Noether symmetries and the quantization of a Liénard-type nonlinear oscillator

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Outline

Classical Quantisation

The classical quantisation is given by the following correspondence between classical quantities and quantum operators:

$$q_i \to q_i, \quad p_i \to -\mathrm{i}\frac{\partial}{\partial q_i}.$$

Problems mainly arise if:

1. non-standard form of the Hamiltonian H,
2. ordering problem,
3. non-linear canonical transformations.

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Using Lie-Noether symmetries to quantise

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Lie-Noether symmetries and Quantum Mechanics

To get a consistent quantisation one has to preserve the Noether symmetries of the classical system embedding them in the finite Lie symmetry algebra of the corresponding Schrödinger equation.
N.B.: Two additional symmetries: $\psi\partial_\psi$ and $\Psi\partial_\psi$ exist for any linear homogeneous partial differential equation.

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New algorithm for quantisation I

The observations made in the cited papers led to an algorithm for quantisation if we are dealing with a system of equations

$$\ddot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}, \dot{\boldsymbol{x}}), \tag{1}$$

that admits the maximum number of Lie symmetries, namely $N^2 + 4N + 3$ [A. González-López, *J. Math. Phys.* (1988)].
Furthermore we shall need a Lagrangian admitting the maximum number of Noether symmetries, namely $(N^2 + 3N + 6)/2$.

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New algorithm for quantisation II

Step 1. Find the linearising transformation *without changing the time t*:

$$t \to t, \quad \boldsymbol{x} \to \boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}). \tag{2}$$

The resulting equation for $\boldsymbol{y}$ will be a linear second-order system (LS).

Step 2. Find the Lagrangian which admits the maximum number of Noether symmetries, namely $(N^2 + 3N + 6)/2$. *A system with the maximum number of Lie symmetries is always Lagrangian.*

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Step 3. Take the Schrödinger equation corresponding to the linear system (LS)
and apply the linearising transformation $y = y(x)$.
The result will be the Schrödinger equation corresponding to the initial system (1).

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The Liénard equation

The Liénard equation is the following second-order class of ODEs:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{3}$$

The only linearisable equation belonging to this class is:

$$\ddot{x} + kx\dot{x} + \frac{k^2}{9}x^3 + \omega^2 x = 0, \tag{4}$$

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Lagrangian of the Liénard equation (4) I

In [M.C. Nucci and K.M. Tamizhmani, *J. Nonlinear Math. Phys.* (2010)], Lagrangians L for (3) were derived using Jacobi Last Multipliers M, i.e. $M = \frac{\partial^2 L}{\partial \dot{x}^2}$.

Since a particular Jacobi Last Multiplier is $M = e^{\int f(x)\mathrm{d}t}$ then the following condition was introduced:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{g(x)}{f(x)}\right) = \alpha(1-\alpha)f(x), \quad \alpha \neq 1, \tag{5}$$

that yields the Lagrangian:

$$L = \left(\dot{x} + \frac{g(x)}{\alpha f(x)}\right)^{2-1/\alpha} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x). \tag{6}$$

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Lagrangian of the Liénard equation (4) II

In particular in the case of equation (4) it was found that:

$$f(x) = kx, \quad g(x) = \frac{k^2}{9}x^3 + \omega^2 x, \quad \alpha = \frac{1}{3}. \tag{7}$$

Thus since $M = e^{\int kx(t)\mathrm{d}t}$ the following Lagrangian was obtained:

$$L_G = \frac{1}{k\dot{x} + \frac{k}{3}x^2 + \frac{3\omega^2}{k}} + \frac{\mathrm{d}}{\mathrm{d}t}G(t,x). \tag{8}$$

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$$G = \frac{3\omega^2}{2k}x - \frac{9\omega^4}{2k^2}t. \tag{9}$$

This position allows to get rid of $\sqrt{-p}$ when the Hamiltonian corresponding to (8) is derived, i.e.:

$$H = \frac{9\omega^4}{2k^2}\left[2 - \frac{2k}{3\omega^2}p - 2\left(1 - \frac{2k}{3\omega^2}p\right)^{\frac{1}{2}} + \frac{k^2x^2}{9\omega^2}\left(1 - \frac{2k}{3\omega^2}p\right)\right]. \tag{10}$$

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Lie symmetries of the Liénard equation (4)

Equation (4) admits an eight-dimensional Lie point symmetry algebra generated by:

$$\Gamma_1 = x\partial_t - \left(\frac{1}{3}x^3 k + \frac{3\omega^2}{k}x\right)\partial_x,$$

$$\Gamma_2 = \sin(\omega t)x\partial_t - \left(\frac{k}{3}\sin(\omega t)x^3 - \omega\cos(\omega t)x^2\right)\partial_x,$$

$$\Gamma_3 = \cos(\omega t)x\partial_t - \left(\frac{k}{3}\cos(\omega t)x^3 + \omega\sin(\omega t)x^2\right)\partial_x,$$

$$\Gamma_4 = \left(\frac{3\omega}{k}\cos(2\omega t) - \sin(2\omega t)x\right)\partial_t + \left[\left(\frac{k}{3}x^3 - \frac{3\omega^2}{k}x\right)\sin(2\omega t) - 2\omega x^2\cos(2\omega t)\right]\partial_x,$$

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$$\Gamma_5 = -\left(\frac{3\omega}{k}\sin(2\omega t) + \cos(2\omega t)x\right)\partial_t$$
$$- \left[\left(\frac{k}{3}x^3 - \frac{3\omega^2}{k}x\right)\cos(2\omega t) + 2\omega x^2\cos(2\omega t)\right]\partial_x,$$
$$\Gamma_6 = \cos(\omega t)\partial_t + \left(\omega\sin(\omega t)x - \frac{3\omega^2}{k}\cos(\omega t)\right)\partial_x,$$
$$\Gamma_7 = -\sin(\omega t)\partial_t + \left(\omega\cos(\omega t)x + \frac{3\omega^2}{k}\sin(\omega t)\right)\partial_x,$$
$$\Gamma_8 = \partial_t.$$

Point transformation to the free particle

Lie's method: the linearising transformation can be found by looking for a two-dimensional abelian intransitive subalgebra L_2 and transforming into its canonical form $\partial_{\tilde{x}},\ \tilde{t}\partial_{\tilde{x}}$.

In the case of equation (4) L_2 is generated by:

$$k\Gamma_2 - 3\omega\Gamma_6, \qquad k\Gamma_3 - 3\omega\Gamma_7. \tag{11}$$

and the corresponding point transformation is:

$$\tilde{t} = \frac{kx\cos(\omega t) + 3\omega\sin(\omega t)}{kx\sin(\omega t) - 3\omega\cos(\omega t)}, \quad \tilde{x} = -\frac{1}{9\omega^2}\frac{x}{kx\sin(\omega t) - 3\omega\cos(\omega t)}. \tag{12}$$

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$$\frac{\mathrm{d}^2\tilde{x}}{\mathrm{d}\tilde{t}^2} = 0$$

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Non-local transformation to the harmonic oscillator

In [V. K. Chandrasekar, M. Senthilvelan, and M. Lakshmanan. *Phys. Rev. E* (2005)], it was found that:

$$U = xe^{\frac{k}{3}\int x(\tau)\mathrm{d}\tau} \Rightarrow \ddot{U} + \omega^2 U = 0. \tag{13}$$

The authors determined the general solution of equation (4) and inserted it into the non-local transformation (13) in order to eliminate the non-locality.

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Canonical transformation to the harmonic oscillator

We show that this can be achieved without solving equation (4).

Using the results in [M.C. Nucci and K.M. Tamizhmani, *J. Nonlinear Math. Phys.* (2010)], we define:

$$q = \frac{x}{\dot{x} + \frac{k}{3}x^2 + \frac{3\omega^2}{k}}. \tag{14}$$

such that $q = xe^{\frac{k}{3}\int x(\tau)\mathrm{d}\tau}$ and satisfies:

$$\ddot{q} + \omega^2 q = 0. \tag{15}$$

Using a generating function of type $f_3 = f_3(q, p)$ we found the canonical transformation:

$$x = \frac{27\omega^6 q}{k^3 s}, \quad p = -\frac{k^3}{54\omega^6}s^2 + \frac{3\omega^2}{2k}$$

Then the Hamiltonian (10) is mapped into:

$$K = \frac{1}{2}\left(\frac{ks}{9\omega^2} + \frac{\omega^2}{k}\right)^2 + \frac{9\omega^2}{2k^2}q^2. \tag{16}$$

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Linearisation and quantisation?

Therefore:

1. The Liénard equation (4) **cannot** be related to the free-particle equation without changing the time.
2. The Liénard equation (4) **cannot** be related to the harmonic oscillator without a nonlinear canonical transformation involving the momentum.

What can be done next?

Remark: the Hamiltonian of the Liénard equation (4) is quadratic in x^2

$$H = \frac{9\omega^4}{2k^2}\left[2 - \frac{2k}{3\omega^2}p - 2\left(1 - \frac{2k}{3\omega^2}p\right)^{\frac{1}{2}} + \frac{k^2x^2}{9\omega^2}\left(1 - \frac{2k}{3\omega^2}p\right)\right].$$

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Classical Hamiltonian in Momentum Representation

In [V. Chithiika Ruby, M. Senthilvelan, and M. Lakshmanan, *J. Phys. A* (2012)], the quantisation was tackled in Momentum Representation.

Classically this corresponds to make the linear canonical transformation:

$$(x, p) \to (X, P) = (p, -x) \tag{17}$$

such that the Hamiltonian (10) becomes:

$$\tilde{H} = \frac{9\omega^4}{2k^2}\left[2 - \frac{2k}{3\omega^2}X - 2\left(1 - \frac{2k}{3\omega^2}X\right)^{\frac{1}{2}} + \frac{k^2P^2}{9\omega^2}\left(1 - \frac{2k}{3\omega^2}X\right)\right]. \tag{18}$$

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The classical Momentum Representation equation

We now apply the Lie-Noether symmetry quantisation to the Hamiltonian (18).

The corresponding Lagrangian of (18) is:

$$\tilde{L} = \frac{\dot{X}^2}{2\omega^2\left(1-\frac{2k}{3\omega^2}X\right)} - \frac{9\omega^4}{2k^2}\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)^2. \tag{19}$$

and the corresponding Lagrange equation is:

$$\ddot{X} = \frac{3\omega^4}{k}\left(1-\frac{2k}{3\omega^2}X-\sqrt{1-\frac{2k}{3\omega^2}X}\right) - \frac{k\dot{X}^2}{3\omega^2\left(1-\frac{2k}{3\omega^2}X\right)}. \tag{20}$$

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Symmetries of (20) I

The Lie symmetries of (20) are:

$$\Xi_1 = \partial_t,$$

$$\Xi_2 = \cos(2\omega t)\partial_t + \sin(2\omega t)\frac{3\omega^3}{k}\left[1 - \frac{2k}{3\omega^2}X - \sqrt{1 - \frac{2k}{3\omega^2}X}\right]\partial_X$$

$$\Xi_3 = \sin(2\omega t)\partial_t - \cos(2\omega t)\frac{3\omega^3}{k}\left[1 - \frac{2k}{3\omega^2}X - \sqrt{1 - \frac{2k}{3\omega^2}X}\right]\partial_X$$

$$\Xi_4 = \sqrt{1 - \frac{2k}{3\omega^2}X}\cos(\omega t)\partial_X$$

$$\Xi_5 = \sqrt{1 - \frac{2k}{3\omega^2}X}\sin(\omega t)\partial_X$$

Symmetries of (20) II

$$
\begin{aligned}
\Xi_6 = \; & \cos(\omega t)\left(\sqrt{1-\frac{2k}{3\omega^2}X}-1\right)\partial_t \\
& + \frac{3\omega^3}{k}\sin(\omega t)\left(\sqrt{1-\frac{2k}{3\omega^2}X}-2\right)\left(1-\frac{2k}{3\omega^2}X\right)\partial_X, \\
\Xi_7 = \; & \sin(\omega t)\left(\sqrt{1-\frac{2k}{3\omega^2}X}-1\right)\partial_t \\
& - \frac{3\omega^3}{k}\cos(\omega t)\left(\sqrt{1-\frac{2k}{3\omega^2}X}-2\right)\left(1-\frac{2k}{3\omega^2}X\right)\partial_X, \\
\Xi_8 = \; & \sqrt{1-\frac{2k}{3\omega^2}X}\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)\partial_X.
\end{aligned}
$$

Quantisation with symmetries: Step 1

Using Lie's theory we find that the linearising transformation is:

$$\begin{cases} \tau = \tan(\omega t), \\ \xi = \omega\sqrt{\dfrac{6}{k}}\dfrac{1-\sqrt{1-\dfrac{2k}{3\omega^2}X}}{\cos(\omega t)} \end{cases} \Rightarrow \frac{\mathrm{d}^2\xi}{\mathrm{d}\tau^2} = 0 \tag{21}$$

Also we can transform (20) into the harmonic oscillator:

$$\eta = \omega\sqrt{\frac{6}{k}}\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right) \Rightarrow \ddot{\eta} + \omega^2\eta = 0. \tag{22}$$

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Quantisation with symmetries: Step 2.

The following Lagrangian admits five Noether symmetries generated by Ξ_i, $i = 1, \ldots, 5$, :

$$\tilde{L} = \frac{\dot{X}^2}{2\omega^2\left(1 - \frac{2k}{3\omega^2}X\right)} - \frac{9\omega^4}{2k^2}\left(1 - \sqrt{1 - \frac{2k}{3\omega^2}X}\right)^2 . \qquad (23)$$

Quantisation with symmetries: Step 3. I

The Schrödinger equation for the harmonic oscillator is:

$$2\mathrm{i}\Phi_t + \Phi_{\eta\eta} - \omega^2\eta^2\Phi = 0. \tag{24}$$

The point transformation (22) takes equation (24) into the following Schrödinger equation:

$$2\mathrm{i}\widetilde{\Phi}_t + \frac{3\omega^2}{2k}\left(1 - \frac{2k}{3\omega^2}X\right)\widetilde{\Phi}_{XX} - \frac{1}{2}\widetilde{\Phi}_X - \frac{6\omega^4}{k}\left(1 - \sqrt{1 - \frac{2k}{3\omega^2}X}\right)^2\widetilde{\Phi}. \tag{25}$$

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Quantisation with symmetries: Step 3. I

We may eliminate the first derivative $\widetilde{\Phi}_X$ by imposing the transformation:

$$\Phi(t,X) = \frac{\widetilde{\Phi}(t,X)}{\left[3\omega^2\left(1-\frac{2k}{3\omega^2}X\right)\right]^{\frac{1}{4}}} \tag{26}$$

Finally:

$$\begin{aligned} &2\mathrm{i}\Phi_t + 9\left(1-\frac{2k}{3\omega^2}X\right)\Phi_{XX} \\ &\quad + \left[\frac{3k^2}{4\omega^4\left(1-\frac{2k}{3\omega^2}X\right)} - \frac{2\omega^6}{k^2}\left(1-\frac{k}{3\omega^2}X\right)\frac{2\omega^6}{k^2}\sqrt{1-\frac{2k}{3\omega^2}X}\right]\Phi. \end{aligned} \tag{27}$$

This is the final form of the Schrödinger equation for (20).

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This is the final form of the Schrödinger equation for (20).

Step 4.

The Lie symmetries of equation (27) are

$$\Omega_1 = \Xi_1,$$

$$\Omega_2 = \Xi_2 + \left[\frac{\omega}{2}\sin(2\omega t) - \mathrm{i}\omega^2\cos(2\omega t)\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)^2\right]\frac{\Phi}{\sqrt{1-\frac{2k}{3\omega^2}X}}\partial_\Phi,$$

$$\Omega_3 = \Xi_3 - \left[\frac{\omega}{2}\cos(2\omega t) + \mathrm{i}\omega^2\sin(2\omega t)\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)^2\right]\frac{\Phi}{\sqrt{1-\frac{2k}{3\omega^2}X}}\partial_\Phi,$$

$$\Omega_4 = \Xi_4 - \left[\frac{k\cos(\omega t)}{6\omega^2\sqrt{1-\frac{2k}{3\omega^2}X}} + 2\mathrm{i}\omega\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)\sin(\omega t)\right]\Phi\partial_\Phi$$

$$\Omega_5 = \Xi_5 - \left[\frac{k\sin(\omega t)}{6\omega^2\sqrt{1-\frac{2k}{3\omega^2}X}} - 2\mathrm{i}\omega\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)\cos(\omega t)\right]\Phi\partial_\Phi$$

$$\Omega_6 = \Phi\partial_\Phi,$$

$$\Omega_\chi = \chi(t, X)\partial_\Phi,$$

Spectrum of the solution I

We expect the spectrum of (27) to be that of the harmonic oscillator, i.e.:

$$E_n = \omega \left(n + \tfrac{1}{2}\right). \tag{29}$$

as was proven in [V. Chithiika Ruby, M. Senthilvelan, and M. Lakshmanan, *J. Phys. A* (2012)].

We show how to use Lie symmetries for this purpose. This was first applied to the harmonic oscillator in [P.G.L. Leach, *J. Phys. A* (2005)] and subsequently to other problems in [M.C. Nucci and P.G.L. Leach, *Phys. Scripta* (2010)] and [M.C. Nucci and P.G.L. Leach, *J. Math. Anal. Appl.* (2013)].

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Spectrum of the solution II

We rewrite the Lie symmetries of (27) in complex form:

$$\Sigma_1 = \mathrm{i}\partial_t,$$

$$\Sigma_{2\pm} = e^{\pm 2\mathrm{i}\omega t}\partial_t \mp \mathrm{i}e^{\pm 2\mathrm{i}\omega t}\frac{3\omega^3}{k}\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)\sqrt{1-\frac{2k}{3\omega^2}X}\partial_X$$

$$+\frac{\mathrm{i}}{2}e^{\pm 2\mathrm{i}\omega t}\left[\frac{24\omega^3 - 24\omega^3\sqrt{1-\frac{2k}{3\omega^2}X} - k + 8kX\omega\sqrt{1-\frac{2k}{3\omega^2}X} - 16k\omega X}{\omega k\sqrt{1-\frac{2k}{3\omega^2}X}}\right]$$

$$\Sigma_{3\pm} = e^{\pm \mathrm{i}\omega}\sqrt{1-\frac{2k}{3\omega^2}X}\partial_X$$

$$\mp e^{\pm \mathrm{i}\omega^3}2\omega\left[\left(1-\sqrt{1-\frac{2k}{3\omega^2}X}\right)+\frac{k}{12\sqrt{3}\omega^4}\frac{1}{\sqrt{1-\frac{2k}{3\omega^2}X}}\right]\Phi\partial_\Phi$$

$$\Sigma_4 = \Phi\partial_\Phi$$

$$\Sigma_\chi = \chi(t,X)\partial_\Phi$$

Spectrum of the solution III

The operators $\Sigma_{3\pm}$ will act as creation and annihilation operators.
The ground state can be found from the invariant solution corresponding to Σ_{3+}, i.e. $\Sigma_{3+}F(t,X,\Phi)=0$.
We obtain:

$$F(t,X,\Phi)=f\left(t,\Phi\frac{e^{-\frac{2\omega}{k}\left(3\omega^2\sqrt{1-\frac{2k}{3\omega^2}X}+kX\right)}}{\left(1-\frac{2k}{3\omega^2}X\right)^{\frac{1}{4}}}\right)=0. \tag{30}$$

that replaced into (27) gives:

$$\Phi_0(t,X)=\left(1-\frac{2k}{3\omega^2}X\right)^{\frac{1}{4}}e^{-\frac{1}{2}\mathrm{i}t+\frac{2\omega}{k}\left(3\omega^2\sqrt{1-\frac{2k}{3\omega^2}X}+kX\right)}. \tag{31}$$

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Spectrum of the solution IV

This ground state satisfies the boundary conditions:

$$\lim_{X\to-\infty} \Phi_0(t,X) = 0, \quad \Phi_0\left(t, \frac{3\omega^2}{2k}\right) = 0. \tag{32}$$

Remark: Σ_{3-} yields solutions unbounded at infinity.
We construct all the eigenfunctions in the following manner:

$$\left[\Sigma_{3-}, \Sigma_{\Phi_{n-1}}\right] = \Phi_n \partial_\Phi \equiv \Sigma_{\Phi_n}, \quad n \geq 1. \tag{33}$$

The operator Σ_1 acts as the eigenvalue operator and yields:

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Comparison with previous results

Using a modified von Roos method [O. von Roos, *Phys. Rev. B*, (1983), O. von Roos and H. Mavromatis, *Phys. Rev. B*, (1985)] in V. Chithiika Ruby, M. Senthilvelan, and M. Lakshmanan, *J. Phys. A*, (2012) the following Schrödinger equation for $\Theta = \Theta(t, p)$ was determined:

$$2\mathrm{i}\Theta_t + \omega^2 \left(1 - \frac{2k}{3\omega^2} p\right) \Theta_{pp} + \left[\frac{k^2}{12\omega^2 \left(1 - \frac{2k}{3\omega^2} p\right)} - \frac{9\omega^4}{k^2} \left(\sqrt{1 - \frac{2k}{3\omega^2} p} - 1\right)^2\right] \Theta = 0,$$

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Final remarks

- Noether symmetries yield the correct Schrödinger equation, although the wave function part of the Lie symmetry operator may have different form (no uniqueness).
- Lie symmetries can be algorithmically used to find the spectrum.
- Work is in progress to apply this method to other classical problems.