{i\theta},
  \]

  \[
  S(x, t) = \nu(x - x_o - 2kt), \quad \theta(x, t) = -kx + (k^2 - \nu^2)t,
  \]

  \( k = \text{Re } z_o = |z_o| \cos \alpha, \quad \nu = \text{Im } z_o = |z_o| \sin \alpha. \)
Defocusing Manakov with NZBC: double-pole solutions

- Scalar focusing NLS with NZBC admits double-pole solutions
  - obtained when the analytic scattering coefficient has a double zero
  - similar to a pair of solitons with same amplitude and velocity, but diverge from each other logarithmically

- Discrete eigenvalues of scalar defocusing NLS with NZBC are simple. But proof does not generalize to Manakov system.
**Defocusing Manakov with NZBC: double-pole solutions**

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- Discrete eigenvalues of scalar defocusing NLS with NZBC are simple. But proof does not generalize to Manakov system.

- New: Defocusing Manakov with NZBC admits double-pole solutions. Need to enlarge the Riemann-Hilbert problem & include the derivatives of the eigenfunctions w.r.t. $z$ as additional unknowns.
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Focusing Manakov with NZBC: IST outline

- Lax pair is essentially the same, but now $\lambda^2 = k^2 + q_o^2$ (as in scalar case).
- Difference from defocusing case:
  - each eigenfunction and diagonal entry of scattering matrix is analytic somewhere,
  - but four different fundamental domains of analyticity are present.
As in defocusing case, in each region, only two eigenfunctions and scattering coefficients are analytic.
- One can still use the adjoint problem, but four auxiliary eigenfunctions are now necessary.

\[
\begin{align*}
\text{Re } z & \\
\text{Im } z & \\
0^- & \\
0^+ & \\
i q_o & \\
i q_o & \\
\end{align*}
\]

\[
\begin{align*}
(\chi_4, \phi_{+,2}, \phi_{-,3}) & \\
a_{3,3}, b_{2,2} & \\
(\chi_3, \phi_{-,2}, \phi_{+,3}) & \\
a_{2,3}, b_{3,3} & \\
& \\
(\phi_{-,1}, \phi_{+,2}, \chi_1) & \\
a_{1,1}, b_{2,2} & \\
(\phi_{+,1}, \phi_{-,2}, \chi_2) & \\
a_{2,2}, b_{1,1} & \\
\end{align*}
\]
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- One can still use the adjoint problem, but four auxiliary eigenfunctions are now necessary.
- Discrete eigenvalues still appear in symmetric quartets (as in the scalar case)
  \[ \{z_n, z_n^*, -q_o^2/z_n, -q_o^2/z_n^*\} \].
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  As in defocusing case, in each region, only two eigenfunctions and scattering coefficients are analytic.
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- Discrete eigenvalues still appear in symmetric quartets (as in the scalar case) \( \{z_n, z_n^*, -q_o^2/z_n, -q_o^2/z_n^*\} \).
- Three kinds of discrete eigenvalues.
  Let \( z_o \in \text{UHP} \cap \{ |z_o| > q_o \} \).
  (i) \( a_{1,1}(z_o) = 0 \) & \( b_{2,2}(z_o) \neq 0 \),
  (ii) \( a_{1,1}(z_o) \neq 0 \) & \( b_{2,2}(z_o) = 0 \),
  (iii) \( a_{1,1}(z_o) = b_{2,2}(z_o) = 0 \).
  Each kind of eigenvalue generates a different kind of solution.
Focusing Manakov with NZBC: soliton solutions

- Eigenvalues of 1st kind yield the same kind of solitons as scalar NLS.
- Eigenvalues of 2nd kind:

- Eigenvalues of 3rd kind:
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Ongoing/future work

- Focusing NLS:
  - detailed comparison with homoclinic instabilities of periodic case
  - use IST to characterize Benjamin-Feir (growth rate, most unstable mode...)
    (this is not possible in the periodic case!)
  - compare with linearization

- Focusing/defocusing NLS:
  - develop IST for ICs with different amplitudes as $x \to \pm \infty$
    (this has not been done even in the scalar defocusing case!)
  - study long-time asymptotics!

- Focusing Manakov: classify all the soliton solutions

- Defocusing $N$-component: obtain and classify all soliton solutions
  (The adjoint problem trick does not generalize to an arbitrary components,
   so a different formulation using tensors/forms is used...)

- Focusing $N$-component NLS with NZBC: completely open!
Conclusions

References

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Thank you for your attention!