Noether symmetries and the quantization of a Liénard-type nonlinear oscillator

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Outline

1. Introduction

2. Classical properties of the Liénard equation

3. Quantisation of Liénard equation
The classical quantisation is given by the following correspondence between classical quantities and quantum operators:

\[ q_i \rightarrow q_i, \quad p_i \rightarrow -i \frac{\partial}{\partial q_i}. \]

Problems mainly arise if:

1. non-standard form of the Hamiltonian \( H \),
2. ordering problem,
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To get a consistent quantisation one has to preserve the Noether symmetries of the classical system embedding them in the finite Lie symmetry algebra of the corresponding Schrödinger equation.

N.B.: Two additional symmetries: $\psi \partial_\psi$ and $\Psi \partial_\psi$ exist for any linear homogeneous partial differential equation.
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The observations made in the cited papers led to an algorithm for quantisation if we are dealing with a system of equations

\[ \ddot{x}(t) = f(t, x, \dot{x}), \quad (1) \]

that admits the maximum number of Lie symmetries, namely \( N^2 + 4N + 3 \) [A. González-López, *J. Math. Phys.* (1988)]. Furthermore we shall need a Lagrangian admitting the maximum number of Noether symmetries, namely \( (N^2 + 3N + 6)/2 \).
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**Step 1.** Find the linearising transformation *without changing the time* $t$:

$$t \rightarrow t, \quad x \rightarrow y = y(x).$$  \hspace{1cm} (2)

The resulting equation for $y$ will be a linear second-order system (LS).

**Step 2.** Find the Lagrangian which admits the maximum number of Noether symmetries, namely $(N^2 + 3N + 6)/2$. A system with the maximum number of Lie symmetries is always Lagrangian.
**New algorithm for quantisation II**

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**Step 3.** Take the Schrödinger equation corresponding to the linear system (LS) and apply the linearising transformation $y = y(x)$. The result will be the Schrödinger equation corresponding to the initial system (1).

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The Liénard equation is the following second-order class of ODEs:

\[ \ddot{x} + f(x)\dot{x} + g(x) = 0. \]  

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The only linearisable equation belonging to this class is:

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In [M.C. Nucci and K.M. Tamizhmani, *J. Nonlinear Math. Phys.* (2010)], Lagrangians $L$ for (3) were derived using Jacobi Last Multipliers $M$, i.e. $M = \frac{\partial^2 L}{\partial x^2}$.

Since a particular Jacobi Last Multiplier is $M = e^{\int f(x) \, dt}$ then the following condition was introduced:

$$\frac{d}{dx} \left( \frac{g(x)}{f(x)} \right) = \alpha (1 - \alpha) f(x), \quad \alpha \neq 1,$$

(5)

that yields the Lagrangian:

$$L = \left( \dot{x} + \frac{g(x)}{\alpha f(x)} \right)^{2-1/\alpha} + \frac{d}{dt} G(t, x).$$

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Lagrangian of the Liénard equation (4) I

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In particular in the case of equation (4) it was found that:

\[ f(x) = kx, \quad g(x) = \frac{k^2}{9} x^3 + \omega^2 x, \quad \alpha = \frac{1}{3}. \]  

Thus since \( M = e^\int kx(t)\,dt \) the following Lagrangian was obtained:

\[ L_G = \frac{1}{k\dot{x} + \frac{3}{2}x^2 + \frac{3\omega^2}{k}} + \frac{d}{dt} G(t, x). \]
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In [V. Chithiika Ruby, M. Senthilvelan, and M. Lakshmanan, *J. Phys. A* (2012)], the gauge function $G$ was taken to be:

$$G = \frac{3\omega^2}{2k} x - \frac{9\omega^4}{2k^2} t. \quad (9)$$

This position allows to get rid of $\sqrt{-p}$ when the Hamiltonian corresponding to (8) is derived, i.e.:

$$H = \frac{9\omega^4}{2k^2} \left[ 2 - \frac{2k}{3\omega^2} p - 2 \left( 1 - \frac{2k}{3\omega^2} p \right)^{\frac{1}{2}} + \frac{k^2x^2}{9\omega^2} \left( 1 - \frac{2k}{3\omega^2} p \right) \right]. \quad (10)$$

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Equation (4) admits an eight-dimensional Lie point symmetry algebra generated by:

\[ \Gamma_1 = x \partial_t - \left( \frac{1}{3} x^3 k + \frac{3 \omega^2}{k} x \right) \partial_x , \]

\[ \Gamma_2 = \sin(\omega t) x \partial_t - \left( \frac{k}{3} \sin(\omega t) x^3 - \omega \cos(\omega t) x^2 \right) \partial_x , \]

\[ \Gamma_3 = \cos(\omega t) x \partial_t - \left( \frac{k}{3} \cos(\omega t) x^3 + \omega \sin(\omega t) x^2 \right) \partial_x , \]

\[ \Gamma_4 = \left( \frac{3 \omega}{k} \cos(2\omega t) - \sin(2\omega t) x \right) \partial_t \]

\[ + \left[ \left( \frac{k}{3} x^3 - \frac{3 \omega^2}{k} x \right) \sin(2\omega t) - 2 \omega x^2 \cos(2\omega t) \right] \partial_x , \]
Lie symmetries of the Liénard equation (4)

\[ \Gamma_5 = - \left( \frac{3\omega}{k} \sin(2\omega t) + \cos(2\omega t)x \right) \partial_t \]
\[ - \left[ \left( \frac{k}{3}x^3 - \frac{3\omega^2}{k}x \right) \cos(2\omega t) + 2\omega x^2 \cos(2\omega t) \right] \partial_x, \]
\[ \Gamma_6 = \cos(\omega t) \partial_t + \left( \omega \sin(\omega t)x - \frac{3\omega^2}{k} \cos(\omega t) \right) \partial_x, \]
\[ \Gamma_7 = - \sin(\omega t) \partial_t + \left( \omega \cos(\omega t)x + \frac{3\omega^2}{k} \sin(\omega t) \right) \partial_x, \]
\[ \Gamma_8 = \partial_t. \]
Lie’s method: the linearising transformation can be found by looking for a two-dimensional abelian intransitive subalgebra $L_2$ and transforming into its canonical form $\partial_{\tilde{x}}, \tilde{t}\partial_{\tilde{x}}$.

In the case of equation (4) $L_2$ is generated by:

$$k\Gamma_2 - 3\omega\Gamma_6, \quad k\Gamma_3 - 3\omega\Gamma_7. \quad (11)$$

and the corresponding point transformation is:

$$\tilde{t} = \frac{kx \cos(\omega t) + 3\omega \sin(\omega t)}{kx \sin(\omega t) - 3\omega \cos(\omega t)}, \quad \tilde{x} = -\frac{1}{9\omega^2} \frac{x}{kx \sin(\omega t) - 3\omega \cos(\omega t)}. \quad (12)$$

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In [V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan. *Phys. Rev. E* (2005)], it was found that:

\[ U = xe^{\frac{k}{3}} \int x(\tau) d\tau \Rightarrow \ddot{U} + \omega^2 U = 0. \] (13)

The authors determined the general solution of equation (4) and inserted it into the non-local transformation (13) in order to eliminate the non-locality.
Non-local transformation to the harmonic oscillator

In [V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan. *Phys. Rev. E* (2005)], it was found that:

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Canonical transformation to the harmonic oscillator

We show that this can be achieved without solving equation (4).


\[ q = \frac{x}{\dot{x} + \frac{k}{3}x^2 + \frac{3\omega^2}{k}}. \]  \hspace{1cm} (14)

such that \( q = xe^{\frac{k}{3} \int x(\tau) \, d\tau} \) and satisfies:

\[ \ddot{q} + \omega^2 q = 0. \]  \hspace{1cm} (15)

Using a generating function of type \( f_3 = f_3(q, p) \) we found the canonical transformation:

\[ x = \frac{27\omega^6 q}{k^3 s}, \quad p = -\frac{k^3}{54\omega^6} s^2 + \frac{3\omega^2}{2k} \]

Then the Hamiltonian (10) is mapped into:

\[ K = \frac{1}{2} \left( \frac{ks}{9\omega^2} + \frac{\omega^2}{k} \right)^2 + \frac{9\omega^2}{2k^2} q^2. \]  \hspace{1cm} (16)

N.B. This canonical transformation is not linear.
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Therefore:

1. The Liénard equation (4) cannot be related to the free-particle equation without changing the time.
2. The Liénard equation (4) cannot be related to the harmonic oscillator without a nonlinear canonical transformation involving the momentum.

What can be done next?

Remark: the Hamiltonian of the Liénard equation (4) is quadratic in $x^2$

$$H = \frac{9\omega^4}{2k^2} \left[ 2 - \frac{2k}{3\omega^2} p - 2 \left( 1 - \frac{2k}{3\omega^2} p \right)^{\frac{1}{2}} + \frac{k^2 x^2}{9\omega^2} \left( 1 - \frac{2k}{3\omega^2} p \right) \right].$$
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Classical Hamiltonian in Momentum Representation

In [V. Chithiika Ruby, M. Senthilvelan, and M. Lakshmanan, J. Phys. A (2012)], the quantisation was tackled in Momentum Representation.

Classically this corresponds to make the linear canonical transformation:

\[(x, p) \rightarrow (X, P) = (p, -x)\] (17)

such that the Hamiltonian (10) becomes:

\[\tilde{H} = \frac{9\omega^4}{2k^2} \left[ 2 - \frac{2k}{3\omega^2} X - 2 \left( 1 - \frac{2k}{3\omega^2} X \right)^{\frac{1}{2}} + \frac{k^2 P^2}{9\omega^2} \left( 1 - \frac{2k}{3\omega^2} X \right) \right].\] (18)
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The classical Momentum Representation equation

We now apply the Lie-Noether symmetry quantisation to the Hamiltonian (18).

The corresponding Lagrangian of (18) is:

\[
\tilde{L} = \frac{\dot{X}^2}{2\omega^2 \left(1 - \frac{2k}{3\omega^2} X\right)} - \frac{9\omega^4}{2k^2} \left(1 - \sqrt{1 - \frac{2k}{3\omega^2} X}\right)^2.
\]  

(19)

and the corresponding Lagrange equation is:

\[
\ddot{X} = \frac{3\omega^4}{k} \left(1 - \frac{2k}{3\omega^2} X - \sqrt{1 - \frac{2k}{3\omega^2} X}\right) - \frac{k\dot{X}^2}{3\omega^2 \left(1 - \frac{2k}{3\omega^2} X\right)}.
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\]
Symmetries of (20) I

The Lie symmetries of (20) are:

\[ \Xi_1 = \partial_t, \]
\[ \Xi_2 = \cos(2\omega t)\partial_t + \sin(2\omega t)\frac{3\omega^3}{k} \left[ 1 - \frac{2k}{3\omega^2} X - \sqrt{1 - \frac{2k}{3\omega^2} X} \right] \partial X \]
\[ \Xi_3 = \sin(2\omega t)\partial_t - \cos(2\omega t)\frac{3\omega^3}{k} \left[ 1 - \frac{2k}{3\omega^2} X - \sqrt{1 - \frac{2k}{3\omega^2} X} \right] \partial X \]
\[ \Xi_4 = \sqrt{1 - \frac{2k}{3\omega^2} X} \cos(\omega t) \partial X \]
\[ \Xi_5 = \sqrt{1 - \frac{2k}{3\omega^2} X} \sin(\omega t) \partial X \]
Symmetries of (20) II

\[ \Xi_6 = \cos(\omega t) \left( \sqrt{1 - \frac{2k}{3\omega^2} X - 1} \right) \partial_t \]

\[ + \frac{3\omega^3}{k} \sin(\omega t) \left( \sqrt{1 - \frac{2k}{3\omega^2} X - 2} \right) \left( 1 - \frac{2k}{3\omega^2} X \right) \partial_X, \]

\[ \Xi_7 = \sin(\omega t) \left( \sqrt{1 - \frac{2k}{3\omega^2} X - 1} \right) \partial_t \]

\[ - \frac{3\omega^3}{k} \cos(\omega t) \left( \sqrt{1 - \frac{2k}{3\omega^2} X - 2} \right) \left( 1 - \frac{2k}{3\omega^2} X \right) \partial_X, \]

\[ \Xi_8 = \sqrt{1 - \frac{2k}{3\omega^2} X} \left( 1 - \sqrt{1 - \frac{2k}{3\omega^2} X} \right) \partial_X. \]
Using Lie’s theory we find that the linearising transformation is:

\[
\begin{align*}
\tau &= \tan(\omega t), \\
\xi &= \omega \sqrt{\frac{6}{k}} \frac{1 - \sqrt{1 - \frac{2k}{3\omega^2} X}}{\cos(\omega t)} \quad \Rightarrow \quad \frac{d^2 \xi}{d\tau^2} = 0 
\end{align*}
\] (21)

Also we can transform (20) into the harmonic oscillator:

\[
\eta = \omega \sqrt{\frac{6}{k}} \left(1 - \sqrt{1 - \frac{2k}{3\omega^2} X}\right) \Rightarrow \ddot{\eta} + \omega^2 \eta = 0. 
\] (22)
Quantisation with symmetries: Step 1

Using Lie’s theory we find that the linearising transformation is:

\[
\begin{cases}
\tau = \tan(\omega t), \\
\xi = \omega \sqrt{\frac{6}{k}} \frac{1 - \sqrt{1 - \frac{2k}{3\omega^2} X}}{\cos(\omega t)} \Rightarrow \frac{d^2 \xi}{d\tau^2} = 0 
\end{cases}
\]  

(21)

Also we can transform (20) into the harmonic oscillator:

\[
\dot{\eta} + \omega^2 \eta = 0.
\]  

(22)
The following Lagrangian admits five Noether symmetries generated by $\Xi_i$, $i = 1, \ldots, 5$,:

$$
\tilde{L} = \frac{\dot{X}^2}{2\omega^2 \left(1 - \frac{2k}{3\omega^2} X\right)} - \frac{9\omega^4}{2k^2} \left(1 - \sqrt{1 - \frac{2k}{3\omega^2} X}\right)^2.
$$

(23)
The Schrödinger equation for the harmonic oscillator is:

\[ 2i \Phi_t + \Phi_{\eta \eta} - \omega^2 \eta^2 \Phi = 0. \]  
\[ (24) \]

The point transformation (22) takes equation (24) into the following Schrödinger equation:

\[ 2i \tilde{\Phi}_t + \frac{3 \omega^2}{2k} \left( 1 - \frac{2k}{3 \omega^2} X \right) \tilde{\Phi}_{XX} - \frac{1}{2} \tilde{\Phi}_X - \frac{6 \omega^4}{k} \left( 1 - \sqrt{1 - \frac{2k}{3 \omega^2} X} \right)^2 \tilde{\Phi}. \]  
\[ (25) \]
Quantisation with symmetries: Step 3. I

The Schrödinger equation for the harmonic oscillator is:

$$2i\Phi_t + \Phi_{\eta\eta} - \omega^2 \eta^2 \Phi = 0.$$  \hspace{1cm} (24)

The point transformation (22) takes equation (24) into the following Schrödinger equation:

$$2i\tilde{\Phi}_t + \frac{3\omega^2}{2k} \left( 1 - \frac{2k}{3\omega^2} X \right) \tilde{\Phi}_{XX} - \frac{1}{2} \tilde{\Phi}_X - \frac{6\omega^4}{k} \left( 1 - \sqrt{1 - \frac{2k}{3\omega^2} X} \right)^2 \tilde{\Phi}.$$  \hspace{1cm} (25)
We may eliminate the first derivative $\tilde{\Phi}_X$ by imposing the transformation:

$$\Phi(t, X) = \frac{\tilde{\Phi}(t, X)}{\left[3\omega^2 \left(1 - \frac{2k}{3\omega^2} X\right)\right]^{\frac{1}{4}}}$$  \hspace{1cm} (26)$$

Finally:

$$2i\Phi_t + 9 \left(1 - \frac{2k}{3\omega^2} X\right) \Phi_{XX}$$

$$+ \left[\frac{3k^2}{4\omega^4 \left(1 - \frac{2k}{3\omega^2} X\right)} - \frac{2\omega^6}{k^2} \left(1 - \frac{k}{3\omega^2} X\right) \frac{2\omega^6}{k^2} \sqrt{1 - \frac{2k}{3\omega^2} X}\right] \Phi.$$  \hspace{1cm} (27)$$

This is the final form of the Schrödinger equation for (20).
Quantisation with symmetries: Step 3. I

We may eliminate the first derivative \( \tilde{\Phi}_x \) by imposing the transformation:

\[
\Phi(t, X) = \frac{\tilde{\Phi}(t, X)}{\left[3\omega^2 \left(1 - \frac{2k}{3\omega^2} X\right)\right]^{\frac{1}{4}}}
\]  

(26)

Finally:

\[
2i\Phi_t + 9 \left(1 - \frac{2k}{3\omega^2} X\right) \Phi_{xx} \\
+ \left[\frac{3k^2}{4\omega^4 \left(1 - \frac{2k}{3\omega^2} X\right)} - \frac{2\omega^6}{k^2} \left(1 - \frac{k}{3\omega^2} X\right) \frac{2\omega^6}{k^2} \sqrt{1 - \frac{2k}{3\omega^2} X}\right] \Phi.
\]

(27)

This is the final form of the Schrödinger equation for (20).
Step 4.

The Lie symmetries of equation (27) are

\[
\begin{align*}
\Omega_1 &= \Xi_1, \\
\Omega_2 &= \Xi_2 + \left[ \frac{\omega}{2} \sin(2\omega t) - i\omega^2 \cos(2\omega t) \left( 1 - \sqrt{1 - \frac{2k}{3\omega^2} X} \right)^2 \right] \frac{\phi}{\sqrt{1 - \frac{2k}{3\omega^2} X}} \partial\phi, \\
\Omega_3 &= \Xi_3 - \left[ \frac{\omega}{2} \cos(2\omega t) + i\omega^2 \sin(2\omega t) \left( 1 - \sqrt{1 - \frac{2k}{3\omega^2} X} \right)^2 \right] \frac{\phi}{\sqrt{1 - \frac{2k}{3\omega^2} X}} \partial\phi, \\
\Omega_4 &= \Xi_4 - \left[ \frac{k \cos(\omega t)}{6\omega^2 \sqrt{1 - \frac{2k}{3\omega^2} X}} + 2i\omega \left( 1 - \sqrt{1 - \frac{2k}{3\omega^2} X} \right) \sin(\omega t) \right] \phi \partial\phi, \\
\Omega_5 &= \Xi_5 - \left[ \frac{k \sin(\omega t)}{6\omega^2 \sqrt{1 - \frac{2k}{3\omega^2} X}} - 2i\omega \left( 1 - \sqrt{1 - \frac{2k}{3\omega^2} X} \right) \cos(\omega t) \right] \phi \partial\phi, \\
\Omega_6 &= \phi \partial\phi, \\
\Omega_\chi &= \chi(t, X) \partial\phi,
\end{align*}
\]
We expect the spectrum of (27) to be that of the harmonic oscillator, i.e.:

\[ E_n = \omega \left( n + \frac{1}{2} \right) . \]  

(29)

as was proven in [V. Chithiika Ruby, M. Senthilvelan, and M. Lakshmanan, *J. Phys. A* (2012)].

We expect the spectrum of (27) to be that of the harmonic oscillator, i.e.:

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as was proven in [V. Chithiika Ruby, M. Senthilvelan, and M. Lakshmanan, J. Phys. A (2012)].

Spectrum of the solution II

We rewrite the Lie symmetries of (27) in complex form:

\[ \Sigma_1 = i \partial_t, \]

\[ \Sigma_{2\pm} = e^{\pm 2i\omega t} \partial_t \mp ie^{\pm 2i\omega t} \frac{3\omega^3}{k} \left( 1 - \sqrt{1 - \frac{2k}{3\omega^2} X} \right) \sqrt{1 - \frac{2k}{3\omega^2} X} \partial_X \]

\[ + \frac{1}{2} e^{\pm 2i\omega t} \left[ \frac{24\omega^3 - 24\omega^3 \sqrt{1 - \frac{2k}{3\omega^2} X} - k + 8kX\omega \sqrt{1 - \frac{2k}{3\omega^2} X} - 16k\omega X}{\omega k \sqrt{1 - \frac{2k}{3\omega^2} X}} \right] \]

\[ \Sigma_{3\pm} = e^{\pm i\omega} \sqrt{1 - \frac{2k}{3\omega^2} X} \partial_X \]

\[ \mp e^{\pm i\omega^3} 2\omega \left[ \left( 1 - \sqrt{1 - \frac{2k}{3\omega^2} X} \right) + \frac{k}{12\sqrt{3}\omega^4} \frac{1}{\sqrt{1 - \frac{2k}{3\omega^2} X}} \right] \Phi \partial_\Phi \]

\[ \Sigma_4 = \Phi \partial_\Phi \]

\[ \Sigma_\chi = \chi(t, X) \partial_\Phi \]
The operators $\Sigma_{3 \pm}$ will act as creation and annihilation operators. The ground state can be found from the invariant solution corresponding to $\Sigma_{3+}$, i.e. $\Sigma_{3+} F(t, X, \Phi) = 0$.

We obtain:

$$F(t, X, \Phi) = f \left( t, \Phi e^{-\frac{2\omega}{k} \left( 3\omega^2 \sqrt{1 - \frac{2k}{3\omega^2} X + kX} \right) \left( 1 - \frac{2k}{3\omega^2} X \right)^{\frac{1}{4}}} \right) = 0. \quad (30)$$

that replaced into (27) gives:

$$\Phi_0(t, X) = \left( 1 - \frac{2k}{3\omega^2} X \right)^{\frac{1}{4}} e^{-\frac{1}{2}it + \frac{2\omega}{k} \left( 3\omega^2 \sqrt{1 - \frac{2k}{3\omega^2} X + kX} \right)}. \quad (31)$$
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$$\Phi_0(t, X) = \left( 1 - \frac{2k}{3\omega^2} X \right)^{\frac{1}{4}} \text{e}^{-\frac{1}{2}it + \frac{2\omega}{k} \left( 3\omega^2 \sqrt{1-\frac{2k}{3\omega^2} X} + kX \right)}. \quad (31)$$
This ground state satisfies the boundary conditions:

\[
\lim_{X \to -\infty} \Phi_0(t, X) = 0, \quad \Phi_0 \left( t, \frac{3\omega^2}{2k} \right) = 0.
\] (32)

Remark: \( \Sigma_3^- \) yields solutions unbounded at infinity.

We construct all the eigenfunctions in the following manner:

\[
[\Sigma_3^-, \Sigma \Phi_{n-1}] = \Phi_n \partial \Phi \equiv \Sigma \Phi_n, \quad n \geq 1.
\] (33)

The operator \( \Sigma_1 \) acts as the eigenvalue operator and yields:

\[
\Sigma_1 \Phi_n = \omega \left( n + \frac{1}{2} \right) \Phi_n.
\] (34)
This ground state satisfies the boundary conditions:

\[
\lim_{X \to -\infty} \Phi_0(t, X) = 0, \quad \Phi_0 \left( t, \frac{3\omega^2}{2k} \right) = 0.
\]  

\(\text{(32)}\)

Remark: \(\Sigma_3^\rightarrow\) yields solutions unbounded at infinity.

We construct all the eigenfunctions in the following manner:

\[
[\Sigma_3^\rightarrow, \Sigma\Phi_{n-1}] = \Phi_n \partial_\phi \equiv \Sigma\Phi_n, \quad n \geq 1.
\]  

\(\text{(33)}\)

The operator \(\Sigma_1\) acts as the eigenvalue operator and yields:

\[
\Sigma_1 \Phi_n = \omega \left( n + \frac{1}{2} \right) \Phi_n.
\]  

\(\text{(34)}\)
This ground state satisfies the boundary conditions:

$$\lim_{X \to -\infty} \Phi_0(t, X) = 0, \quad \Phi_0 \left( t, \frac{3\omega^2}{2k} \right) = 0.$$  \hspace{1cm} (32)

Remark: $\Sigma_3$ yields solutions unbounded at infinity.
We construct all the eigenfunctions in the following manner:

$$[\Sigma_3, \Sigma \Phi_{n-1}] = \Phi_n \partial \Phi \equiv \Sigma \Phi_n, \quad n \geq 1.$$  \hspace{1cm} (33)

The operator $\Sigma_1$ acts as the eigenvalue operator and yields:

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We construct all the eigenfunctions in the following manner:

$$\left[\Sigma_3, \Sigma \Phi_{n-1}\right] = \Phi_n \partial \Phi \equiv \Sigma \Phi_n, \quad n \geq 1. \quad (33)$$

The operator $\Sigma_1$ acts as the eigenvalue operator and yields:

$$\Sigma_1 \Phi_n = \omega \left( n + \frac{1}{2} \right) \Phi_n. \quad (34)$$
**Comparison with previous results**


$$
2i\Theta_t + \omega^2 \left(1 - \frac{2k}{3\omega^2} p\right) \Theta_{pp} + \left[\frac{k^2}{12\omega^2 \left(1 - \frac{2k}{3\omega^2} p\right)} - \frac{9\omega^4}{k^2} \left(\sqrt{1 - \frac{2k}{3\omega^2} p} - 1\right)^2\right] \Theta = 0,
$$

Although this Schrödinger equation is formally different from ours, it still admits the Noether symmetries of (20) as Lie symmetries. Then this equation is classically consistent according to our quantisation method.
Comparison with previous results


$$2i\Theta_t + \omega^2 \left( 1 - \frac{2k}{3\omega^2 p} \right) \Theta_{pp} + \left[ \frac{k^2}{12\omega^2 \left( 1 - \frac{2k}{3\omega^2 p} \right)} - \frac{9\omega^4}{k^2} \left( \sqrt{1 - \frac{2k}{3\omega^2 p}} - 1 \right)^2 \right] \Theta = 0,$$

Although this Schrödinger equation is formally different from ours, it still admits the Noether symmetries of (20) as Lie symmetries. Then this equation is classically consistent according to our quantisation method.

\[2i\Theta_t + \omega^2 \left(1 - \frac{2k}{3\omega^2} p\right) \Theta_{pp} + \left[\frac{k^2}{12\omega^2 \left(1 - \frac{2k}{3\omega^2} p\right)} - \frac{9\omega^4}{k^2} \left(\sqrt{1 - \frac{2k}{3\omega^2} p} - 1\right)^2\right] \Theta = 0,\]

Although this Schrödinger equation is formally different from ours, it still admits the Noether symmetries of (20) as Lie symmetries. Then this equation is classically consistent according to our quantisation method.
Final remarks

- Noether symmetries yield the correct Schrödinger equation, although the wave function part of the Lie symmetry operator may have different form (no uniqueness).
- Lie symmetries can be algorithmically used to find the spectrum.
- Work is in progress to apply this method to other classical problems.