Conservation Laws as Consequences of Fundamental Properties of Space and Time. Noether's Theorem

Elena Miroshnikova

Luleå University of Technology,
Sweden

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Content

- Lagrangian systems
- Noether's theorem
- Symmetry breaking in quantum physics

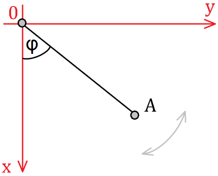
Generalized coordinates

Let s be a number of degrees of freedom for some mechanical system A

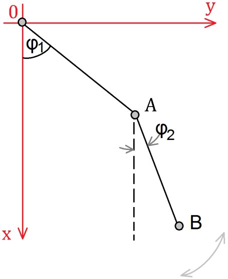
$$s = dN - r,$$

where d — dimension of the space, N — number of particles, r — number of constrains.

Any s independent parameters $q_1, \ldots, q_s$ which characterize the location of the system A in the space are called *generalized coordinates* of A.



a) $s = 2 \times 1 - 1 = 1$



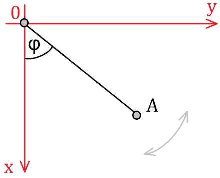
b) $s = 2 \times 2 - 2 = 2$

Generalized coordinates and velocities

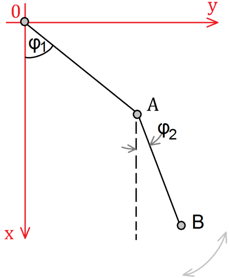
However, the set $\{q_i\}$ does not fully define the mechanical state of the system.
Let $\dot{q}_i = \frac{dq_i}{dt}$, $i = 1, ..., s$.

The time-derivatives $\dot{q}_1, ..., \dot{q}_s$ of generalized coordinates $q_1, ..., q_s$ are called *generalized velocities* of A.

Together $\{q_i\}$ and $\{\dot{q}_i\}$ provide a full description of A (allow to predict it's state in time $t_0 + \Delta t$)



a) $\{\varphi, \dot{\varphi}\}$



b) $\{\varphi_1, \varphi_2, \dot{\varphi}_1, \dot{\varphi}_2\}$

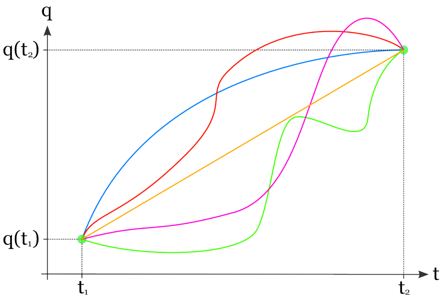
Principle of least (stationary) action

Every mechanical system A is characterized by certain function

$$L = L(q, \dot{q}, t) = L(q_1, ..., q_s, \dot{q}_1, ..., \dot{q}_s, t),$$

and the true evolution $q(t)$ of A between two specified states $q(t_1) = q^{(1)}$ and $q(t_2) = q^{(2)}$ at two specified times t_1 and t_2 is a stationary point of the action functional

$$S = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt.$$



Principle of least action

The true evolution $q(t)$ of A between two specified states $q(t_1)$ and $q(t_2)$ is a stationary point of the action functional

$$S = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt.$$

That means

$$\delta S = \delta \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt = 0.$$

Then L is called Lagrangian of A and S – action of A. The requirement $\delta S = 0$ leads to the Lagrange (Euler–Lagrange) equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

Properties of Lagrangian

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

1 If $A = A_1 + A_2$, then

$$\lim_{\mathrm{dist}(A_1, A_2) \to \infty} L_A = L_{A_1} + L_{A_2}$$

(additivity).

2 L is defined up to factor C ($L \to CL$ does not change the equation).

1+2 All Lagrangians are defined up to the same constant C.

3 L is defined up to total derivative of any "good" function $F = F(q, t)$:

$$L = L \pm \frac{d}{dt} F(q, t).$$

Examples (one particle in potential V)

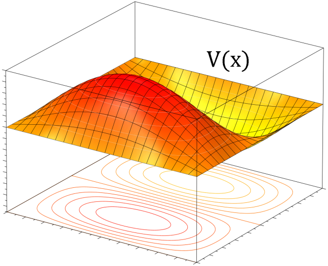
Lagrange equation: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

- One particle moving in some potential $V = V(x)$

$$L(x, \dot{x}, t) = \frac{m\dot{x}^2}{2} - V(x),$$

equations of motion

$$m\ddot{x} = -\frac{\partial V}{\partial x}.$$



Examples (single pendulum)

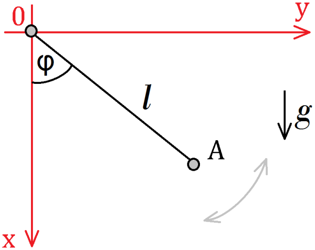
Lagrange equation: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

- Simple pendulum

$$L(\varphi, \dot{\varphi}, t) = \frac{1}{2} ml\dot{\varphi}^2 - mg(1 - \cos\varphi),$$

equations of motion

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0.$$



Lagrangian in field theory

$$\text{Lagrange equation:} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

$N, s \to \infty$, $\{x_i\} \to \varphi(x)$, $\{\dot{x}_i\} \to \partial^\mu \varphi(x)$, $d\boldsymbol{x} = dx\, dy\, dz\, dt$,

$$S = \int \mathfrak{L}(\varphi(\boldsymbol{x}), \partial^\mu \varphi(\boldsymbol{x})) d\boldsymbol{x}, \qquad \partial^\nu \frac{\partial \mathfrak{L}(\varphi, \partial^\mu \varphi)}{\partial \partial^\mu \varphi} - \frac{\partial \mathfrak{L}(\varphi, \partial^\mu \varphi)}{\partial \varphi} = 0.$$

- Electromagnetic field

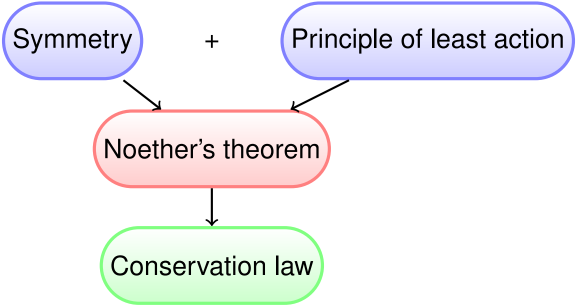
$$\mathfrak{L} = \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu},$$

$$F_{\mu\nu}^{(\mu\nu)} = \begin{pmatrix} 0 & (-)E_x & (-)E_y & (-)E_z \\ (+)-E_x & 0 & -B_z & B_y \\ (+)-E_y & B_z & 0 & -B_x \\ (+)-E_z & -B_y & B_x & 0 \end{pmatrix}$$

Noether's theorem

Noether's theorem

There is a conserved quantity associated with every symmetry of the Lagrangian of a system.



“Proof” $\star$

Consider transformation of the type $q_i \to q_i' = q_i + \delta q_i$. Then

$$\delta L = \frac{dF(q, \dot{q}, t)}{dt} \quad \text{also} \quad \delta L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right).$$

By writing δq as an infinitesimal variation of the form

$$q_i' = q_i + \delta q_i = q_i + \varepsilon f_i, \qquad \varepsilon \ll 1,$$

we get

$$\lim_{\varepsilon \to 0} q_i' = q_i \Rightarrow \lim_{\varepsilon \to 0} \delta L = 0 \Rightarrow F = \varepsilon \tilde{F}$$

and finally

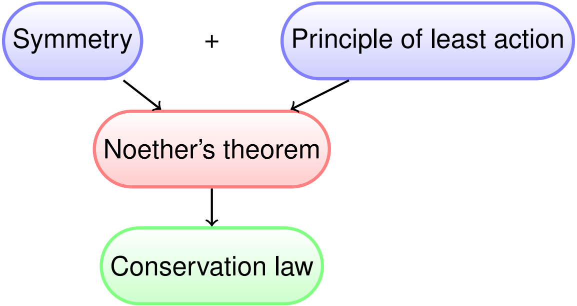
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) = \varepsilon \frac{d\tilde{F}}{dt},$$

$$C = \frac{\partial L}{\partial \dot{q}_i} f_i - \tilde{F}.$$

Noether's theorem

Noether's theorem (Emmy Noether, 1918)

If the integral I is invariant under a [group] $\mathfrak{G}_\rho$, then there are ρ linearly independent combinations among the Lagrangian expressions which become divergences and conversely, this implies the invariance of I under a [group] $\mathfrak{G}_\rho$. The theorem remains valid in the limiting case of an infinite number of parameters.



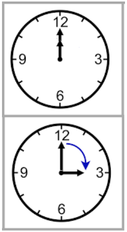
[1]

[1] *Noether E.*, "Invariante Variationsprobleme," Gott. Nachr., **1918**, 235–257 (1918)

Time translations $C = \frac{\partial L}{\partial \dot{q}_i} f_i - \tilde{F}$

- Infinitesimal time shift $t \to t + \varepsilon$.

The first order Taylor expansions give us



$$\delta q_i = q_i(t+\varepsilon) - q_i(t) = \varepsilon \dot{q}_i \quad \left(+O(\varepsilon^2)\right),$$

$$\delta \dot{q}_i = \dot{q}_i(t+\varepsilon) - \dot{q}_i(t) = \varepsilon \ddot{q}_i = \frac{d}{dt}(\delta q_i) \quad \left(+O(\varepsilon^2)\right).$$

If $\frac{\partial L}{\partial t} = 0$ then

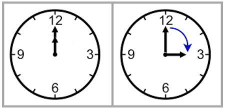
$$\delta L = \varepsilon \frac{\partial L}{\partial q_i} \delta \dot{q}_i + \varepsilon \frac{\partial L}{\partial \dot{q}_i} \delta \ddot{q}_i = \varepsilon \frac{dL}{dt} \Rightarrow \tilde{F} = L$$

$$\Downarrow$$

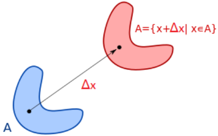
$$C = \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = E = \text{const} \qquad \text{(Conservation of energy)}$$

Symmetry –> conservation law

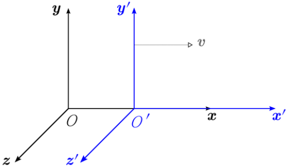
- Time invariance — Conservation of total energy E
- Invariance under translations in space — Conservation of total momentum $\vec{P}$
- Rotation invariance — Conservation of total angular momentum $\vec{L}$
- Invariance under Galilean transformation — Conservation of union motion of the center of mass $\vec{P}t - M\vec{R}$



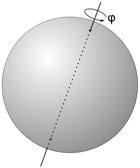
$E = \text{const}$



$\vec{P} = \text{const}$



$\vec{P}t - M\vec{R} = \text{const}$



$\vec{L} = \text{const}$

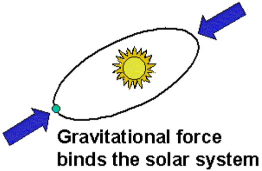


Figure: Gravitation

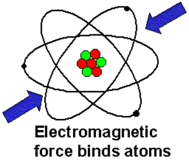


Figure: Electromagnetism

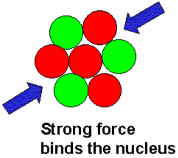


Figure: Strong interaction

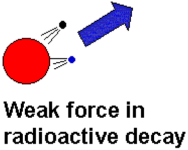
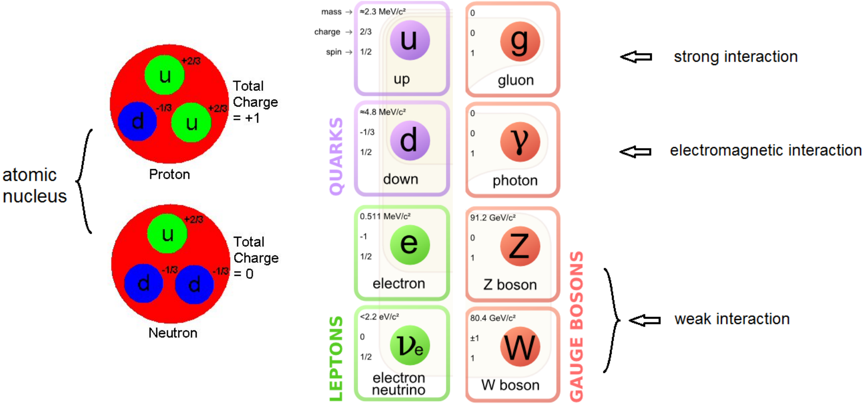


Figure: Weak interaction

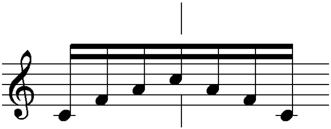
Standard Model



Symmetries in quantum physics. T-symmetry

Time inverse

$$t \to -t$$

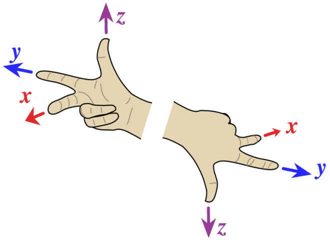


Corresponding constant — parity

Even	Odd
$\vec{x}$ — position of a particle	t – time when an event occurs
$\vec{a}$ — acceleration of a particle	$\vec{v}$ — velocity of a particle
$\vec{f}$ — force on a particle	$\vec{p}$ — linear momentum of a particle
E — energy of a particle	$\vec{l}$ — angular momentum of a particle

Coordinate inverse

$$\vec{x} \to -\vec{x}$$

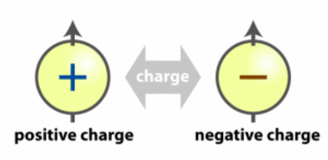


Corresponding constant — “p”–parity

- Gravitation $+$
- Electromagnetism $+$
- Strong interaction $+$
- Weak interaction $-$

Charge conjugation transformations

$$q \to -q$$



Corresponding constant — “c”–parity

- Gravitation $+$
- Electromagnetism $+$
- Strong interaction $+$
- Weak interaction $-$

Symmetries in quantum physics

- T-symmetry $t \to -t$
- P-symmetry $\vec{x} \to -\vec{x}$
- C-symmetry $q \to -q$

- Gravitation T, P, C
- Electromagnetism T, P, C
- Strong interaction T, P, C
- Weak interaction CPT

Lüders-Pauli theorem

$\mathfrak{L}_{\text{SM}} = \mathfrak{L}_{\text{Quarks}} + \mathfrak{L}_{\text{Leptons}} + \mathfrak{L}_{\text{Gauge}}$ is CPT–symmetric.

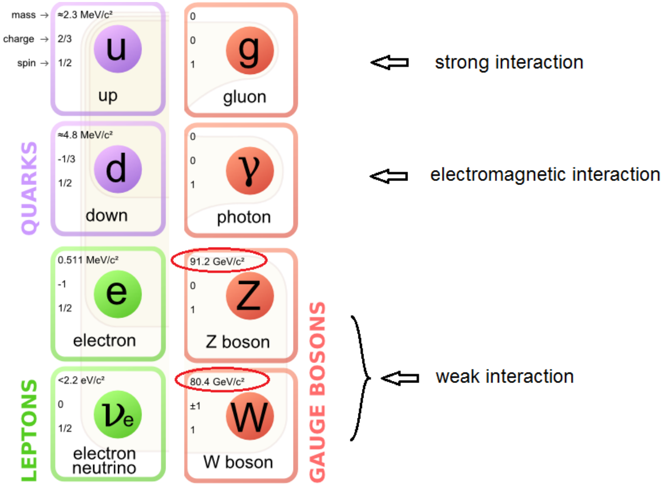
Any Lorentz invariant local quantum field theory with a self-adjoint Hamiltonian must have CPT-symmetry.

[2] [3]

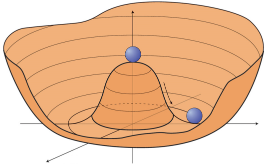
[2] *Lüders, G.,* "On the Equivalence of Invariance under Time Reversal and under Particle-Antiparticle Conjugation for Relativistic Field Theories," Kongelige Danske Videnskabernes Selskab, Matematisk-Fysiske Meddelelser, **28**(5), 1–17 (1954).

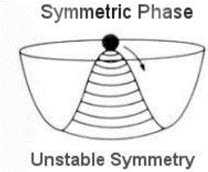
[3] *Pauli W.,* Niels Bohr and the development of physics; essays dedicated to Niels Bohr on the occasion of his seventieth birthday (Ed. by Pauli W., with the assistance of Rosenfeld L. and Weisskopf V.), London, Pergamon Press, 1955.

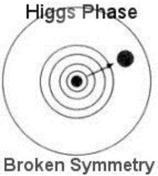
Standard Model



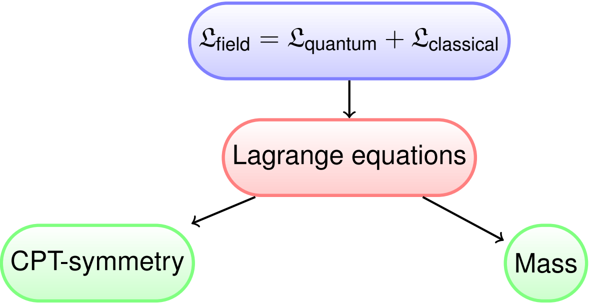
Spontaneous symmetry breaking



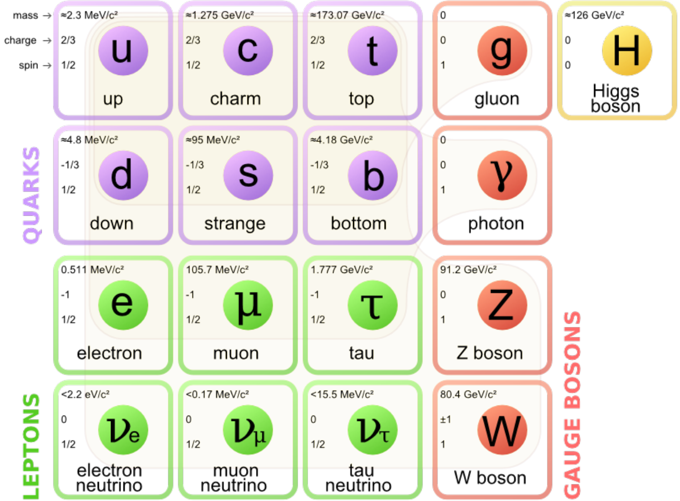




By this we define $\langle \varphi \rangle$ — average value of classical field.



Standard Model



Thank you for your attention!