On Bianchi permutability of Bäcklund transformations for asymmetric quad-equations

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Discretization in Geometry and Dynamics
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Based on:

R. Boll, *On Bianchi permutability of Bäcklund transformations for asymmetric quad-equations*, arXiv:1211.4374 [nlin.SI]

extending the results of

R. Boll, *Classification of 3D consistent quad-equations*, J. Nonlin. Math. Phys., 2011, 18, No. 3, pp. 337-365.

Quad-equations

Definition (Quad-equation)

An equation

$$Q\left(x_1, x_2, x_3, x_4\right) = 0$$

with

- $Q \in \mathbb{C}\left[x_1, x_2, x_3, x_4\right]$,
- Q irreducible and multi-affine.

Simplest situation: Quad-equation is an elementary building block of a discrete system

$$Q\left(x_{m,n}, x_{m+1,n}, x_{m+1,n+1}, x_{m,n+1}\right) = Q\left(x, x_1, x_2, x_{12}\right) = 0$$

for a function $x : \mathbb{Z}^2 \to \mathbb{C}$.

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Reflecting quad-equations

Construction (Reflecting one quad-equation)

For a quad-equation

$$Q\left(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2\right) = 0$$

the three other quad-equations of an elementary cell are given by $|Q = 0$, $\underline{Q} = 0$ and $\left|\underline{Q}\right. = 0$ with

$$\begin{aligned} |Q\left(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2\right) &:= Q\left(x_1, x, x_{12}, x_2; \alpha_1, \alpha_2\right), \\ \underline{Q}\left(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2\right) &:= Q\left(x_2, x_{12}, x, x_1; \alpha_1, \alpha_2\right) \quad \text{and} \\ \left|\underline{Q}\right.\left(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2\right) &:= Q\left(x_{12}, x_2, x_1, x; \alpha_1, \alpha_2\right). \end{aligned}$$

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A quad-equation is called *asymmetric* if $Q \neq |Q$ or $Q \neq \underline{Q}$.

Example:

$$Q := (x - x_1)(x_2 - x_{12}) - \alpha_1 \left(1 + \varepsilon^2 x_2 x_{12}\right) = 0$$

Here, $Q = |Q$ but $Q \neq \underline{Q}$:

$$\underline{Q} = (x - x_1)(x_2 - x_{12}) - \alpha_1 \left(1 + \varepsilon^2 x x_1\right) = 0.$$

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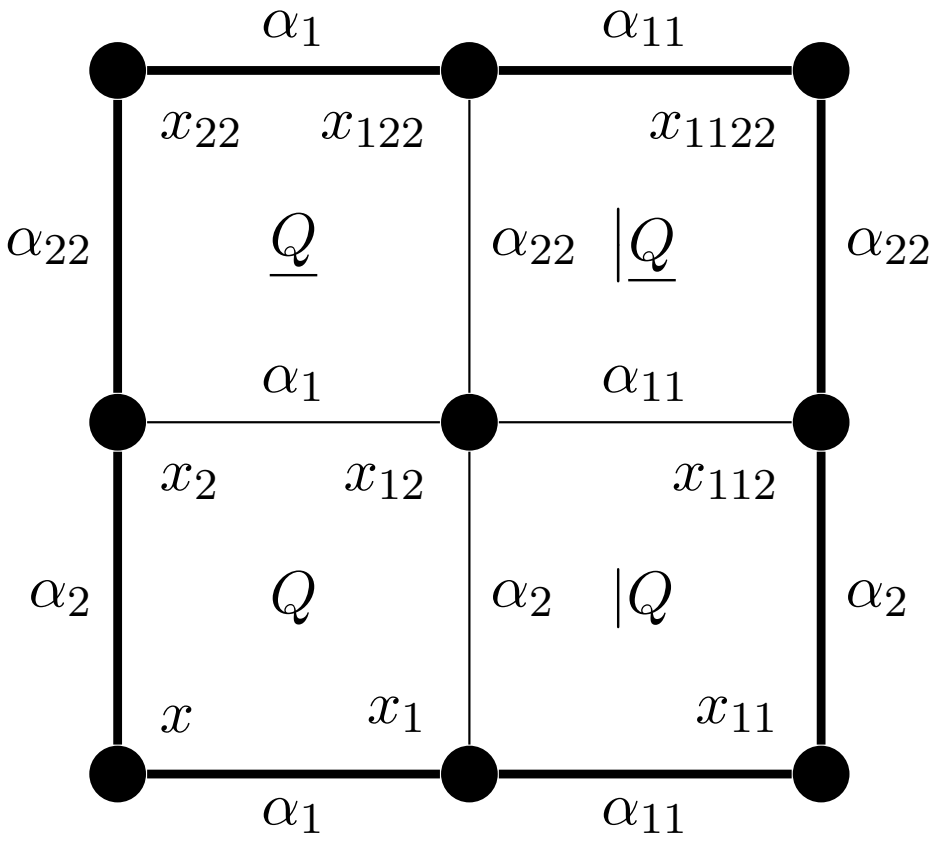
Alternative embedding in $\mathbb{Z}^2$

Elementary cell:



Translation of elementary cells:

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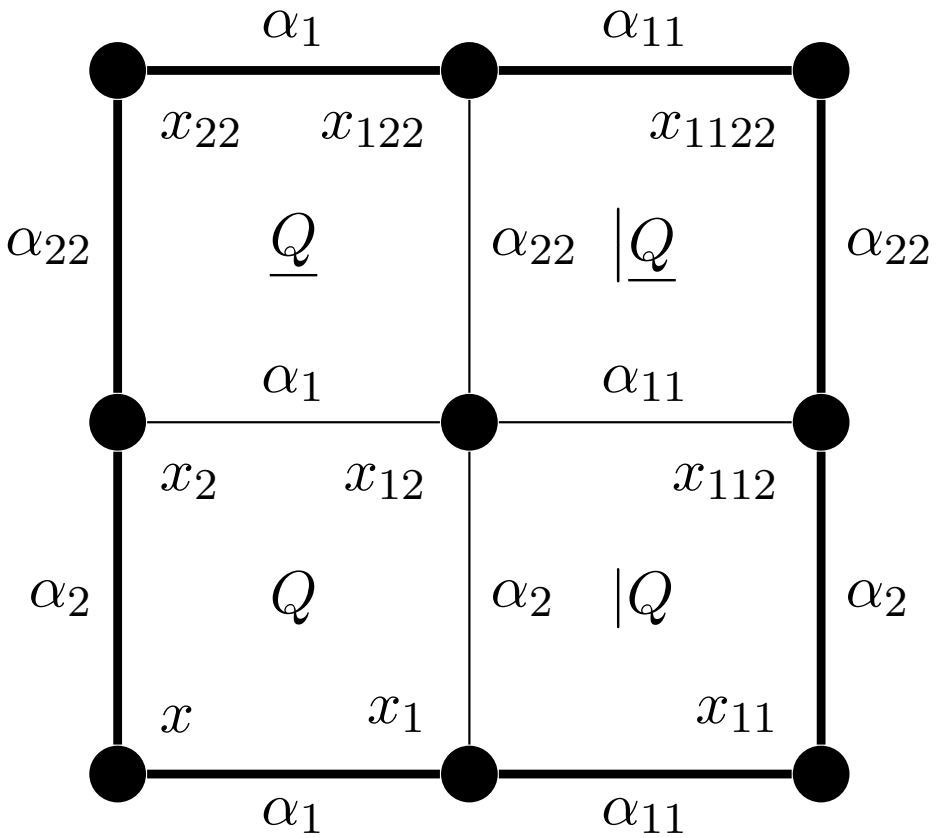

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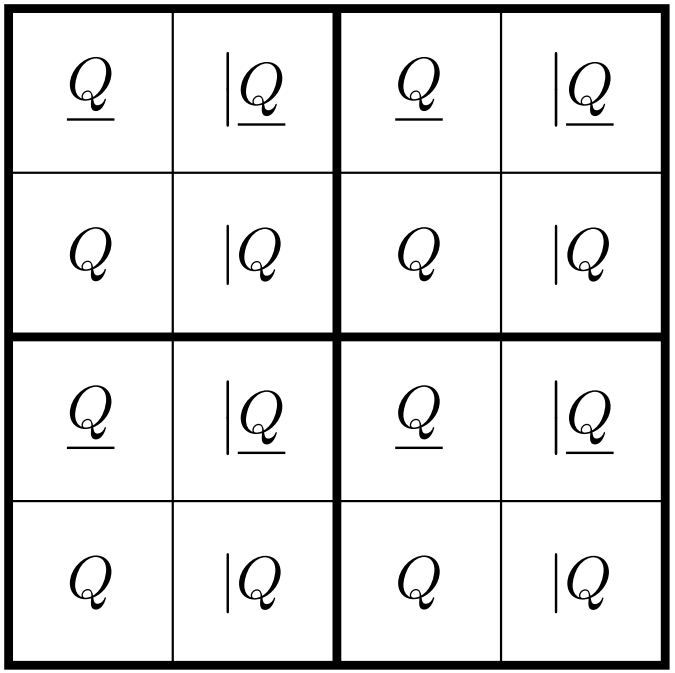

Biquadratics and types of quad-equations

- For every quad-equation $Q(x_1, x_2, x_3, x_4) = 0$ one can compute 6 so-called biquadratics which can be assigned to the 4 edges and 2 diagonals of a quadrilateral.
- We distinguish between non-degenerate and degenerate biquadratics.
- There are 3 different types of quad-equations:
  - type Q: all 6 biquadratics are non-degenerate
  - type H^4: 4 out of 6 biquadratics are degenerate
  - type H^6: all 6 biquadratics are degenerate (We will not consider them here.)

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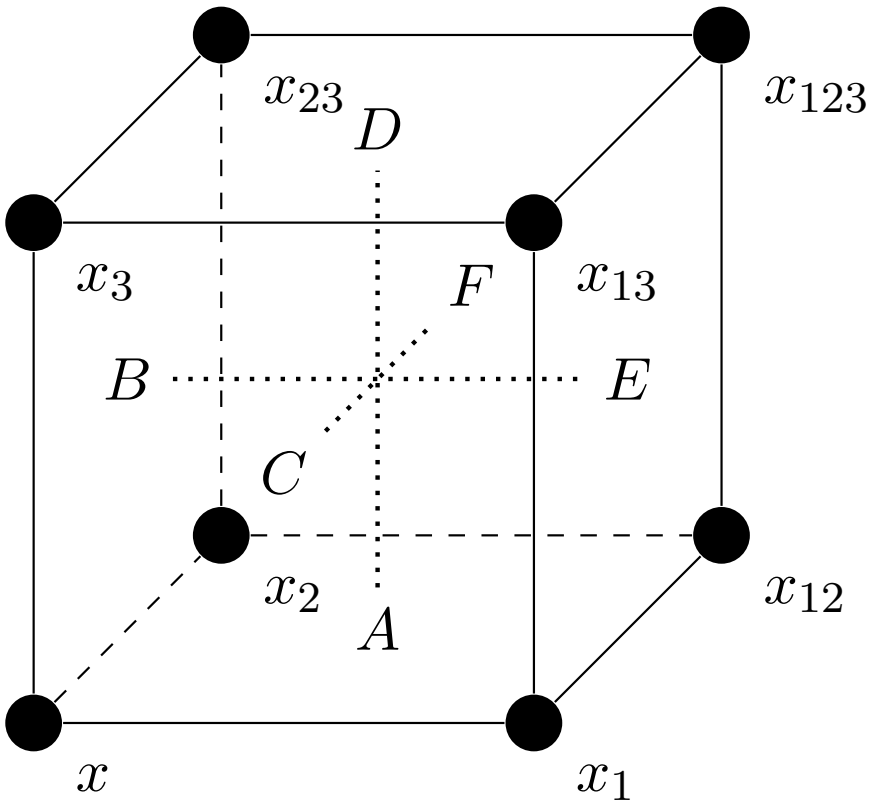
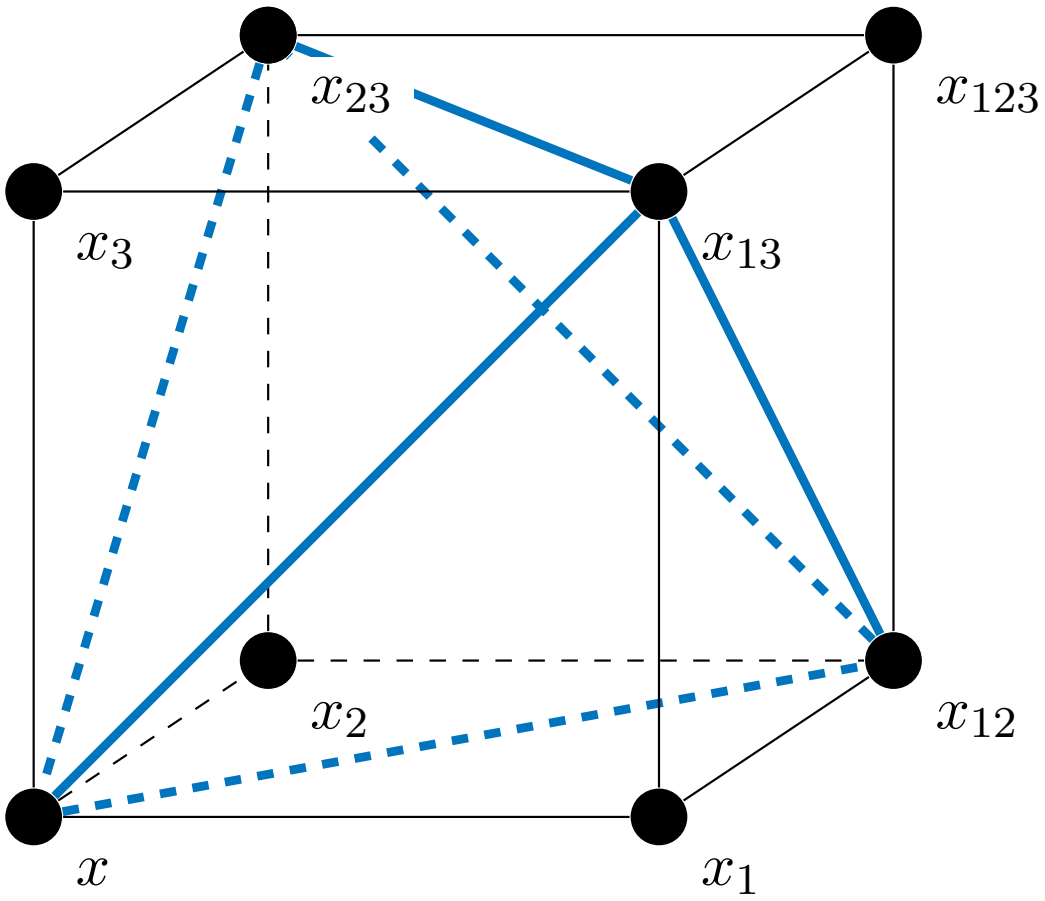
A Bäcklund cube – 6 quad-equations on the faces of a 3D cube



$$A\left(x, x_1, x_2, x_{12}\right) = 0,$$
$$B\left(x, x_2, x_3, x_{23}\right) = 0,$$
$$C\left(x, x_1, x_3, x_{13}\right) = 0,$$
$$D\left(x_3, x_{13}, x_{23}, x_{123}\right) = 0,$$
$$E\left(x_1, x_{12}, x_{13}, x_{123}\right) = 0,$$
$$F\left(x_2, x_{12}, x_{23}, x_{123}\right) = 0$$

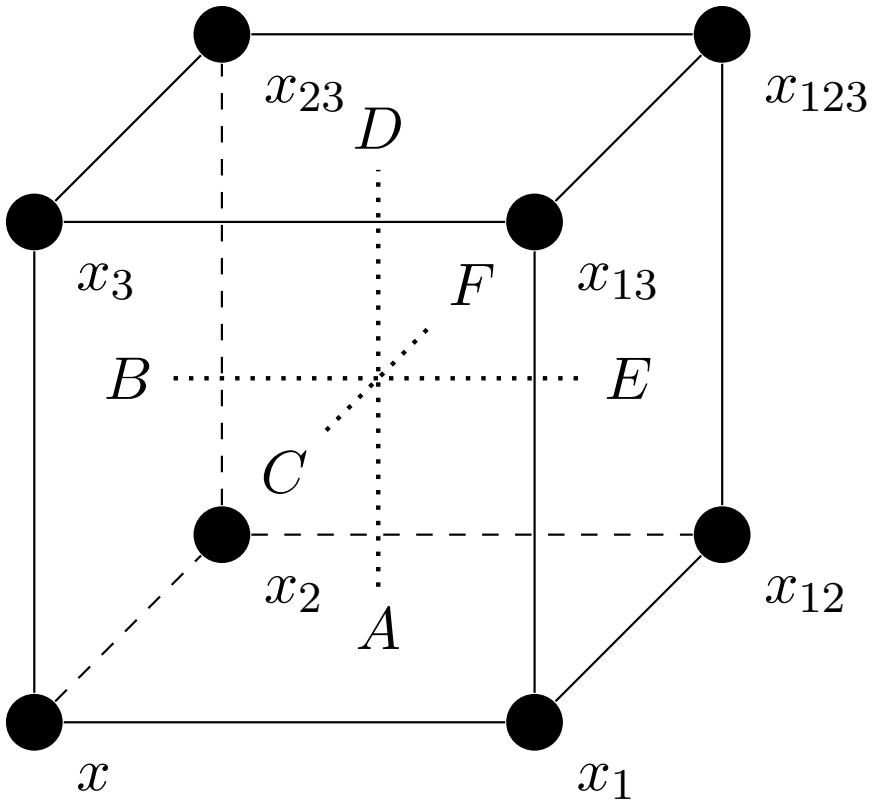
Tetrahedron property



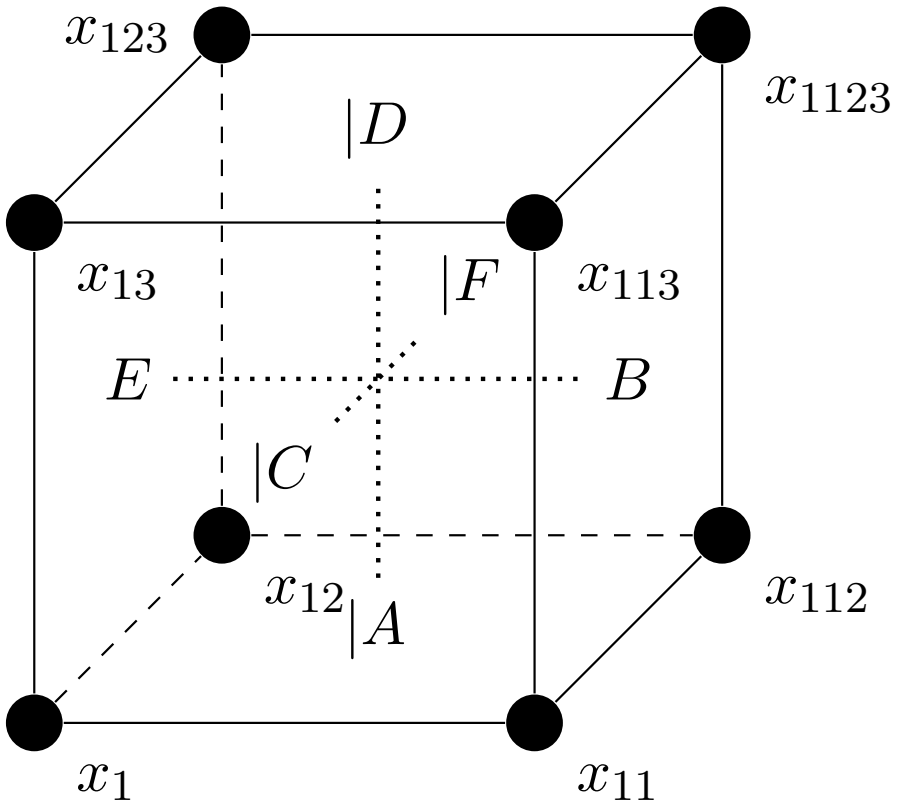
$$K\left(x, x_{12}, x_{13}, x_{23}\right) = 0,$$
$$L\left(x_1, x_2, x_3, x_{123}\right) = 0$$

Reflecting a Bäcklund cube

Original Bäcklund cube:



Neighbor on the right hand side:



Bäcklund transformation

Definition (Bäcklund transformation $x \overset{\lambda}{\mapsto} y$)

Consider a system of quad-equations on the lattice $\mathbb{Z}^2 \times \{0,1\}$ generated by reflection and a solution x of the system composed of the equations $A = 0$, $|A = 0$, $\underline{A} = 0$ and $|\underline{A} = 0$ on the "horizontal" sublattice $\mathbb{Z}^2 \times \{0\}$.

Then we say that a solution y of the system composed of the equations $D = 0$, $|D = 0$, $\underline{D} = 0$ and $|\underline{D} = 0$ on the sublattice $\mathbb{Z}^2 \times \{1\}$ is a *Bäcklund transform* of x if the combination of x and y build a solution of the system on the whole lattice $\mathbb{Z}^2 \times \{0,1\}$, i.e., also the equations on "vertical" faces are satisfied.

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Example

Consider a solution of system of quad-equations on $\mathbb{Z}^2 \times \{0\}$ with

$$A := (x - x_1)(x_2 - x_{12}) - \alpha_1 \left(1 + \varepsilon^2 x_2 x_{12}\right) = 0.$$

Then there are two essentially different Bäcklund transformations:

- $BT_1 : x \overset{\lambda_1}{\mapsto} y^{(1)}$ with

$$D := \left(y^{(1)} - y_1^{(1)}\right)\left(y_2^{(1)} - y_{12}^{(1)}\right) - \alpha\left(1 + \varepsilon^2 y_2^{(1)} y_{12}^{(1)}\right) = 0$$

- $BT_2 : x \overset{\lambda_2}{\mapsto} y^{(2)}$ with

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Two layers of Bäcklund cubes

Construction (Two layers of Bäcklund cubes)

Consider a layer of Bäcklund cubes, i.e., a 3D consistent system of quad-equations on $\mathbb{Z}^2 \times \{0,1\}$ generated by reflection.

The second layer can be derived from the original one on the lattice $\mathbb{Z}^2 \times \{0,1\}$ by reflecting the Bäcklund cubes in the plane $\mathbb{Z}^2 \times \{1\}$ in order to get a 3D consistent system of quad-equations on the lattice $\mathbb{Z}^2 \times \{0,1,2\}$.

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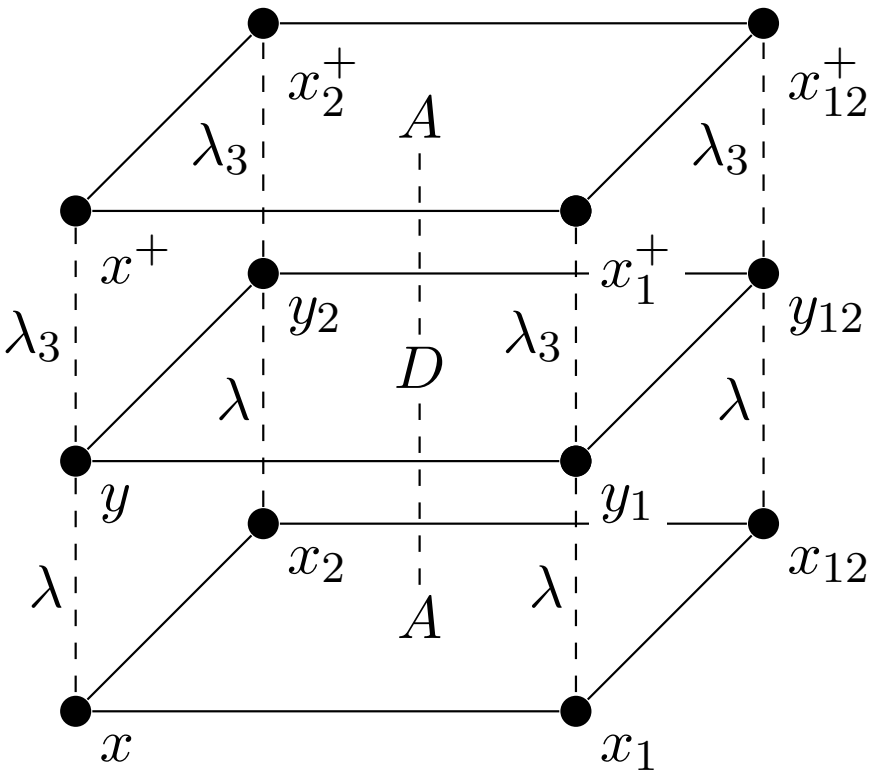

Auto-Bäcklund transformation

Construction (Two-stage auto-Bäcklund transformation $x \overset{\lambda}{\mapsto} y \overset{\lambda_3}{\mapsto} x^+$)

Consider a 3D consistent system of quad-equations on the lattice $\mathbb{Z}^2 \times \{0, 1, 2\}$ generated by reflection.

Furthermore, consider a solution x of the system composed of the equations $A = 0$, $|A = 0$, $\underline{A} = 0$ and $|\underline{A} = 0$ on the sublattice $\mathbb{Z}^2 \times \{0\}$ and one of its Bäcklund transforms y satisfying the system composed of the equations $D = 0$, $|D = 0$, $\underline{D} = 0$ and $|\underline{D} = 0$ on the sublattice $\mathbb{Z}^2 \times \{1\}$ with Bäcklund parameter λ. Then a Bäcklund transform x^+ of y satisfying the system composed of the equations $A = 0$, $|A = 0$, $\underline{A} = 0$ and $|\underline{A} = 0$ on the sublattice $\mathbb{Z}^2 \times \{2\}$ with Bäcklund parameter λ_3 is a (non-elementary) auto-Bäcklund transform of x.

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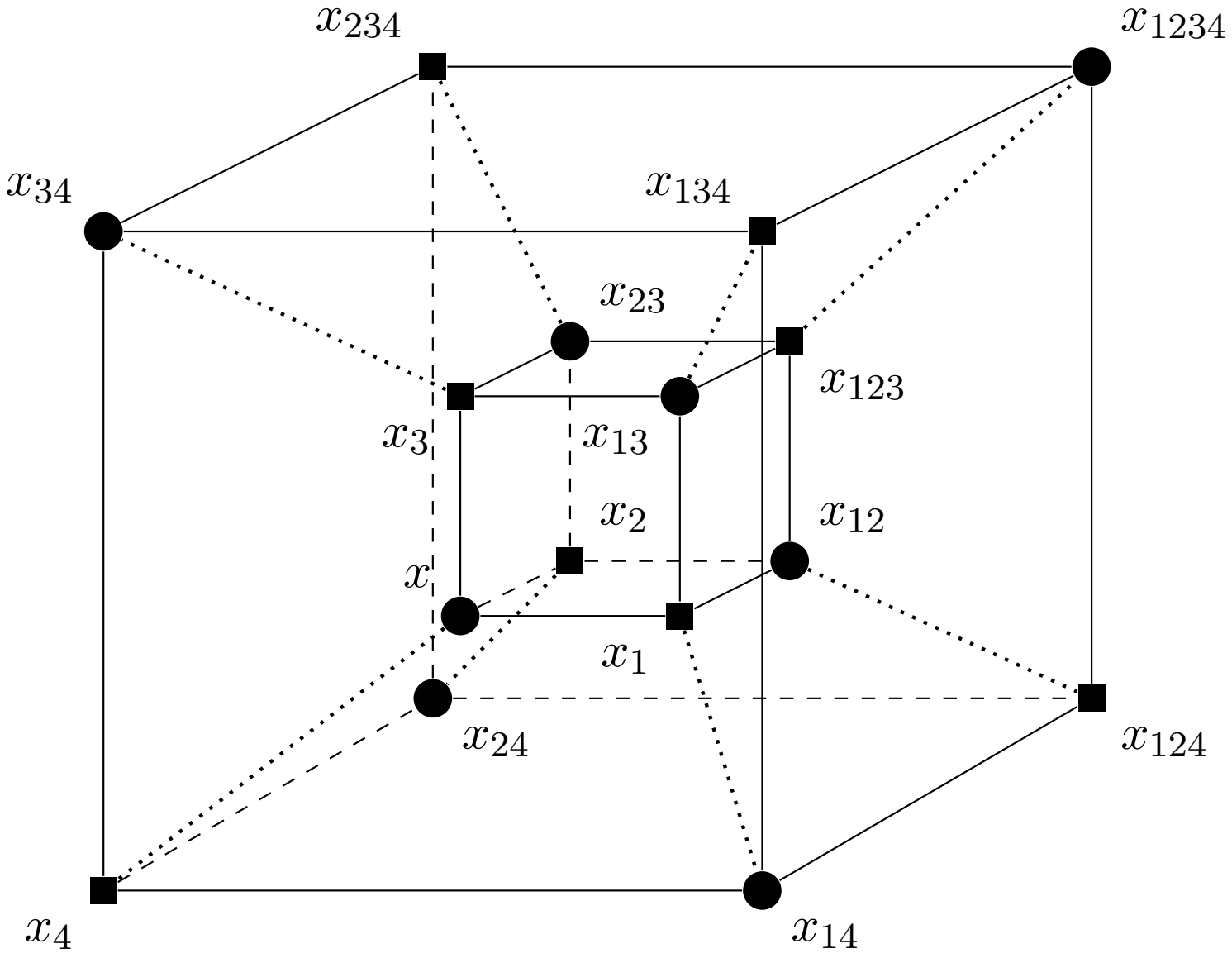
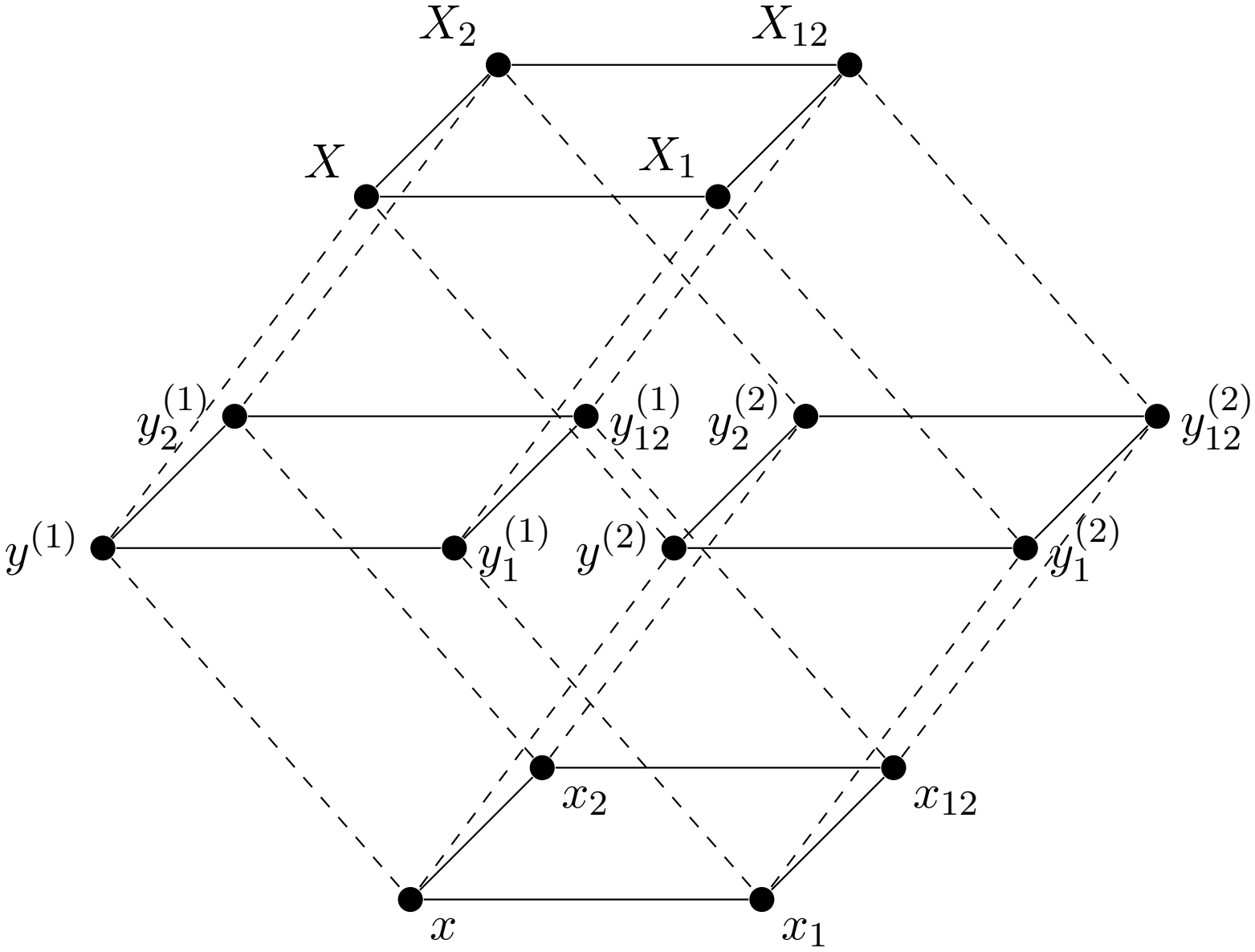
A Bianchi cube



The extension of two Bäcklund cubes to a Bianchi cube



Theorem

Given two Bäcklund cubes sharing one quad-equation, there exists an extension to a Bianchi cube. This extension is unique up to Möbius transformations of fields not belonging to the original Bäcklund cubes (the fields X, X_1, X_2 and X_{12}).

Bianchi diagram:

$$\begin{tikzcd}
 & y^{(2)} \arrow[dr, "\lambda_1"] & \\
x \arrow[ur, "\lambda_2"] \arrow[dr, "\lambda_1"'] & & X \\
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Bianchi permutability of elementary Bäcklund transformations

Corollary

Let x be a solution of a system of quad-equations on $\mathbb{Z}^2 \cong \mathbb{Z}^2 \times \{0\} \times \{0\}$ generated from the equation $A(x, x_1, x_2, x_{12}) = 0$ by reflection and two of its Bäcklund transformations $x \overset{\lambda_1}{\mapsto} y^{(1)}$ and $x \overset{\lambda_2}{\mapsto} y^{(2)}$ (depending on one value $y^{(1)}(0)$ respectively on $y^{(2)}(0)$ each).

Then there exists a unique solution X on the system of quad-equations on $\mathbb{Z}^2 \times \{1\} \times \{1\}$ which is obtained from an equation $G(X, X_1, X_2, X_{12}) = 0$ by reflection which is simultaneously a Bäcklund transform of $y^{(1)}$ via a Bäcklund transformation $y^{(1)} \overset{\lambda_2}{\mapsto} X$ and of $y^{(2)}$ via a Bäcklund transformation $y^{(2)} \overset{\lambda_1}{\mapsto} X$.

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Bianchi permutability of auto-Bäcklund transformations

$$
\begin{tikzcd}
 & & x^{(2)} \arrow[dr, "\lambda_1"] & & \\
 & y^{(2)} \arrow[ur, "\lambda_{22}"] \arrow[dr, "\lambda_1"] & & y^{(21)} \arrow[dr, "\lambda_{11}"] & \\
x \arrow[ur, "\lambda_2"] \arrow[dr, "\lambda_1"'] & & X \arrow[ur, "\lambda_{22}"] \arrow[dr, "\lambda_{11}"'] & & x^{(12)} = x^{(21)} \\
 & y^{(1)} \arrow[ur, "\lambda_2"] \arrow[dr, "\lambda_{11}"'] & & y^{(12)} \arrow[ur, "\lambda_{22}"'] & \\
 & & x^{(1)} \arrow[ur, "\lambda_2"'] & &
\end{tikzcd}
$$

Theorem

Consider a solution x of a system of quad-equations on $\mathbb{Z}^2 \cong \mathbb{Z}^2 \times \{0\} \times \{0\}$ generated by reflection and two two-stage auto-Bäcklund transformations

$$aBT_1 : x \overset{\lambda_1}{\mapsto} y^{(1)} \overset{\lambda_{11}}{\mapsto} x^{(1)} \quad \text{and} \quad aBT_2 : x \overset{\lambda_2}{\mapsto} y^{(2)} \overset{\lambda_{22}}{\mapsto} x^{(2)}$$

(depending on two values $y^{(1)}(0)$ and $x^{(1)}(0)$ respectively on $y^{(2)}(0)$ and $x^{(2)}(0)$ each).

Then there exists a unique solution $x^{(12)} = x^{(21)}$ which is simultaneously an auto-Bäcklund transform of $x^{(1)}$ and of $x^{(2)}$ via

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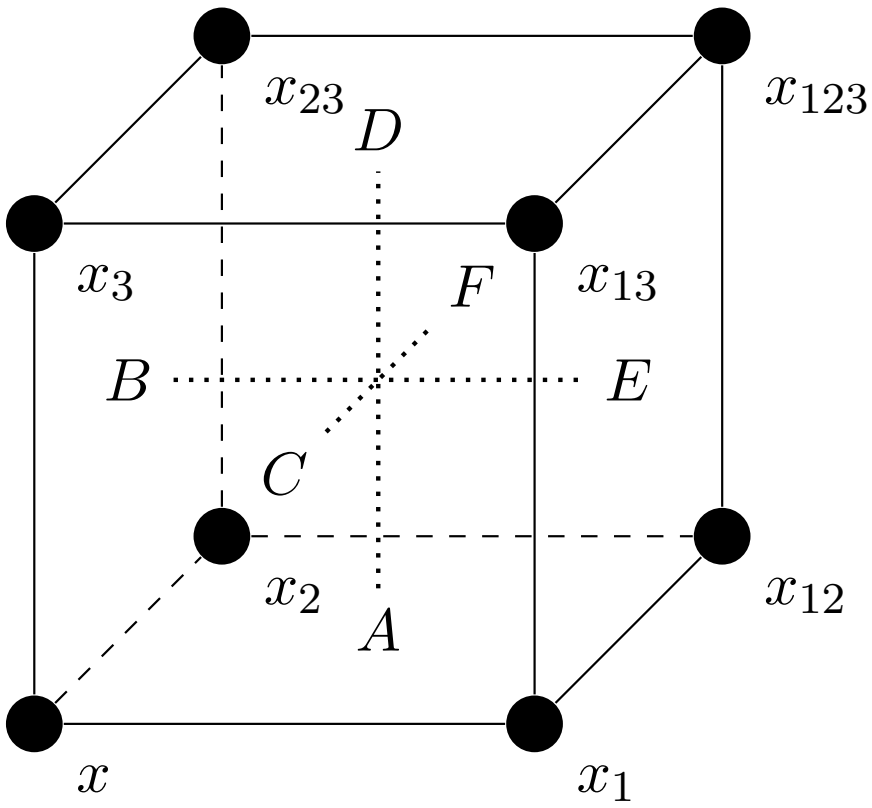
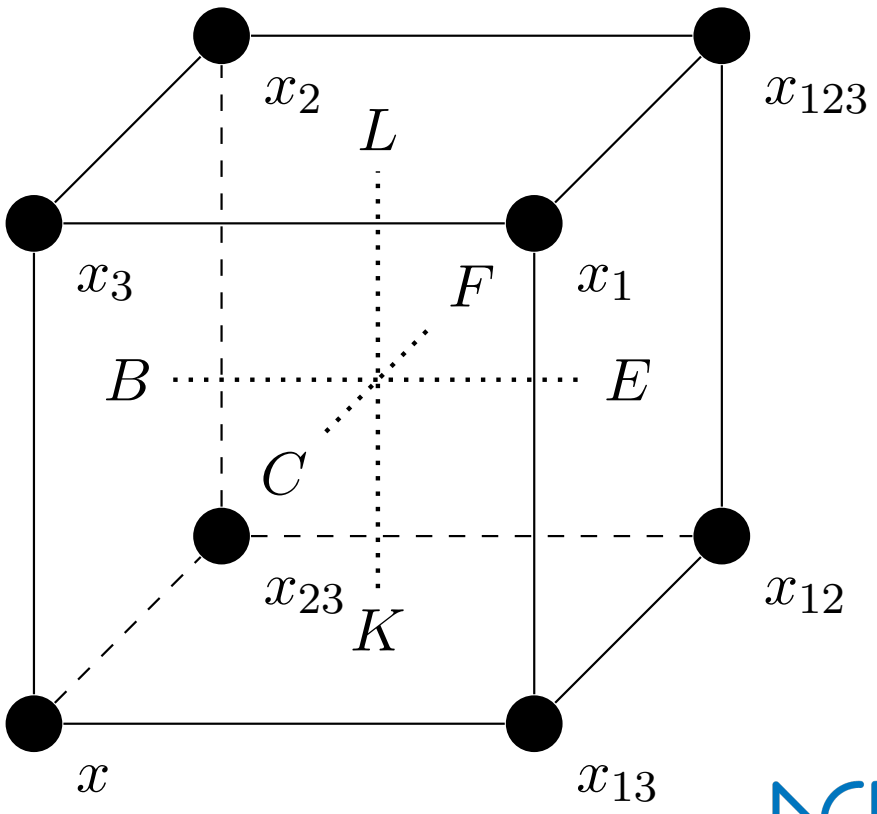
respectively.

Making tetrahedra to faces – Super-consistent eight-tuples on decorated cubes

Normal case:



Flipped case:



Lemma

Consider a 3D consistent system possessing the tetrahedron property.
Then, the system

$$\begin{aligned} K\left(x, x_{12}, x_{13}, x_{23}\right) &= 0, & L\left(x_1, x_2, x_3, x_{123}\right) &= 0, \\ B\left(x, x_2, x_3, x_{23}\right) &= 0, & E\left(x_1, x_{12}, x_{13}, x_{123}\right) &= 0, \\ C\left(x, x_1, x_3, x_{13}\right) &= 0, & F\left(x_2, x_{12}, x_{23}, x_{123}\right) &= 0 \end{aligned}$$

is 3D consistent and possesses the tetrahedron property. 3D consistency of this system is understood as the property of the initial value problem with initial data x, x_3, x_{13} and x_{23}.

We call the eight-tuple $A = D = B = E = C = F = K = L = 0$ a *super-consistent eight-tuple on a decorated cube.*

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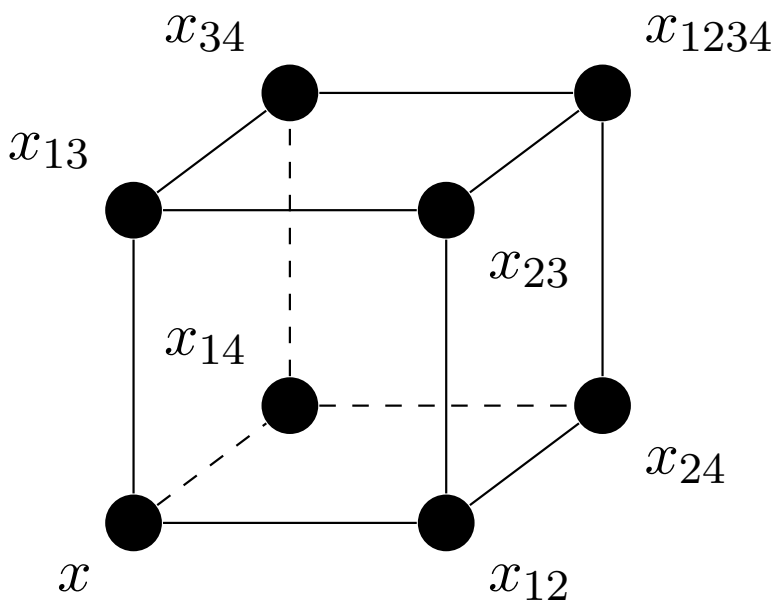
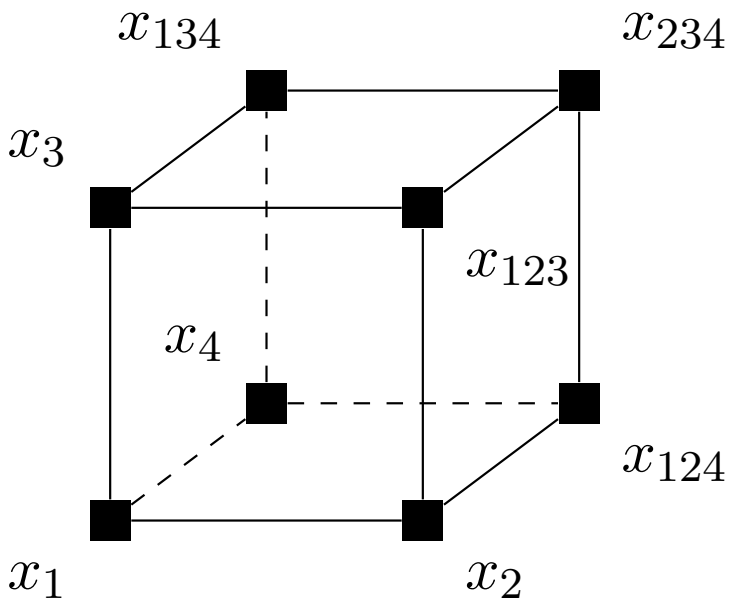
Super-consistent eight-tuples composed from tetrahedron equations of the 3D facets

Decorated 3D cube of vertices whose indices have an even number of digits:



Decorated 3D cube of vertices whose indices have an odd number of digits:



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