Functional analysis and continuum mechanics

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Presentation

CV:

- Lektor, LTU, 2016
- Bitr. lektor, LTU, 2012
- Tekn. dr. i matematik, LTU, 2011
- Civ. ing. i teknisk fysik, Uppsala universitet, 2006
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Research profile: partial differential equations, homogenization theory, lubrication theory, fluid mechanics

What is a Hilbert space?

An abstract concept with many concrete examples.

- H is a linear space over the real (or complex) numbers.
- The elements of H are called “vectors”.
- The elements of $\mathbb{R}$ (or $\mathbb{C}$) are called scalars.
- H is equipped with a scalar product denoted as (u, v).
- The scalar product induces a norm $\|u\| = \sqrt{(u, u)}$ that makes H *complete*.

For a complete axiomatic definition, see any textbook on functional analysis.

Example 1

The n-dimensional Euclidian space $\mathbb{R}^n$ consists of all vectors

$$x = (x_1, x_2, \ldots, x_n) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

$\mathbb{R}^n$ is a Hilbert space for the scalar product

$$(x, y) = x^T y = x \cdot y = \sum_{i=1}^{n} x_i \, y_i.$$

The corresponding norm is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example 2

The space $L^2(a, b)$ of real-valued functions on the interval (a, b) such that

$$\int_a^b |f(x)|^2 \, dx < \infty. \tag{1}$$

is a Hilbert space for the scalar product

$$(f, g) = \int_a^b f(x) \, g(x) \, dx.$$

The corresponding norm is

$$\|f\| = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2}.$$

Calculus notation

The boundary value problems of physics are usually posed in a bounded domain in Euclidian space $\mathbb{R}^n$ ($n = 1, 2$ or 3).

Notation:

Ω	(bounded domain)
$\partial\Omega$	(boundary of Ω)
$\Gamma_N \cup \Gamma_D = \partial\Omega$	(partition of boundary)
$\hat{n}$	(outward unit normal)
$x = (x_1, \ldots, x_n)$	(a point in space)
t	(time variable)
$D_i f = \dfrac{\partial f}{\partial x_i}$	(partial derivative)
$\nabla f = (D_1 f, \ldots, D_n f)$	(gradient)
$\int_\Omega f \, dx$	("volume" integral)
$\int_{\partial\Omega} f \, dS$	("surface" integral)

Heat conduction

Let Ω be the region in space occupied by a conducting body (e.g. a solid cylinder).

u	(temperature distribution)
Φ	(heat flow vector)
ρ	(density)
c	(specific heat capacity)
k	(thermal conductivity)
f	(heat source)
g	(heat flux)

Φ and u are related through

$$\Phi = -k \nabla u \quad \text{(Fourier's law).}$$

For simplicity we assume that ρ, c and k are given constants.

Conservation of energy

The internal energy of the heated body is

$$E_{\text{int}} = \int_\Omega c\rho\, u\, dx.$$

For any subdomain V of Ω we have

$$\frac{d}{dt} E_{\text{int}} = \int_V f\, dx - \int_{\partial V} \Phi \cdot \hat{n}\, dS.$$

Differentiating under the integral sign and using the divergence theorem give

$$\frac{\partial}{\partial t}(c\rho\, u) + \nabla \cdot \Phi = f \quad \text{in } \Omega$$

Boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}(c\rho\, u) + \nabla \cdot \Phi = f & \text{in } \Omega \quad \text{(2a)} \\ \Phi \cdot \hat{n} = g & \text{on } \Gamma_N \quad \text{(2b)} \\ u = 0 & \text{on } \Gamma_D \quad \text{(2c)} \\ u\big|_{t=0} = u_0 & \text{in } \Omega \quad \text{(2d)} \end{cases} \tag{2}$$

Observe that (2b–c) is a so called mixed boundary condition.

Boundary value problem

Alternative formulation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \kappa \Delta u = \tilde{f} & \text{in } \Omega \quad \text{(3a)} \\ \dfrac{\partial u}{\partial \hat{n}} = \tilde{g} & \text{on } \Gamma_N \quad \text{(3b)} \\ u = 0 & \text{on } \Gamma_D \quad \text{(3c)} \\ u|_{t=0} = u_0 & \text{in } \Omega \quad \text{(3c)} \end{cases}, \qquad (3)$$

where

$$\kappa = \frac{k}{c\rho}, \quad \tilde{f} = \frac{1}{c\rho} f, \quad \tilde{g} = -\frac{1}{k} g.$$

and

$$\Delta = D_1^2 + \cdots + D_n^2 \quad \text{(Laplacian)}.$$

Energy equality

Let the total "kinetic" energy in Ω be defined as

$$E_{\text{kin}} = \int_{\Omega} \frac{1}{2} c\rho\, u^2 dx.$$

Differentiating with respect to t gives

$$\begin{aligned} \frac{d}{dt} E_{\text{kin}} &= \int_{\Omega} c\rho \, \frac{\partial u}{\partial t} \, u \, dx \\ &= \cdots \quad (\text{using (3)}) \\ &= -\int_{\Omega} k \, |\nabla u|^2 \, dx + \int_{\Omega} f u \, dx + \int_{\Gamma_N} g u \, dS \end{aligned}$$

Physical restrictions

If we impose

$$E_{\text{kin}} < \infty \quad \text{and} \quad \frac{d}{dt} E_{\text{kin}} < \infty$$

we must have

$$\int_\Omega |u|^2 \, dx < \infty \quad \text{and} \quad \int_\Omega |\nabla u|^2 \, dx < \infty.$$

This corresponds to the Hilbert space $H^1(\Omega)$.

Function spaces

The space of finite “kinetic” energy is the Hilbert space

$$H = L^2(\Omega).$$

The space of finite “kinetic” energy dissipation is

$$V = \left\{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \right\}.$$

We shall seek our solution u in the space V.

$$\|u\|_H = \left(\int_\Omega |u|^2 \, dx \right)^{1/2}, \quad \|u\|_V = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2}.$$

Weak formulation

From the identity

$$\int_{\partial\Omega} v\,\Phi\cdot\hat{n}\,dS = \int_{\Omega} \Phi\cdot\nabla v + (\nabla\cdot\Phi)v\,dx.$$

we deduce the so called *weak formulation*

$$\int_{\Omega} c\rho\,\frac{\partial u}{\partial t} v + k\nabla u\cdot\nabla v\,dx = \int_{\Omega} f\,v\,dx + \int_{\Gamma_N} g\,v\,dS \qquad (4)$$

for any “test function” v in V. Complemented with the initial condition

$$u|_{t=0} = u_0.$$

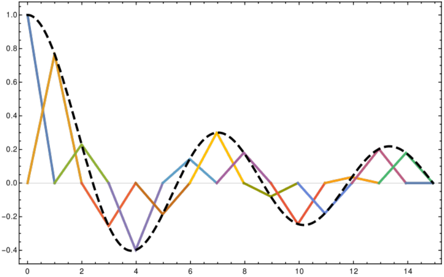
The Finite Element Method

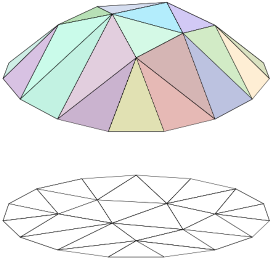
Three steps:

1. Rephrase the BVP in a weak form.
2. Discretize the weak formulation in a finite dimensional space.
3. Solve the resulting ODE (or system of algebraic equations).

There are many possible discretizations. Which is the most clever one? The resulting ODE should be simple!

Discretization





A simple way is to divide the domain Ω into a mesh of *polyhedra* and use *piecewise polynomial* functions as basis functions.

Spectral decomposition

Recall the following result from linear algebra.

Theorem (Spectral theorem)

Let A be a real symmetric $n \times n$ matrix. Then there exists an orthonormal basis consisting of eigenvectors of A. Each eigenvalue of A is real.

$$Q^T A Q = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad Q^T Q = I.$$

This result generalizes to compact symmetric operators on a Hilbert space. In particular it can be applied to the operator

$$-\Delta^{-1} \colon H \to V.$$

Eigenfunctions of the Laplacian (mixed BC)

According to the spectral theorem there exists a sequence of positive real numbers $\Sigma = (\lambda_m)_{m=1}^{\infty}$ such that for any $\lambda \in \Sigma$, the BVP

$$\begin{cases} -\Delta\psi = \lambda\,\psi & \text{in } \Omega \quad \text{(5a)} \\ \dfrac{\partial\psi}{\partial\hat{n}} = 0 & \text{on } \Gamma_N \quad \text{(5b)} \\ \psi = 0 & \text{on } \Gamma_D \quad \text{(5c).} \end{cases} \tag{5}$$

has a non-trivial solution ψ in V. Any such function ψ is called an eigenfunction of $-\Delta$.

Spectral decomposition

Assume

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots .$$

Then

1. The eigenvalues tend to infinity:

$$\lim_{m\to\infty} \lambda_m = \infty.$$

2. Each eigenspace has finite dimension.
3. There exists a sequence of eigenfunctions $(\psi_m)_{m=1}^{\infty}$ that forms an orthonormal basis in H.
4. The sequence $(\lambda_m^{-1/2}\psi_m)_{m=1}^{\infty}$ is an orthonormal basis in V.

Orthogonality

Orthogonality in H:

$$(\psi_m, \psi_k) = \int_\Omega \psi_m \, \psi_k \, dx = \begin{cases} 1 & (m = k) \\ 0 & (m \neq k), \end{cases}$$

Orthogonality in V:

$$((\psi_m, \psi_k)) = \int_\Omega \nabla \psi_m \cdot \nabla \psi_k \, dx = \begin{cases} \lambda_m & (m = k) \\ 0 & (m \neq k), \end{cases}$$

In other words, both "mass" and "stiffness" matrices are diagonal.

Completeness

For any v in H (or V) the Fourier series

$$\sum_{m=1}^{\infty} (v, \psi_m)\psi_m \tag{6}$$

converges to v in H (or V). The numbers

$$c_m = (v, \psi_m) \quad (m \geq 1)$$

are the "Fourier coefficients" of v. Moreover

$$\|v\|_H = \left(\sum_{m=1}^{\infty} c_m^2\right)^{1/2}, \quad \|v\|_V = \left(\sum_{m=1}^{\infty} \lambda_m c_m^2\right)^{1/2}$$

Riemann–Lebesgue

The Fourier coefficients tend to zero:

$$\lim_{m\to\infty} c_m = 0 \quad (v \in H)$$

$$\lim_{m\to\infty} \sqrt{\lambda_m} c_m = 0 \quad (v \in V)$$

Moral: The faster they tend to zero, the more regular is the function.

Variational principle for the smallest eigenvalue

We have

$$\lambda_1 = \min_{v \neq 0} \frac{\int_\Omega |\nabla v|^2 \, dx}{\int_\Omega |v|^2 \, dx}.$$

This implies the following lower bound for the energy dissipation

$$\int_\Omega |\nabla v|^2 \, dx \geq \lambda_1 \int_\Omega |v|^2 \, dx \quad \text{(Friedrichs' inequality)}$$

valid for all v in V.

Fourier–Ritz–Galerkin approximation

Let us now "solve" the BVP (2).

1. Let $(\psi_m)_{m=1}^{\infty}$ be a basis of eigenfunctions of $-\Delta$.
2. We seek an *approximate solution* u_N of the form

$$u_N(x,t) = \sum_{m=1}^{N} c_m(t)\,\psi_m(x). \tag{7}$$

This means that (for fixed t) u_N belongs to the N-dimensional subspace

$$V_N = \mathrm{Span}\{\psi_1, \ldots, \psi_N\} \subset V.$$

Discretized weak formulation

Inserting (7) into the weak formulation (4) gives

$$\sum_i \int_\Omega c\rho \frac{dc_i}{dt}\psi_i\, v + c_i\, k\, \nabla\psi_i \cdot \nabla v\, dx = \int_\Omega f v\, dx + \int_{\Gamma_N} gv\, dS$$

for all v in V_N. For brevity we write the right-hand side as

$$\langle F(t), v\rangle = \int_\Omega f(x,t)v(x)\, dx + \int_{\Gamma_N} g(x,t)v(x)\, dS$$

and consider $F(t)$ as an element in V', i.e. a bounded linear functional.

ODE

Taking $v = \psi_m$ gives the uncoupled linear system

$$c\rho \frac{dc_m}{dt} + k\lambda_m c_m = F_m \quad (1 \leq m \leq N), \tag{8}$$

where

$$F_m(t) = \langle F(t), \psi_m \rangle.$$

This system can be easily integrated

$$c_m(t) = e^{-\kappa\lambda_m t} \left(c_m(0) + \frac{1}{c\rho} \int_0^t e^{\kappa\lambda_m s} F_m(s)\, ds \right), \tag{9}$$

where $\kappa = k/(c\rho)$ and $c_m(0)$ are the Fourier coefficients of u_0.

Summary of the method

1. We derive the weak formulation of our BVP.
2. We discretize using a finite number of the eigenfunctions $(\psi_m)_{m=1}^N$ that "diagonalize" the operator $-\Delta$ in a finite dimensional space.
3. We find an approximate solution

$$u_N(x,t) = \sum_{m=1}^{N} c_m(t)\,\psi_m(x)$$

by solving an uncoupled linear ODE (easy!).

Does the approximate solution u_N converge to an exact solution u?

Energy equality

Each approximate solution satisfies an energy equality:

$$\frac{d}{dt}\int_\Omega \frac{1}{2} c\rho\, u_N^2 dx + \int_\Omega k\, |\nabla u_N|^2 \; dx = \int_\Omega f u_N \, dx + \int_{\Gamma_N} g u_N \, dS.$$

From this identity we can derive various estimates.

Energy estimates

For any $T > 0$ we have

$$\|u_N(T)\|_H^2 \leq e^{-\kappa\lambda_1 T}\|u_0\|_H^2 + \frac{1}{c\rho k}\int_0^T e^{\kappa\lambda_1(t-T)}\|F\|_{V'}^2\,dt \tag{10}$$

$$k\|u_N\|_{L^2(0,T;V)}^2 \leq c\rho\|u_0\|_H^2 + \frac{1}{k}\|F\|_{L^2(0,T;V')}^2 \tag{11}$$

$$(c\rho)^2\|\partial u_N/\partial t\|_{L^2(0,T;V')}^2 \leq 2\left(\|F\|_{L^2(0,T;V')}^2 + k^2\|u_N\|_{L^2(0,T;V)}^2\right). \tag{12}$$

This allows us to pass to the limit as $N \to \infty$.

Existence and uniqueness of weak solutions

Theorem

For any

$$u_0 \in H, \quad f \in L^2(0,T;H), \quad g \in L^2(0,T;L^2(\Gamma_N))$$

there exists a weak solution u *of the BVP* (2) *in the class*

$$u \in L^2(0,T;V), \quad \frac{\partial u}{\partial t} \in L^2(0,T;V'), \quad u|_{t=0} = u_0$$

that satisfies

$$\frac{d}{dt}\int_\Omega c\rho\, u\, v\, dx + \int_\Omega k\, \nabla u \cdot \nabla v\, dx = \int_\Omega f v\, dx + \int_{\Gamma_N} g v\, dS \qquad (13)$$

for all test functions v *in* V*. Moreover* u *is unique (in its class!) and depends continuously on boundary and initial data.*

Comments

1. The BVP (2) has all the characteristics of a good (well-posed) mathematical model (for $T > 0$!).
2. Are *all* weak solutions acceptable solutions?
3. When are solutions smooth? Two times differentiable? Three times differentiable?
4. Can we estimate the approximation errors $\|u - u_N\|_H$, $\|u - u_N\|_V$? Rate of convergence?
5. Qualitative properties: smoothing of initial data, infinite speed of propagation, stationary solutions, maximum principle

Simple example

Let us consider heat conduction in a thin rod:

$$\Omega = \{x : 0 < x < L\}, \quad \Gamma_N = \{x = 0\}, \quad \Gamma_D = \{x = L\}.$$

Then (2) becomes

$$\begin{cases} c\rho \dfrac{\partial u}{\partial t} - k \dfrac{\partial^2 u}{\partial x^2} = f & \text{in } \Omega \quad \text{(14a)} \\ -k \dfrac{\partial u}{\partial x}\Big|_{x=0} = g & \text{on } \Gamma_N \quad \text{(14b)} \\ u|_{x=L} = 0 & \text{on } \Gamma_D \quad \text{(14c)} \\ u|_{t=0} = u_0 & \text{in } \Omega \quad \text{(14c)} \end{cases} \tag{14}$$

Weak formulation

Find u (in the admissible class) such that

$$u|_{t=0} = u_0$$

and

$$\int_0^L c\rho \, \frac{\partial u}{\partial t} v + k \, \frac{\partial u}{\partial x} \, \frac{dv}{dx} \, dx = \int_0^L f \, v \, dx - g(t) v(0) \tag{15}$$

for all v in

$$V = \left\{ v \in H^1(0, L) : v(L) = 0 \right\}.$$

Eigenfunctions

$$\begin{cases} -\dfrac{d^2}{dx^2}\psi = \lambda\,\psi & \text{in } \Omega \quad \text{(16a)} \\ \dfrac{d\psi}{dx}(0) = 0 & \text{on } \Gamma_N \quad \text{(16b)} \\ \psi(L) = 0 & \text{on } \Gamma_D \quad \text{(16c).} \end{cases} \tag{16}$$

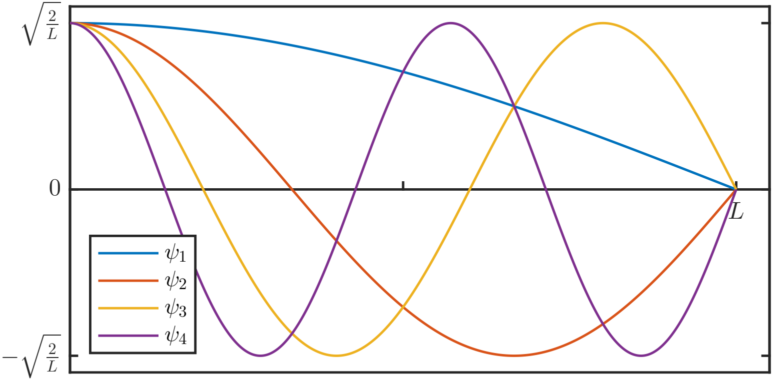
The general solution of (16a) is

$$\psi(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x).$$

Taking (16b–c) into account gives

$$\begin{cases} \lambda_m = \dfrac{\pi^2}{L^2}(m - 1/2)^2 \\ \psi_m(x) = \sqrt{\dfrac{2}{L}}\cos(\sqrt{\lambda_m}\,x) \end{cases} \qquad (m \geq 1) \tag{17}$$

Plot of eigenfunctions



ODE

Assuming the Fourier expansions

$$f(x,t) = \sum_{m=1}^{\infty} f_m(t)\,\psi_m(x)$$

$$u_0(x) = \sum_{m=1}^{\infty} c_m(0)\,\psi_m(x)$$

we obtain the ODE

$$c\rho\,\frac{dc_m}{dt} + k\lambda_m\,c_m = f_m - \sqrt{\frac{2}{L}}\,g \quad (m \geq 1).$$

Solution

Thus, the solution of BVP (14) is given by

$$u(x,t) = \sum_{m=1}^{\infty} c_m(t)\,\psi_m(x)$$

where

$$c_m(t) = e^{-\kappa\lambda_m t}\left(c_m(0) + \frac{1}{c\rho}\int_0^t e^{\kappa\lambda_m s}\left(f_m(s) - \sqrt{\frac{2}{L}}\,g(s)\right)\,ds\right),$$

and $\kappa = k/(c\rho)$.

Numerical solutions

Set

$$L = 2, \quad \rho = c = k = 1.$$

Example 1: Evolution of initial data

Let u_0 be a given a function and set $f = g = 0$.

Example 2: Stationary heat source

Let $f = f(x)$ be a given a function and set $u_0 = g = 0$.

Example 3: Prescribed heat flux at $x = 0$

Let $g = g(t)$ be a given a function and set $u_0 = f = 0$.

Continuum mechanics

The Fourier–Ritz–Galerkin method can be applied to BVPs for other PDEs such as the wave equation, the Schrödinger equation or

Navier-Stokes system

$$\begin{cases} \rho\left(\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u}\right) = \nabla\cdot\boldsymbol{\sigma} + \rho\boldsymbol{f} & \text{(momentum equation)} \\ \dfrac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\boldsymbol{u}) = 0 & \text{(mass conservation)} \\ \boldsymbol{\sigma} = (-p + \lambda\nabla\cdot\boldsymbol{u})\,\mathbf{I} + 2\mu\, e(\nabla\boldsymbol{u}) & \text{(constitutive law I)} \\ \rho = f(p) & \text{(constitutive law II)} \end{cases}$$

Reynolds equation

$$\frac{\partial}{\partial t}(h\rho) + \nabla\cdot\left(-\frac{h^3\rho}{12\mu}\nabla p + h\rho\,\boldsymbol{v}\right) = 0.$$

References

[1] W. A. Strauss. *Partial differerential equations: An introduction*. John Wiley & Sons, 2008.

[2] F.-J. Sayas. *A gentle introduction to the finite element method*, 2008.

[3] L. C. Evans. *Partial differerential equations*. American Mathematical Society, 2010.

[4] O. A. Ladyzhenskaya. *The boundary value problems of mathematical physics*. Springer-Verlag, 1985.

[1] is a gentle introduction the mathematical theory, rich in examples and applications. [3] and [4] are for the mathematically inclined reader.

THE END