On Bianchi permutability of Bäcklund transformations for asymmetric quad-equations

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Based on:


extending the results of

1. 2D Systems of Quad-equations

2. Bäcklund transformations – 3D Consistent Systems of Quad-Equations

3. Bianchi Permutability – 4D Consistent Systems of Quad-Equations

4. Necessary tools for proofs
Quad-equations

Definition (Quad-equation)
An equation

\[ Q(x_1, x_2, x_3, x_4) = 0 \]

with

- \( Q \in \mathbb{C}[x_1, x_2, x_3, x_4] \),
- \( Q \) irreducible and multi-affine.

Simplest situation: Quad-equation is an elementary building block of a discrete system

\[ Q(x_{m,n}, x_{m+1,n}, x_{m+1,n+1}, x_{m,n+1}) = Q(x, x_1, x_2, x_{12}) = 0 \]

for a function \( x : \mathbb{Z}^2 \to \mathbb{C} \).
Quad-equations

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for a function \( x : \mathbb{Z}^2 \rightarrow \mathbb{C} \).
Reflecting quad-equations

Construction (Reflecting one quad-equation)

For a quad-equation

\[ Q (x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) = 0 \]

the three other quad-equations of an elementary cell are given by

\( |Q| = 0, \underline{Q} = 0 \) and \( \underline{|Q|} = 0 \) with

\[ \begin{align*} 
|Q (x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) :& = Q (x_1, x, x_{12}, x_2; \alpha_1, \alpha_2), \\
Q (x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) :& = Q (x_1, x_{12}, x_1, x; \alpha_1, \alpha_2) \quad \text{and} \\
\underline{|Q} (x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) :& = Q (x_{12}, x_2, x_1, x; \alpha_1, \alpha_2). 
\end{align*} \]
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the three other quad-equations of an elementary cell are given by

\[ |Q| = 0, \quad _Q = 0 \quad \text{and} \quad |Q| = 0 \]

with

\[ |Q(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) := Q(x_1, x, x_{12}, x_2; \alpha_1, \alpha_2) , \]
\[ _Q(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) := Q(x_2, x_{12}, x, x_1; \alpha_1, \alpha_2) \quad \text{and} \]
\[ |Q(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) := Q(x_{12}, x_2, x_1, x; \alpha_1, \alpha_2) . \]
Asymmetric quad-equations

Definition (Asymmetric quad-equation)

A quad-equation is called *asymmetric* if $Q \neq |Q|$ or $Q \neq \overline{Q}$.

Example:

$$Q := (x - x_1)(x_2 - x_{12}) - \alpha_1 (1 + \varepsilon^2 x_2 x_{12}) = 0$$

Here, $Q = |Q|$ but $Q \neq \overline{Q}$:

$$\overline{Q} = (x - x_1)(x_2 - x_{12}) - \alpha_1 (1 + \varepsilon^2 xx_1) = 0.$$
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Alternative embedding in $\mathbb{Z}^2$
Alternative embedding in $\mathbb{Z}^2$

Elementary cell:

Translation of elementary cells:
Biquadratics and types of quad-equations

- For every quad-equation $Q(x_1, x_2, x_3, x_4) = 0$ one can compute 6 so-called biquadratics which can be assigned to the 4 edges and 2 diagonals of a quadrilateral.
- We distinguish between non-degenerate and degenerate biquadratics.
- There are 3 different types of quad-equations:
  - type Q: all 6 biquadratics are non-degenerate
  - type $H^4$: 4 out of 6 biquadratics are degenerate
  - type $H^6$: all 6 biquadratics are degenerate (We will not consider them here.)
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A Bäcklund cube – 6 quad-equations on the faces of a 3D cube

\[ A(x, x_1, x_2, x_{12}) = 0, \]
\[ B(x, x_2, x_3, x_{23}) = 0, \]
\[ C(x, x_1, x_3, x_{13}) = 0, \]
\[ D(x_3, x_{13}, x_{23}, x_{123}) = 0, \]
\[ E(x_1, x_{12}, x_{13}, x_{123}) = 0, \]
\[ F(x_2, x_{12}, x_{23}, x_{123}) = 0 \]
Tetrahedron property

\[ K(x, x_{12}, x_{13}, x_{23}) = 0, \]
\[ L(x_1, x_2, x_3, x_{123}) = 0 \]
Reflecting a Bäcklund cube

Original Bäcklund cube:

Neighbor on the right hand side:
Bäcklund transformation

Definition (Bäcklund transformation $x \xrightarrow{\lambda} y$)

Consider a system of quad-equations on the lattice $\mathbb{Z}^2 \times \{0, 1\}$ generated by reflection and a solution $x$ of the system composed of the equations $A = 0$, $|A| = 0$, $\underline{A} = 0$ and $|\underline{A}| = 0$ on the “horizontal” sublattice $\mathbb{Z}^2 \times \{0\}$.

Then we say that a solution $y$ of the system composed of the equations $D = 0$, $|D| = 0$, $\underline{D} = 0$ and $|\underline{D}| = 0$ on the sublattice $\mathbb{Z}^2 \times \{1\}$ is a Bäcklund transform of $x$ if the combination of $x$ and $y$ build a solution of the system on the whole lattice $\mathbb{Z}^2 \times \{0, 1\}$, i.e., also the equations on “vertical” faces are satisfied.
Bäcklund transformation

Definition (Bäcklund transformation $\lambda \mapsto y$)

Consider a system of quad-equations on the lattice $\mathbb{Z}^2 \times \{0, 1\}$ generated by reflection and a solution $x$ of the system composed of the equations $A = 0$, $|A| = 0$, $A = 0$ and $|A| = 0$ on the “horizontal” sublattice $\mathbb{Z}^2 \times \{0\}$.

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Example

Consider a solution of system of quad-equations on $\mathbb{Z}^2 \times \{0\}$ with

$$A := (x - x_1)(x_2 - x_{12}) - \alpha_1 \left(1 + \varepsilon^2 x_2 x_{12}\right) = 0.$$

Then there are two essentially different Bäcklund transformations:

- $BT_1 : x \xrightarrow{\lambda_1} y^{(1)}$ with
  $$D := \left(y^{(1)} - y^{(1)}_1\right) \left(y^{(1)}_2 - y^{(1)}_{12}\right) - \alpha \left(1 + \varepsilon^2 y^{(1)}_2 y^{(1)}_{12}\right) = 0$$

- $BT_2 : x \xrightarrow{\lambda_2} y^{(2)}$ with
  $$D := \left(y^{(2)} - y^{(2)}_1\right) \left(y^{(2)}_2 - y^{(2)}_{12}\right) - \alpha \left(1 + \varepsilon^2 y^{(2)}_2 y^{(2)}_{12}\right) = 0$$
Example

Consider a solution of system of quad-equations on $\mathbb{Z}^2 \times \{0\}$ with

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- $BT_2 : x \mapsto y^{(2)}$ with
  $$D := \left(y^{(2)} - y_1^{(2)}\right)\left(y_2^{(2)} - y_{12}^{(2)}\right) - \alpha \left(1 + \varepsilon^2 y_2^{(2)} y_1^{(2)}\right) = 0$$
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- $BT_2 : x \overset{\lambda_2}{\mapsto} y^{(2)}$ with
  $$D := \left(y^{(2)} - y_{1}^{(2)}\right)\left(y_{2}^{(2)} - y_{12}^{(2)}\right) - \alpha \left(1 + \varepsilon^2 y_{2}^{(2)} y_{1}^{(2)}\right) = 0$$
Two layers of Bäcklund cubes

Construction (Two layers of Bäcklund cubes)

Consider a layer of Bäcklund cubes, i.e., a 3D consistent system of quad-equations on $\mathbb{Z}^2 \times \{0, 1\}$ generated by reflection.

The second layer can be derived from the original one on the lattice $\mathbb{Z}^2 \times \{0, 1\}$ by reflecting the Bäcklund cubes in the plane $\mathbb{Z}^2 \times \{1\}$ in order to get a 3D consistent system of quad-equations on the lattice $\mathbb{Z}^2 \times \{0, 1, 2\}$. 
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Auto-Bäcklund transformation

Construction (Two-stage auto-Bäcklund transformation
\( x \xleftarrow{\lambda} y \xleftarrow{\lambda^3} x^+ \))

Consider a 3D consistent system of quad-equations on the lattice \( \mathbb{Z}^2 \times \{0, 1, 2\} \) generated by reflection. Furthermore, consider a solution \( x \) of the system composed of the equations \( A = 0, \mid A \mid = 0, \overline{A} = 0 \) and \( \overline{\mid A \mid} = 0 \) on the sublattice \( \mathbb{Z}^2 \times \{0\} \) and one of its Bäcklund transforms \( y \) satisfying the system composed of the equations \( D = 0, \mid D \mid = 0, \overline{D} = 0 \) and \( \overline{\mid D \mid} = 0 \) on the sublattice \( \mathbb{Z}^2 \times \{1\} \) with Bäcklund parameter \( \lambda \). Then a Bäcklund transform \( x^+ \) of \( y \) satisfying the system composed of the equations \( A = 0, \mid A \mid = 0, \overline{A} = 0 \) and \( \overline{\mid A \mid} = 0 \) on the sublattice \( \mathbb{Z}^2 \times \{2\} \) with Bäcklund parameter \( \lambda_3 \) is a (non-elementary) auto-Bäcklund transform of \( x \).
Auto-Bäcklundi transformation

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\(x \xrightarrow{\lambda} y \xrightarrow{\lambda^3} x^+\))

Consider a 3D consistent system of quad-equations on the lattice \(\mathbb{Z}^2 \times \{0, 1, 2\}\) generated by reflection. Furthermore, consider a solution \(x\) of the system composed of the equations \(A = 0, |A| = 0, \bar{A} = 0\) and \(|\bar{A}| = 0\) on the sublattice \(\mathbb{Z}^2 \times \{0\}\) and one of its Bäcklundi transforms \(y\) satisfying the system composed of the equations \(D = 0, |D| = 0, \bar{D} = 0\) and \(|\bar{D}| = 0\) on the sublattice \(\mathbb{Z}^2 \times \{1\}\) with Bäcklundi parameter \(\lambda\). Then a Bäcklundi transform \(x^+\) of \(y\) satisfying the system composed of the equations \(A = 0, |A| = 0, \bar{A} = 0\) and \(|\bar{A}| = 0\) on the sublattice \(\mathbb{Z}^2 \times \{2\}\) with Bäcklundi parameter \(\lambda^3\) is a (non-elementary) auto-Bäcklundi transform of \(x\).
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Then a Bäcklund transform \( x^+ \) of \( y \) satisfying the system composed of the equations \( A = 0, \overline{A} = 0, \underline{A} = 0 \) and \( \overline{A} = 0 \) on the sublattice \( \mathbb{Z}^2 \times \{2\} \) with Bäcklund parameter \( \lambda_3 \) is a (non-elementary) auto-Bäcklund transform of \( x \).
A Bianchi cube
The extension of two Bäcklund cubes to a Bianchi cube
Theorem

Given two Bäcklund cubes sharing one quad-equation, there exists an extension to a Bianchi cube. This extension is unique up to Möbius transformations of fields not belonging to the original Bäcklund cubes (the fields $X$, $X_1$, $X_2$ and $X_{12}$).

Bianchi diagram:
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Bianchi diagram:
Bianchi permutability of elementary Bäcklund transformations

Corollary

Let $x$ be a solution of a system of quad-equations on $\mathbb{Z}^2 \cong \mathbb{Z}^2 \times \{0\} \times \{0\}$ generated from the equation $A(x, x_1, x_2, x_{12}) = 0$ by reflection and two of its Bäcklund transformations $x \xrightarrow{\lambda_1} y^{(1)}$ and $x \xrightarrow{\lambda_2} y^{(2)}$ (depending on one value $y^{(1)}(0)$ respectively on $y^{(2)}(0)$ each).

Then there exists a unique solution $X$ on the system of quad-equations on $\mathbb{Z}^2 \cong \mathbb{Z}^2 \times \{1\} \times \{1\}$ which is obtained from an equation $G(X, X_1, X_2, X_{12}) = 0$ by reflection which is simultaneously a Bäcklund transform of $y^{(1)}$ via a Bäcklund transformation $y^{(1)} \xrightarrow{\lambda_2} X$ and of $y^{(2)}$ via a Bäcklund transformation $y^{(2)} \xrightarrow{\lambda_1} X$. 
Bianchi permutability of elementary Bäcklund transformations

Corollary

Let \( x \) be a solution of a system of quad-equations on \( \mathbb{Z}^2 \cong \mathbb{Z}^2 \times \{0\} \times \{0\} \) generated from the equation \( A(x, x_1, x_2, x_{12}) = 0 \) by reflection and two of its Bäcklund transformations \( x \xrightarrow{\lambda_1} y^{(1)} \) and \( x \xrightarrow{\lambda_2} y^{(2)} \) (depending on one value \( y^{(1)}(0) \) respectively on \( y^{(2)}(0) \) each).

Then there exists a unique solution \( X \) on the system of quad-equations on \( \mathbb{Z}^2 \times \{1\} \times \{1\} \) which is obtained from an equation \( G(X, X_1, X_2, X_{12}) = 0 \) by reflection which is simultaneously a Bäcklund transform of \( y^{(1)} \) via a Bäcklund transformation \( y^{(1)} \xrightarrow{\lambda_2} X \) and of \( y^{(2)} \) via a Bäcklund transformation \( y^{(2)} \xrightarrow{\lambda_1} X \).
Bianchi permutability of auto-Bäcklund transformations

\[ x(12) = x(21) \]
**Theorem**

Consider a solution \( x \) of a system of quad-equations on \( \mathbb{Z}^2 \cong \mathbb{Z}^2 \times \{0\} \times \{0\} \) generated by reflection and two two-stage auto-Bäcklund transformations

\[
\text{aBT}_1 : x^{(1)} \xrightarrow{\lambda_1} y^{(1)} \xrightarrow{\lambda_{11}} x^{(1)} \quad \text{and} \quad \text{aBT}_2 : x^{(2)} \xrightarrow{\lambda_2} y^{(2)} \xrightarrow{\lambda_{22}} x^{(2)}
\]

(depending on two values \( y^{(1)}(0) \) and \( x^{(1)}(0) \) respectively on \( y^{(2)}(0) \) and \( x^{(2)}(0) \) each).

Then there exists a unique solution \( x^{(12)} = x^{(21)} \) which is simultaneously an auto-Bäcklund transform of \( x^{(1)} \) and of \( x^{(2)} \) via

\[
\text{aBT}_2 : x^{(1)} \xrightarrow{\lambda_2} y^{(12)} \xrightarrow{\lambda_{22}} x^{(12)} \quad \text{and} \quad \text{aBT}_1 : x^{(2)} \xrightarrow{\lambda_1} y^{(21)} \xrightarrow{\lambda_{11}} x^{(21)},
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Consider a solution \( x \) of a system of quad-equations on \( \mathbb{Z}^2 \cong \mathbb{Z}^2 \times \{0\} \times \{0\} \) generated by reflection and two two-stage auto-Bäcklund transformations

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respectively.
Making tetrahedra to faces – Super-consistent eight-tuples on decorated cubes
Lemma

Consider a 3D consistent system possessing the tetrahedron property.

Then, the system

\[ K(x, x_{12}, x_{13}, x_{23}) = 0, \quad L(x_1, x_2, x_3, x_{123}) = 0, \]
\[ B(x, x_2, x_3, x_{23}) = 0, \quad E(x_1, x_{12}, x_{13}, x_{123}) = 0, \]
\[ C(x, x_1, x_3, x_{13}) = 0, \quad F(x_2, x_{12}, x_{23}, x_{123}) = 0 \]

is 3D consistent and possesses the tetrahedron property. 3D consistency of this system is understood as the property of the initial value problem with initial data \( x, x_3, x_{13} \) and \( x_{23} \).

We call the eight-tuple \( A = D = B = E = C = F = K = L = 0 \) a super-consistent eight-tuple on a decorated cube.
Lemma

Consider a 3D consistent system possessing the tetrahedron property.

Then, the system

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Super-consistent eight-tuples composed from tetrahedron equations of the 3D facets

Decorated 3D cube of vertices whose indices have an even number of digits:

Decorated 3D cube of vertices whose indices have an odd number of digits:
http://discretization.de/

http://page.math.tu-berlin.de/~boll/