Nonlinear Mathematical Physics and Differential Equations: A Personal Perspective and a First Step in Symmetry Analysis

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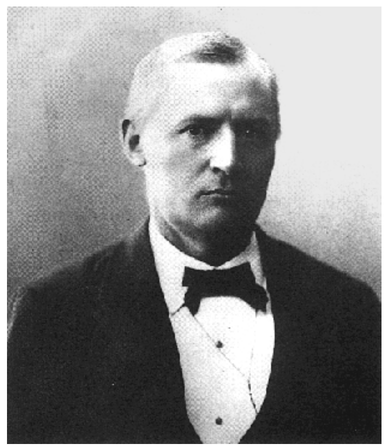


Figure 1. Marius Sophus Lie (1842–1899) Albert Victor Bäcklund (1845–1922)

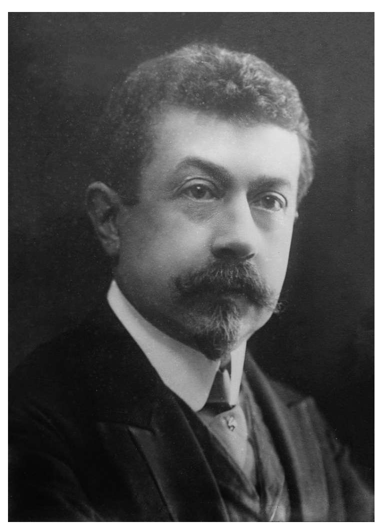



Figure 2. Sofia Kovalevskaya (1850–1891) Paul Painlevé (1863–1933)

Contents:

PART I:

General Overview

Nonlinear Mathematical Physics

Nonlinear Equations:
Differential, Difference,
Differential-Difference
Integro-Differential, ...
Supersymmetries, Quantization, ...

My interests: Nonlinear Differential Equations

Criteria for Integrability

Symmetry Analysis:
[**Lie Groups, Lie Algebras, Diff. Geometry**]

- **Map solutions**
- **Map equations**
- **Reduce order**
- **Linearization**
- **Integrating factors**
- **Conservation laws**
- **Exact solutions**
- **Reduce dimensions**
- **Bäcklund transformations**
- **Recursion operators**
- **Hierarchies**
- **Discretization**

Painlevé Analysis:
[**Complex Analysis**]

- **Singularity structure**
- **Poles as movable singulatrities**
- **Transcendental meromorphic functions**
- **Natural boundaries**
- **Integrability test**
- **Bäcklund transformations**
- **Chaos**
- **Non-integrability**
- **Special functions and solutions**
- **Ziglin's Theorem**

Historical Remarks on Integrability

- **Classical integrable systems:**
  18th+19th century:

  Integrable rigid body motion under gravity:
  Lagrange Top (1750's?), Euler Top (about 1750's?) and Kovalevskaya Top (1889)

  Sophus Lie: continuous symmetry transformation groups (1888):
  If sufficient symmetries then complete integrability is possible.

  Paul Painlevé's classification: Transcendental functions by 2nd-order ODEs:
  Only poles as movable singularities.
  50 equations, 6 transcendental functions.
  Émile Picard (1889), Paul Painlevé (1900, 1902), Richard Fuchs (1906), Bertrand Gambier (1910).

  Henri Poincaré (1857 – 1912)
  "... complete integrability is a highly exceptional property in classical mechanics ..."

  Emmy Noether for Lagrangian systems:
  Noether's First and Second Theorems (1915)

- **New Interest in Integrability in 1965:**

  The Soliton Phenomena:
  Norman Zabusky and Martin Kruskal (1965):
  Extraordinary stability properties of solitary waves:
  Soliton: localized, after collision same speed and amplitude, only phase shift.
  Inverse Scattering Transform: Gardner-Green-Kruska-Miura. Phys. Rev. Lett. (1967)

  E.g: Korteweg-de Vries equation

  $$u_t + 6uu_x + u_{xxx} = 0$$

  John Scott Russell (1834):
  A hump of water moving with constant speed and shape along a canal.

- **Infinite number of conservation laws.**
  IST: A linearizing transformation which maps the initial vlue $u(0,x)$ of the KdV Cauchy problem to spectral and scattering data of the Schrödinger operator $-d^2/dx^2 - u(0,x)$.
  KdV is viewed as an infinite-dimensional classical integrable system (Vladimir E. Zhakarov and Ludvig D. Faddeev, 1971).

  [action-angle variables $->$ spectral and scattering data;

  inverse of the action-angle map $->$ IST ;

  Poisson commuting Hamiltonians $->$ infinity of conservations;

  Forward scattering: Lax pairs (Peter Lax);

  Inverse scattering: linear integral equation (Gelfand-Levitan- Marchenko)]

Connection with the Painlevé Analysis for PDEs

1980: M.J. Ablowitz, A. Ramani and H. Segur, *A connection between nonlinear evolution equations and ordinary differential equations of P-type I*, **J. Math. Phys. 21, 715–721**

Lie point symmetry reductions of some soliton equations lead to Painlevé transcendental equations

1983: J. Weiss, M. Tabor and G Carneval, *The Painlevé Property for partial differential equations*, **J. Math. Phys. 24522–526.**

Painlevé Expansion for the solution (nth-order PDEs):

$$u(x,t) = \phi^m(x,t) \sum_{j=0}^{\infty} u_j(x,t)\, \phi^j(x,t)$$

where ϕ and $n-1$ coefficient functions u_j must be arbitrary analytic functions in the neighbourhood of the sigularity manifold (necessary condition).

If the expansion is consistent, we truncate the series to prove the sufficient condition (Bäcklund transformations, Lax pairs, ...)
[More about this in the Appendix in the last section of this paper]

Applications of soliton PDEs:

Soliton theory has revolutionized nonlinear applied science!

Modelling of wave phenomena in Physics:
hydrodynanics, acoustics, nonlinear optics, plasma physics, solid-state physics

Other areas:
molecular biology (energy transport along proteins), ecology (predator-prey equations), chemistry (charge density waves in organic conductors), electronics (network equations)...

Example: The Sine-Gordon eqation $u_{xx} - u_{tt} = \sin u$ **models propagation of dislocations in crystals, phase difference across Josephson junctions, torsion waves in strings and pendulas, waves along membranes lipids, ...**
Also in pseudo-spherical surfaces (led Bäcklund to his transformation!).

I recommend Chris Eilbeck's Computer Animated Films (1973–1982) Hariot Watt University, Edinburgh.

www.chilton-computing.org.uk/acl/applications/animation/p023.htm

Keywords for related theories:

- **Liouville-Arnold Theorem (action-angle variables)**
- **Frobenius integrability (maximal integral manifolds)**
- **Inverse scattering transform (1967)**
- **Lax pairs**
- **Bi-Hamiltonian formalism for PDEs**
- **Hirota's bilinear method**
- **Darboux transformations**
- τ**-Functions**
- $\bar{\partial}$**-Dressing method**
- **Gelfand-Dickey Theorem**
- **Zero-Curvature formalism**
- **Riemann-Hilbert Factorization (instead of IST)**
- **Donaldson Theory**
- **Quantum integrable systems (Yang-Baxter eqs)**
- **Integrable lattice models (Toda lattice, 1967)**

A journal for this subject:
Journal of Nonlinear Mathematical Physics
(Taylor & Francis, Atlantis)

Some of the nonlinear equations:

ODEs:

Riccati Equation:

$$u' = f(x)u^2 + g(x)u + h(x)$$

Elliptic functions:

$$(u')^2 = 4u^3 - \alpha_1 u - \alpha_2 \quad \textbf{Weierstrass: } \wp(x)$$
$$(u')^2 = (1-u^2)(1-k^2u^2) \quad \textbf{Jacobi: sn}(x)$$
$$(u')^2 = (1-u^2)(1-k^2+k^2u^2) \quad \textbf{Jacobi: cn}(x)$$
$$(u')^2 = (u^2-1)(1-k^2-u^2) \quad \textbf{Jacobi: dn}(x)$$

Painlevé Equations I-VI (Painlevé transcendents):

I: $u'' = 6u^2 + x$

II: $u'' = 2u^3 + xu + \alpha$

III: $xuu'' = x(u')^2 - uu' + \delta u' + \beta u + \alpha u^3 + \gamma x u^4$

etc

Chazy Equation (natural boundaries, Hausdorff dim):

$$u''' = 2uu'' - 3(u')^2$$

Lorenz System (chaos):

$$u' = \sigma(v-u)$$
$$v' = \rho u - uw - v$$
$$w' = uv - \beta w$$

PDEs in 1+1 dimensions:

Korteweg-de Vries Equation [Soliton eq]:

$$u_t + 6uu_x + u_{xxx} = 0$$

Sine-Gordon Equation [Soliton eq]:

$$u_{tt} - u_{xx} + \sin u = 0$$

Nonlinear Schrödinger Equation [Soliton eq]

$$iu_t = -\frac{1}{2}u_{xx} + \kappa|u|^2 u$$

AKNS System:

$$u_t = iu^2 v - \frac{i}{2}u_{xx}$$

$$v_t = -iv^2 u + \frac{i}{2}v_{xx}$$

[Note: For $v = \bar{u}$, AKNS becomes Nonlinear Schrödinger]

Camassa-Holm Equation [Solitons]:

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

Krichever-Novikov Equation:

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x}. \qquad P^{(5)} = 0$$

PDEs in 1+2 dimensions:

Kadomtsev-Petviashvili Equation [dromions]:

$$\left(u_t + uu_x + \epsilon^2 u_{xxx}\right)_x + \lambda u_{yy} = 0$$

Davey-Stewartson Equation:

$$iu_t + c_0 u_{xx} + u_{yy} = c_1 |u|^2 u + c_2 u \phi_x$$
$$\phi_{xx} + c_3 \phi_{yy} = \left(|u|^2\right)_x$$

PDEs in 1+3 dimensions:

Jimbo-Miwa Equation:

$$2u_{ty} - 3u_{xz} + 3u_y u_{xx} + 3u_x u_{xy} + u_{xxxy} = 0$$

Symmetry analysis in a nutshell: Computationally

For scalar PDE:
$E(x, t, u, u^{(1)}, u^{(2)}, \ldots, u^{(n)}) = 0$

Lie-Bäcklund Symmetry Generator: $Z = Q(x, t, u, u^{(1)}, u^{(2)}, \ldots)\dfrac{\partial}{\partial u}$

Symmetry condition:

$$\boxed{L_E[u]Q[u]\Big|_{D^{(k)}E=0} = 0}$$

$\hat{x} = x, \quad \hat{t} = t, \quad \hat{u} = u + \varepsilon\, Q$ **(infinitesimal transformation)**

Conservation law: $D_t\Phi^t[u] + D_x\Phi^x[u] = 0$

Integrating factor: $\Lambda(x, t, u, u_x, u_{xx}, \ldots)$**:** $\quad \Lambda[u] = \hat{E}[u]\,\Phi^t[u]$

$\hat{E}[u]\,(\Lambda\ E) = 0$

$\Longleftrightarrow$

$$\boxed{L_E^*[u]\Lambda\Big|_{E=0} = 0 \quad \textbf{and} \quad L_\Lambda[u]E = L_\Lambda^*[u]E}$$

Linear operator: $L_E[u] := \dfrac{\partial E}{\partial u} + \dfrac{\partial E}{\partial u_t}D_t + \dfrac{\partial E}{\partial u_x}D_x + \dfrac{\partial E}{\partial u_{xx}}D_x^2 + \cdots$

Adjoint operator: $L_E^*[u] := \dfrac{\partial E}{\partial u} - D_t \circ \dfrac{\partial E}{\partial u_t} - D_x \circ \dfrac{\partial E}{\partial u_x} + D_x^2 \circ \dfrac{\partial E}{\partial u_{xx}} - \cdots$

Euler operator: $\hat{E}[u] := \dfrac{\partial}{\partial u} - D_t \circ \dfrac{\partial}{\partial u_t} - D_x \circ \dfrac{\partial}{\partial u_x} + D_x^2 \circ \dfrac{\partial E}{\partial u_{xx}} - \cdots$

Connection to the integrability of PDEs:
For evolution equations $u_t = F(x, t, u, u_x, u_{xx}, \ldots, u_{px})$

- **Infinitely many (local) higher-order symmetries.**
- **Infinitely many conservation laws (some exceptions).**
- **Hierarchies of evolution equations.**

The symmetries Q_j **can be generated by a recursion operator** $R[u]$**:**

$$R[u] = \sum_i G_i D_x^i + \sum_j I_j D_x^{-1} \circ \Lambda_j$$

where

$$\boxed{[L_F,\ R]\varphi = (D_t R)\,\varphi.}$$

Then

$$Q_{j+1} = R[u] Q_j \quad \textbf{and} \quad u_t = R[u]^n F[u] \quad \textbf{(hierarchy)}$$

Recall: $Z_j = Q_j \dfrac{\partial}{\partial u}$

Classification of Evolution Equations:

Potentialisation of evolution equations:

$u_t = F(t, x, u, u_x, u_{xx}, \ldots, u_{px})$

Potentialisation: $v_x = \Phi^t[u], \quad v_t = -\Phi^x[u], \quad v(x,t)$

Proposition: **(Recall:** $\Lambda = \hat{E}\,\Phi^t$**)**

Potentialisation

$$\boxed{u_t = F(x, u, u_x, \ldots, u_{px})}$$

$$\Big\downarrow{\scriptstyle v_x = \Phi^t[u],} \qquad D_x\Phi^t[u] + D_x\Phi^x[u] = 0$$

$$\boxed{v_t = G(x, v_x, v_{xx}, \ldots, v_{px})}$$

can only be achieved if

$\Phi^t(x, t, u, u_x, \ldots, u_{qx})$ **is of order** $\boxed{q \leq p - 1}$

with corresponding integrating factor,

$\Lambda(x, t, u, u_x, \ldots, u_{rx})$ **of order** $\boxed{r \leq p + 1 \quad \textbf{(even)}}$.

Reference:

N Euler and M Euler, The converse problem for the multi-potentialisation of evolution equations and systems,
J. Nonlinear Math. Phys. **18, 77–105, 2011.**

Exaples: The Schwarzian KdV equation

$$\boxed{u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x}}$$

$$\Lambda = \alpha\left(\frac{uu_{4x}}{u_x^2} - \frac{4u_{xx}u_{3x}}{u_x^3} + \frac{3u_{xx}^3}{u_x^4}\right) \quad \downarrow \quad v_x = \frac{\alpha}{2}\left(\frac{u_{xx}}{u_x}\right)^2$$

$$\boxed{v_t = v_{xxx} - \frac{3}{4}\frac{v_{xx}^2}{v_x} - \left(\frac{3}{2\alpha}\right)v_x^2}$$

Auto-Bäcklund transformation:

$$\boxed{u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x}}$$

$$\Lambda = -\frac{2u^2u_{xx}}{u_x^3} + \frac{4u}{u_x} \quad \downarrow \quad v_x = \frac{u^2}{u_x}$$

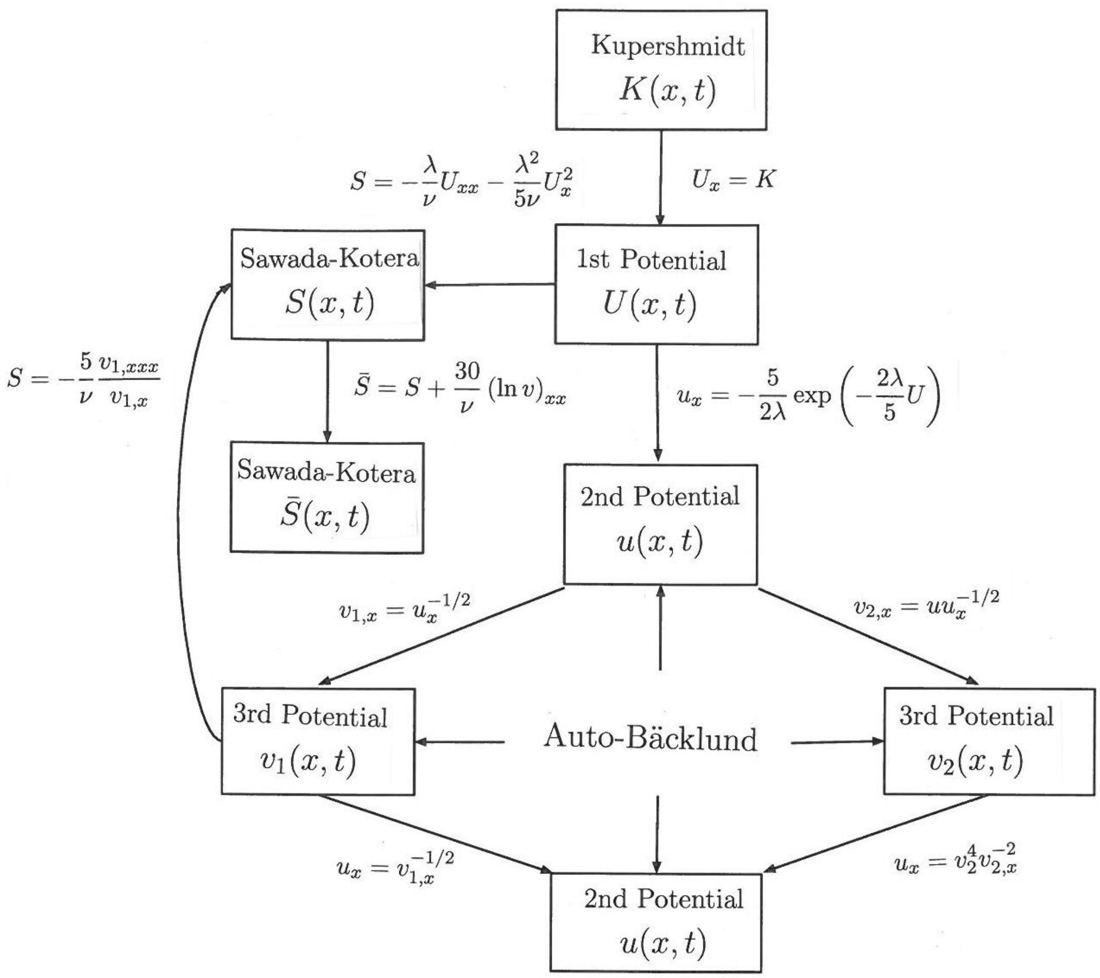
$$\boxed{v_t = v_{xxx} - \frac{3}{2}\frac{v_{xx}^2}{v_x}}$$

Generate solutions:

$$u(x,t) = x \mapsto \frac{x^3}{3} - 4t + a_1 \mapsto \frac{x^5}{45} + \frac{(6a_1 - 24t)x^2}{18} + \frac{72a_1t - 9a_1^2 - 144t^2}{9x} + a_2$$

N Euler and M Euler, Multipotentialisation and iterating-solution formulae: the Krichever-Novikov equation,
J. Nonlinear Math. Phys. **16, 93–106, 2009.**

A new result on multipotentialization (ME, NE; 2017)



$$K_t = K_{xxxxx} + \lambda\left(K_x K_{xxx} + K_{xx}^2\right) - \frac{\lambda^2}{5}\left(K^2 K_{xxx} + 4K K_x K_{xx} + K_x^3\right) + \frac{\lambda^4}{125}K^4 K_x$$

$$v_t = v_{xxxxx} - 5\frac{v_{xx}v_{xxxx}}{v_x} + 5\frac{v_{xx}^2 v_{xxx}}{v_x^2}$$

$$S_t = S_{xxxxx} + \nu S S_{xxx} + \nu S_x S_{xx} + \frac{\nu^2}{5}S^2 S_x$$

Linearization:

$$v_t = v_{xxxxx} + \nu\, v_{xx} S_x, \qquad v_{xxx} = -\frac{\nu}{5} S v_x$$

PART II:

Lie Point Symmetry Transformation Groups

The derivation of the invariance condition for $E = 0$:

$$\boxed{L_E[u]Q\Big|_{D_x^k E=0} = 0, \qquad k \in \{0, 1, 2, \ldots\}}$$

$$Q = \eta(x, u) - \sum_i \xi_i(x, u)\frac{\partial u}{\partial x_i}$$

Lie Point Symmetry transformations

Continuous coordinate transformations

$$\Gamma_\varepsilon \colon \mathbf{z} \mapsto \hat{\mathbf{z}} = (\phi_1(\mathbf{z}, \varepsilon), \ldots, \phi_N(\mathbf{z}, \varepsilon))$$

Group

Group composition=composition of transformations
Axioms: **closure**, **associativity**, **identity**, **inverse**.

Representation in Vector Space

First order linear differential operators $\{X, Y, Z, \ldots, \}$

$$X = \sum_{j=1}^{N} \xi_j(\mathbf{z}) \frac{\partial}{\partial z_j}$$

Lie Algebra

Binary operation (Lie bracket): $[X, Y] = XY - YX$
Axioms:
Bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
Alternativity: $[X, X] = 0$
(implies anti-commutativity: $[X, Y] = -[Y, X]$**)**
Jacobi Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

We study Lie Groups in terms of Lie Algebras:

"Lie Group = EXP (Lie Algebra)"

Some Classical Lie transformation Groups in Physics:

- **SO(3) – Full rotations**
- **SO(4) – Kepler problem**
- **SO(4,1) – Yang-Mills theory of quantum gravity**
- **SO(10) – Great Unified Theories (GUTs)**
- **SO(32) – Heterotic Superstring Theory**
- **U(1) – Gauge group in electromagnetism**
- **SU(2) – Isospin; Gauge in Yang Mills theory**
- **SU(3) – Gauge, quantum electrodynamics**
- **U(3) – Symmetry of 3-dim harmonic oscillator**
- **O(1,3) – Lorentz group in special relativity**
- **Sp(2n) – Hamiltonian canonical transformations**

One-Parameter Continuous Symmetry Transformations

Local Lie point transformation group:

$\Gamma_\varepsilon \colon \mathbb{U} \subseteq \mathbb{R}^N \to \mathbb{R}^N$,

$\Gamma_\varepsilon \colon \mathbf{z} \mapsto \hat{\mathbf{z}} = (\phi_1(\mathbf{z};\varepsilon), \phi_2(\mathbf{z};\varepsilon), \ldots, \phi_N(\mathbf{z};\varepsilon))$

small ε: group parameter

Group law: composition of transformations Γ_ε.

- **Closure:** $\Gamma_\varepsilon \circ \Gamma_\delta = \Gamma_{\varepsilon+\delta}$
- **Identity:** **at** $\varepsilon = 0$**, i.e.** Γ_0
- **Inverse:** **at** $-\varepsilon$**, i.e.** $\Gamma_{-\varepsilon}$
- **Associativity:** $\Gamma_\varepsilon \circ (\Gamma_\delta \circ \Gamma_\gamma) = (\Gamma_\varepsilon \circ \Gamma_\delta) \circ \Gamma_\gamma$
- **Locallity:** ϕ_j **analytic, Taylor expansion about** $\varepsilon = 0$**:**

$$\boxed{\hat{z}_i = z_i + \varepsilon \xi_i(\mathbf{z}) + O(\varepsilon^2) = z_i + \varepsilon Z z_i + O(\varepsilon^2) = e^{\varepsilon Z} z_i}$$

where

$$\xi_i(\mathbf{z}) = \left.\frac{\partial \phi_i}{\partial \varepsilon}\right|_{\varepsilon=0}, \qquad i = 1, 2, \ldots, N$$

$$Z = \sum_{j=1}^{N} \xi_j(\mathbf{z}) \frac{\partial}{\partial z_j}, \qquad Z z_i = \xi_i(\mathbf{z}).$$

Z: **Lie transformation group generator.**

Given

$$Z = \sum_{j=1}^{N} \xi_j(\mathbf{z}) \frac{\partial}{\partial z_j},$$

we can find the Lie transformation group by either

- **The Lie series:** $\hat{z}_i = e^{\varepsilon Z} z_i, \quad i = 1, 2, \ldots, N$

  or

- **Solving the initial-value problem of the Lie equations:**

$$\frac{d\hat{z}_i}{d\varepsilon} = \xi_i(\hat{\mathbf{z}})$$

$$\hat{z}_i(\varepsilon = 0) = z_i, \qquad i = 1, 2, \ldots, N.$$

This is Lie's First Fundamental Theorem.

Properties:

- $\dfrac{df(\hat{\mathbf{z}})}{d\varepsilon} = Z[\hat{\mathbf{z}}] f(\hat{\mathbf{z}})$
- $f(\hat{\mathbf{z}}) = e^{\varepsilon Z[\mathbf{z}]} f(\mathbf{z})$

Exercise: Counter-clockwise Rotations by angle ε about $(0,0)$ in $\mathbb{R}^2$:

$$R_\varepsilon : \begin{cases} \hat{x} = x\cos\varepsilon - y\sin\varepsilon \\ \hat{y} = x\sin\varepsilon + y\cos\varepsilon. \end{cases}$$

a) **Show that R_ε is a Lie transformation group.**

b) **Find the Lie transformation group generator Z.**

c) **Rederive the transformation R_ε from your Lie transformation generator Z.**

d) **Consider $f(\hat{x},\hat{y}) = \hat{x}^2+\hat{y}^2$. Show that $\dfrac{df}{d\varepsilon} = Zf = 0$. These are the Invariants (orbits) of R_ε.**

Note: The Lie transformation group generator is

$$Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Group Invariants and the Invariant Surface Condition:

$$\Gamma_\varepsilon : \begin{cases} \hat{x}_i = \varphi_i(\mathbf{x}, u; \varepsilon), & i = 1, 2, \dots, p \\ \hat{u} = \psi(\mathbf{x}, u; \varepsilon) \ : & \textbf{(dependent variable)} \end{cases}$$

$$Z = \sum_{i=1}^{p} \xi_i(\mathbf{x}, u) \frac{\partial}{\partial x_i} + \eta(\mathbf{x}, u) \frac{\partial}{\partial u}, \qquad \left. \frac{\partial \varphi_i}{\partial \varepsilon} \right|_{\varepsilon=0} = \xi_i, \ \left. \frac{\partial \psi}{\partial \varepsilon} \right|_{\varepsilon=0} = \eta.$$

Lie equations:

$$\frac{d\hat{x}_i}{d\varepsilon} = \xi_i(\hat{\mathbf{x}}, \hat{u}), \qquad i = 1, 2, \dots, p \tag{2a}$$

$$\frac{d\hat{u}}{d\varepsilon} = \eta(\hat{\mathbf{x}}, \hat{u}). \tag{2b}$$

The p first integrals of (2a)–(2b) are the p invariants for Γ_ε:

$$\boxed{\{I_1(\mathbf{x}, u),\ I_2(\mathbf{x}, u), \dots, I_p(\mathbf{x}, u)\},} \tag{3}$$

where

$$\frac{dI_i}{d\varepsilon} = Z I_i(\mathbf{x}, u) = 0, \qquad i = 1, 2, \dots, p.$$

The set (3) gives the general solution

$$\Phi\left(I_1(\mathbf{x}, u),\ I_2(\mathbf{x}, u), \dots, I_p(\mathbf{x}, u)\right) = 0$$

of the Invariant Surface Condition for Γ_ε:

$$\boxed{Q(\mathbf{x}, u, u^{(1)}) := \eta(\mathbf{x}, u) - \sum_{i=1}^{p} \xi_i(\mathbf{x}, u) \frac{\partial u}{\partial x_i} = 0.}$$

Q: Characteristic of Γ_ε.

TheVertical Lie transformation generator:

Consider a curve $u = f(x)$ that maps to the curve $u = g(x;\varepsilon)$ in $\mathbb{R}^2$ by

$$\Gamma_\varepsilon\colon \begin{cases} \hat{x} = \varphi(x,u;\varepsilon) \\ \\ \hat{u} = \psi(x,u;\varepsilon) \end{cases} \qquad Z = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u}.$$

We obtain the same curve $u = g(x;\varepsilon)$ by the vertical transformation

$$\Gamma_{\varepsilon,v}\colon \begin{cases} \hat{x} = x \\ \\ \hat{u} = g(x;\varepsilon) = \left[u + \varepsilon Q + \frac{1}{2}\varepsilon^2\left(Q\frac{\partial}{\partial u} + (D_x Q)\frac{\partial}{\partial u_x}\right)Q + O(\varepsilon^3)\right]_{u=f(x)} \\ \\ \qquad\qquad\;\; = \exp\left\{\varepsilon Z_v^{(\infty)} u\right\}\Big|_{u=f(x)} \end{cases}.$$

with

$$Q(x,u,u^{(1)}) = \eta(x,u) - \xi(x,u)u_x.$$

The vertical Lie transformation generator for $\Gamma_{\varepsilon,v}$ is

$$Z_v = Q(x,u,u^{(1)})\frac{\partial}{\partial u}, \qquad Z_v^{(\infty)} = \sum_{j=0}^{\infty}\left(D_x^j Q\right)\frac{\partial}{\partial u_{jx}}$$

Exercise: Consider the scaling transformation Γ_ε, namely

$$\hat{x} = e^{\varepsilon} x, \qquad \hat{u} = e^{2\varepsilon} u, \tag{5}$$

with Lie transformation group generator

$$Z = x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \tag{6}$$

Consider now the curve

$$u = x^3 \tag{7}$$

and map this curve by the transformation (5). Show then that the same curve can be obtained by mapping (7) with the associated vertical transformation $\Gamma_{\varepsilon,v}$, namely

$$\Gamma_{\varepsilon,v} : \begin{cases} \hat{x} = x \\ \\ \hat{u} = e^{\varepsilon Z_v^{(\infty)}} u\Big|_{u=x^3} . \end{cases} \tag{8a}$$

Note: The vertical Lie transformation generator is

$$Z_v = (2u - xu_x)\frac{\partial}{\partial u}$$

and the given curve maps to the curve $u = e^{-\varepsilon}x^3$ under Γ_ε and under $\Gamma_{\varepsilon,v}$.

Proplongations for $(x, u, u_x, u_{xx}, \dots u_{nx})$:

The nth prolongation of $\Gamma_{\varepsilon,v}$ in the coordinates (x, u) is

$$\Gamma^{(n)}_{\varepsilon,v} : \begin{cases} \hat{x} = x \\ \hat{u} = u + \varepsilon Q + O(\varepsilon^2) \equiv u + \varepsilon Z^{(n)}_v u + O(\varepsilon^2) \\ \hat{u}_{\hat{x}} = u_x + \varepsilon D_x Q + O(\varepsilon^2) \equiv u_x + \varepsilon Z^{(n)}_v u_x + O(\varepsilon^2) \\ \qquad \vdots \\ \hat{u}_{n\hat{x}} = u_{nx} + \varepsilon D^n_x Q + O(\varepsilon^2) \equiv u_{nx} + \varepsilon Z^{(n)}_v u_{nx} + O(\varepsilon^2), \end{cases}$$

The nth prolongation of Z_v is

$$\boxed{Z^{(n)}_v = Q\frac{\partial}{\partial u} + D_x Q\frac{\partial}{\partial u_x} + D^2_x Q\frac{\partial}{\partial u_{xx}} + \cdots + D^n_x Q\frac{\partial}{\partial u_{nx}}.}$$

Here Q is the characteristic

$$Q(x, u_x) = \eta(x, u) - \xi(x, u)u_x$$

for Γ_ε and Z the Lie transformation group generator

$$Z = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}.$$

Invariance condition for scalar ODEs:

An nth-order ODE in the variables $(x, u(x))$:

$$E(x, u, u_x, \dots, u_{nx}) = 0 \ . \tag{10}$$

A Lie point symmetry of equation (10) is a Lie point transformation group Γ_ε, such that the nth prolongation $\Gamma_\varepsilon^{(n)} : (x, u, u_x, \dots, u_{nx}) \mapsto (\hat{x}, \hat{u}, \hat{u}_{\hat{x}}, \dots, \hat{u}_{n\hat{x}})$ maps (10) to the same equation

$$E(\hat{x}, \hat{u}, \hat{u}_{\hat{x}}, \dots, \hat{u}_{n\hat{x}}) = 0. \tag{11}$$

Γ_ε is a Lie point symmetry for (10) IFF the corresponding vertical transformation $\Gamma_{\varepsilon,v}$ and its prolongations $\Gamma_{\varepsilon,v}^{(n)}$ keeps (10) invariant.

From

$$\frac{dI(x,u)}{d\varepsilon} = ZI(x,u) = 0$$

it follows that $E = 0$ **is invariant under** $\Gamma_\varepsilon^{(n)}$ **if and only if**

$$\frac{d}{d\varepsilon}E(x,u,u_x,\ldots,u_{nx}) = Z^{(n)}E(x,u,u_x,\ldots,u_{nx}) = 0$$

for all solutions u **of** $E = 0$ **and its prolongations** $D_xE = 0,\ D_x^2E = 0, \ldots D_x^kE = 0$**.**

Z **is a Lie point symmetry generator for** $E = 0$**, if and only if**

$$Z^{(n)}E\Big|_{D_x^kE=0} = 0, \qquad k \in \{0,1,2,\ldots\}. \tag{12}$$

This is the invariance condition for $E = 0$ **for a Lie point symmetry generator** Z**. Equivalently for** Z_v**, we have the invariance condition**

$$Z_v^{(n)}E\Big|_{D_x^kE=0} = 0, \qquad k \in \{0,1,2,\ldots\} \tag{13}$$

or

$$\boxed{L_E[u]Q\Big|_{D_x^kE=0} = 0, \qquad k \in \{0,1,2,\ldots\}} \tag{14}$$

$$L_E[u] := \frac{\partial E}{\partial u} + \frac{\partial E}{\partial u_x}D_x + \frac{\partial E}{\partial u_{2x}}D_x^2 + \cdots + \frac{\partial E}{\partial u_{nx}}D_x^n$$

$$Q = \eta(x,u) - \xi(x,u)u_x$$

Facts about the invariance condition for ODEs:

- **To calculate the Lie symmetry generators of any given ODE, $E = 0$, we need to find ξ and η that defines Z or Z_v from the invariance condition (12), (13), or (14) known as the determining equations. This is a set of linear PDEs in ξ and η.**

- **The set of determining equations are, for any $n > 1$ over-determined. This means that, for second- and higher-order ODEs, the conditions on ξ and η given by the invariance conditions are in principle not difficult to solve. For $n = 1$ the determining equations for ξ and η are under-determined.**

- **Every ξ and η that satisfy the invariance condition for a give ODE $E = 0$, results in a one paramenter Lie symmetry transformation group Γ_ε, as generated by the corresponding Lie symmetry generator Z.**

- **To find all r symmetry generators Z for a given ODE, we need to find the general solution of the determining equations. For ODEs of order $n > 1$, the set of all solutions for ξ and η is finite and the set of symmetry generators $\{Z_1, Z_2, \ldots, Z_r\}$ spans a vector space and a Lie algebra under the Lie bracket.**

- **Every Lie point symmetry Γ_ε for a given ODE $E = 0$ will map any solution $u = f(x)$ to another solution $u = g(x)$ for the same ODE. If the solution $u = f(x)$ is an invariant of Γ_ε, then Γ_ε will of course map to the same solution. Such solutions are known as symmetry-invariant solutions of the given ODE.**
  The symmetry-invariant solutions for the ODE $E = 0$ can be obtained by solving the compatibility problem

$$E(x, u, u_x, u_{xx}, \ldots, u_{nx}) = 0$$

$$\eta(x, u) - \xi(x, u)u_x = 0,$$

  where $\eta(x, u) - \xi(x, u)u_x = 0$ is the invariant surface condition of the corresponding Lie point symmetry of the given ODE.

- **Regarding the number of Lie point symmetries that a scalar ordinary differential equation can admit, the following is known:**

  - **Any first-order ordinary differential equation admits, in general, an infinite number of Lie point symmetries. However, the problem to find a Lie point symmetry for a first-order ordinary differential equation is equivalent to the problem of finding an integrating factor for the equation by which the equation becomes exact. The problem is that the determining equations are under-determined!**

  - **A second-order ordinary differential equation may admit 0, 1, 2, 3 or 8 Lie point symmetries** [cite: **Sophus Lie, Differentialgleichungen**, Chelsea scientific books, New York, 1967]

  - **An nth-order ordinary differential equation, with $\mathbf{n \geq 3}$ may admit $\mathbf{n+1}$, $\mathbf{n+2}$ or $\mathbf{n+4}$ Lie point symmetries.**

Exercise: Consider the following second-order nonlinear differential equation:

$$u_{xx} + \alpha u u_x + \frac{\alpha^2}{9} u^3 = 0, \qquad \alpha \in \mathbb{R} \tag{16}$$

where α is an arbitrary real constants. Find all Lie point symmetry generators for this equation.

Note: The equations admits 8 Lie point symmetries for arbitrary α. The first three are

$$Z_1 = \frac{\partial}{\partial x}, \qquad Z_2 = u\frac{\partial}{\partial x} - \frac{\alpha}{3}u^3\frac{\partial}{\partial u}$$

$$Z_3 = xu\frac{\partial}{\partial x} + \left(u^2 - \frac{\alpha}{3}xu^3\right)\frac{\partial}{\partial u}, \ \ldots, \ Z_8$$

Invariance condition for systems of PDEs in p independent variables:

System of nth-order PDEs consisting of r equations in p independent variables and q dependent variables:

$$E_\mu(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(n)}) = 0, \qquad \mu = 1, 2, \ldots, r,$$

admits a Lie point symmetry generator

$$Z = \sum_{i=1}^{p} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{j=1}^{q} \eta_j(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_j}.$$

if and only if

$$Z^{(n)} E_\nu \Big|_{E_\mu = 0} = 0, \qquad \mu,\ \nu = 1, 2, \ldots, r, \tag{17}$$

or alternatively, for the vertical Lie symmetry generator

$$Z_v = \sum_{j=1}^{q} Q_j \frac{\partial}{\partial u_j},$$

we have

$$Z_v^{(n)} E_\nu \Big|_{E_\mu = 0} = 0, \qquad \mu,\ \nu = 1, 2, \ldots, r. \tag{18}$$

where

$$Q_j := \eta_j(\mathbf{x}, \mathbf{u}) - \sum_{i=1}^{p} \xi_i(\mathbf{x}, \mathbf{u}) u_{j,i}.$$

The invariance condition (18) can equivalently be written in terms of linear operators:

$$\left.\begin{pmatrix} L_{E_1}[u_1] & L_{E_1}[u_2] & \cdots & L_{E_1}[u_q] \\ L_{E_2}[u_1] & L_{E_2}[u_2] & \cdots & L_{E_2}[u_q] \\ \vdots & \vdots & \vdots & \vdots \\ L_{E_r}[u_1] & L_{E_r}[u_2] & \cdots & L_{E_r}[u_q] \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_q \end{pmatrix}\right|_{D^{(k)}E_\mu=0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $\mu = 1, 2, \ldots, r$ **and** $L_{E_k}[u_j]$ **are linear operators defined by**

$$L_{E_k}[u_j] := \frac{\partial E_k}{\partial u_j} + \sum_{i=1}^{p} \frac{\partial E_k}{\partial u_{j,i}} D_i + \sum_{\{i_1 i_2\}=1}^{p} \frac{\partial E_k}{\partial u_{j,i_1 i_2}} D_{i_1 i_2} + \cdots$$

$$+ \sum_{\{i_1 i_2 \cdots i_n\}=1}^{p} \frac{\partial E_k}{\partial u_{j,i_1 i_2 \cdots i_n}} D_{i_1 i_2 \cdots i_n} ,$$

and

$$Q_j := \eta_j(\mathbf{x}, \mathbf{u}) - \sum_{i=1}^{p} \xi_i(\mathbf{x}, \mathbf{u}) u_{j,i}.$$

The vertical Lie symmetry generator:

$$Z_v = \sum_{j=1}^{q} Q_j \frac{\partial}{\partial u_j},$$

and the Lie point symmetry generator

$$Z = \sum_{i=1}^{p} \xi_i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{j=1}^{q} \eta_j(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_j}.$$

Exercise: Consider the following system of PDEs:

$$\frac{\partial u_1}{\partial x_1} = \frac{1}{(u_1+k)^2}\ \frac{\partial u_2}{\partial x_2}$$

$$\frac{\partial u_2}{\partial x_1} = u_1\ ,$$

where k is an arbitrary constants. Find all Lie point symmetry generators for this system.

Note: The general solution of the determining equations gives the following five Lie point symmetry generators:

$$Z_1 = \frac{\partial}{\partial x_1}, \qquad Z_2 = \frac{\partial}{\partial x_2}, \qquad Z_3 = \frac{\partial}{\partial u_2}$$

$$Z_4 = x_1\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2} + u_2\frac{\partial}{\partial u_2}$$

$$Z_5 = x_1\frac{\partial}{\partial x_1} + (u_1+k)\frac{\partial}{\partial u_1} + (2u_2 + kx_1)\,\frac{\partial}{\partial u_2}.$$

Example: The Navier-Stokes equation for an incompressible viscous fluid in Cartesian coordinates, with velocity field (u, v, w) and pressure p, is given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial p}{\partial y} - \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 0$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} - \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

This system admits infinitely-many Lie point symmetry generators.

The Lie point symmetry generators for the Navier-Stockes equation, where f, g_1, g_2 and g_3 are arbitrary functions:

$$
\begin{aligned}
Z_1 &= \frac{\partial}{\partial t} \\
Z_2 &= f(t)\frac{\partial}{\partial p} \\
Z_3 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} - 2p\frac{\partial}{\partial p} \\
Z_4 &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u} \\
Z_5 &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} + v\frac{\partial}{\partial w} - w\frac{\partial}{\partial v} \\
Z_6 &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w} \\
Z_7 &= g_1(t)\frac{\partial}{\partial x} + g_1'(t)\frac{\partial}{\partial u} - xg_1''(t)\frac{\partial}{\partial p} \\
Z_8 &= g_2(t)\frac{\partial}{\partial y} + g_2'(t)\frac{\partial}{\partial v} - yg_2''(t)\frac{\partial}{\partial p} \\
Z_9 &= g_3(t)\frac{\partial}{\partial z} + g_3'(t)\frac{\partial}{\partial w} - zg_3''(t)\frac{\partial}{\partial p}.
\end{aligned}
$$

Possible future lectures on Symmetry Analysis:

- **Lie point symmetry algebras**: The Lie algebra structure of Lie point symmetries.
- **Applications of Lie point symmetries for ODEs and PDEs**: Solutions, order reductions, dimension reductions, etc
- **Extension of Lie point symmetries to Lie-Bäcklund symmetries**: Conservation laws, Bäcklund transformations, integrability, etc
- **Classification of integrable ODEs and PDEs**: Recursion operators, hierarchies, potentialisation, etc
- **Our current research**:

  – Multipotentialisatons vs higher-order invariance.

  – Nonlocal Symmetry Generators:

$$Z = \Psi\left(\int f(x)u\,dx\right)\frac{\partial}{\partial u}$$

Some references on symmetry analysis:

- **Sophus Lie, Theorie der Transformationsgruppen,** B.G. Teubner, 1888, 1890, 1893.
- **Sophus Lie, Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen,** B.G. Teubner, 1891.
- **Sophus Lie, Differentialgleichungen,** Chelsea Scientific Books, New York, 1967
- P.E. Hydon, **Symmetry Methods for Differential Equations. A Beginner's Guide,** Cambridge University Press, 2000.
- G.W. Bluman and S. Kumei, **Symmetries and Differential Equations,** Springer, 1989.
- P.J. Olver, **Applications of Lie Groups to Differential Equations,** Springer, 1986.
- N. Euler and W.-H. Steeb, **Continuous Symmetries, Lie Algebras and Differential Equations,** B.I. Wissenschaftsverlag, 1992.
- M. Euler and N. Euler, **Symmetry Analysis of Differential Equations: Problems, Theory and Solutions. Part 1. Lie Point Symmetries,** (book in preparation), 2017.

Appendix:
On the Painlevé Analysis

Painlevé Analysis in a Nutshell:

ODEs: **Find a closed form expression for its general solution.**

Linear ODEs: **All ODEs define functions since their general solutions have no movable singularities (which depends on the constants of integration).**

Nonlinear ODEs: **Sigularities of their solutions have the following properties:**
Critical **(local multivaluedness) or** **Noncritical** **(singlevaluedness);** **Movable** **or** **Fixed.**

Painlevé Property: **Its general solution has no movable critical singulatities** (the singularities should be nothing worse than poles).

Necessary condition: **For nth-order ODE (or system of n first order ODEs): The Laurent expansion**

$$u(t) = (z - z_0)^m \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

satisfies the eq. with $n - 1$ arbitrary expansion coefficients and the pole position z_0 arbitrary.
(known as the Painlevé Test)

Transcendental Functions by Nonlinear ODEs

First-Order ODEs: **Only the elliptic functions, e.g.:**

$$\left(\frac{du}{dx}\right)^2 = (1-u^2)(1-ku^2)$$

Second-Order ODEs **(Painlevé, Gambier, Fuchs):**
The six Painlevé transcendents (soliton theory!).

I: $u'' = 6u^2 + x$

II: $u'' = 2u^3 + xu + \alpha, \qquad \cdots\ VI:$

Third-Order ODEs: **No transcendental functions.**

Fourth-Order ODEs **(Christopher Cosgrove):**
Irreducibility is still unproven.

Higher-Order ODEs: **Open problem.**

Painlevé Test for Nonlinear PDEs

$$F(x, t, u, u_t, u_x, u_{xx}, \ldots, u_{nx}) = 0$$

The solutions of the nth order PDE should be single-valued about the movable singularity manifolds[1] $\phi(x, t) = 0$

Painlevé Expansion for the solution:

$$u(x, t) = \phi^m(x, t) \sum_{j=0}^{\infty} u_j(x, t)\, \phi^j(x, t)$$

where ϕ and $n - 1$ coefficient functions u_j must be arbitrary analytic functions in the neighbourhood of the sigularity manifold (necessary condition).

If the expansion is consistent, we truncate the series to prove the sufficient condition.

[1] For analytic functions of several complex variables the singularities cannot be isolated!

Example of the Painleve Analysis for Burgers' Equation:

$$u_t + uu_x = u_{xx}$$

Applying the Painlevé expansion

$$u(x,t) = \phi^{-1} \sum_{j=0}^{\infty} u_j \phi^j$$

we obtain

$$\begin{aligned} j = 0: &\qquad u_0 = -2\phi_x \\ j = 1: &\qquad \phi_t + u_1\phi_x - \phi_{xx} = 0 \\ j = 2: &\qquad \frac{\partial}{\partial x}\left(\phi_t + u_1\phi_x - \phi_{xx}\right) = 0 \cdot u_2 \end{aligned}$$

Expansion consistent at $m = -1$ and $j = 2$.
Trancated Painlevé expansion ($u_j = 0,\ j = 2, 3, \ldots$):

$$u(x,t) = u_0\phi^{-1} + u_1 = -2\phi_x\phi^{-1} + u_1$$

we obtain

$$u_{1t} + u_1u_{1x} = u_{1xx}, \qquad \phi_t + u_1\phi_x = \phi_{xx}.$$

For $u_1 = 0$: the Cole-Hopf transformation

$$u = -2\phi^{-1}\phi, \qquad \phi_t = \phi_{xx}.$$

Some references on the Painlevé analysis:

- **Paul Painlevé, Mémoire sur les équations différentielles du premier ordre,** Gauthier-Villars et fils, 1892 **(222 pages)**
- R. Conte (ed)**, The Painlevé Property: One Century Later,** Springer, 1999.
- W.-H. Steeb and N. Euler**, Nonlinear Evolution Equations and Painlevé Test,** World Scientific, 1988.