On the Lax pairs of the continuous and discrete sixth Painlevé equations

Runliang LIN † [1], *Robert CONTE* † [2] *and Micheline MUSETTE* ‡

† *Service de physique de l'état condensé (Unité de recherche associée au CNRS no. 2464)*
CEA–Saclay, F–91191 Gif-sur-Yvette Cedex, France
E-mail: Conte@drecam.saclay.cea.fr

‡ *Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel*
Pleinlaan 2, B–1050 Brussels, Belgium
E-mail: MMusette@vub.ac.be

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Abstract

Among the recently found discretizations of the sixth Painlevé equation P6, only the one of Jimbo and Sakai admits a discrete Lax pair, which does establish its integrability. However, a subtle restriction in this Lax pair prevents the possibility to generalize it in order to find the other missing Lax pairs. It happens that the same restriction already exists in the matrix Lax pair of Jimbo and Miwa for the continuous P6. In this preliminary article, we remove this last restriction and give a matrix Lax pair for P6 which is traceless, rational in the dependent and independent variables, holomorphic in the monodromy exponents, with four Fuchsian singularities in the complex plane of the spectral parameter. Its only minor drawback is the presence of the apparent singularity which always exists in the scalar Lax pair.

Contents

[1]Permanent address: Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China. E-mail: RLin@math.tsinghua.edu.cn
[2]Corresponding author RC.

1 Introduction

Since the discovery [11] of a discrete version of the sixth Painlevé equation P6, many other equations have been proposed as candidate discrete P6 equations, see the diagram below (courtesy of the authors of Ref. [19], to which we refer for the notation).

$$
\begin{array}{ccccccccccccccc}
\mathrm{E}_8^{\mathrm{e}} & & & & & & & & & & & & & & \\
\downarrow & & & & & & & & & & & & & & \\
\mathrm{E}_8^q & \longrightarrow & \mathrm{E}_7^q & \longrightarrow & \mathrm{E}_6^q & \longrightarrow & \mathrm{D}_5^q & \longrightarrow & \mathrm{A}_4^q & \longrightarrow & (\mathrm{A}_2+\mathrm{A}_1)^q & \longrightarrow & (\mathrm{A}_1+\mathrm{A}_1)^q & \longrightarrow & \mathrm{A}_1^q \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \mid \quad \downarrow & & \mid \quad \downarrow & & \\
\mathrm{E}_8^\delta & \longrightarrow & \mathrm{E}_7^\delta & \longrightarrow & \mathrm{E}_6^\delta & \longrightarrow & \mathrm{D}_4^{\mathrm{c}} & \longrightarrow & \mathrm{A}_3^{\mathrm{c}} & \longrightarrow & \mid \quad (2\mathrm{A}_1)^{\mathrm{c}} & \longrightarrow & \mid \quad \mathrm{A}_1^{\mathrm{c}} & & \\
 & & & & & & & & & \searrow & \downarrow & \searrow & \downarrow & & \\
 & & & & & & & & & & \mathrm{A}_2^{\mathrm{c}} & \longrightarrow & \mathrm{A}_1^{\mathrm{c}} & &
\end{array}
$$

However, as opposed to the $\mathrm{q}-\mathrm{P6}$ of Ref. [11] which admits a Lax pair by construction, they cannot be considered as discrete P6 equations until one has found a Lax pair for each of them. One can think of two methods to find such a discrete Lax pair:

1. Define a discrete isomonodromy problem (in scalar or matrix form) and solve it. Since one must probably introduce at least as many singular points as parameters in the candidate discrete equation, i.e. between 4 and 8, this task is certainly difficult.

2. Choose a continuous pair of P6 (in scalar or matrix form), then discretize both P6 and its Lax pair. The probable drawback is the little hope to obtain more than 4 parameters in the result.

If one chooses to apply the second method, a prerequisite is to select which continuous Lax pair to discretize. In this article, we perform a critical examination of the existing Lax pairs, conclude to the necessity to first build a better Lax pair than the existing ones, and provide a new matrix Lax pair for P6.

The outline of the paper is as follows. In section 2, we point out a restriction in the discrete Lax pair of $\mathrm{q}-\mathrm{P6}$ given in Ref. [11], which makes it inexistent for some values of the four parameters.

In section 3, we remark that the same restriction already exists in the continuous matrix Lax pair as given by Jimbo and Miwa [10].

In section 4, starting from the two original isomonodromy problems (the scalar one of Fuchs, the matrix one of Jimbo and Miwa), we define two ways out of this situation, a difficult one and an easier one.

In section 5, following the second way, initiated by two of us [3], we give two new matrix Lax pairs of P6, without any restriction on $\alpha, \beta, \gamma, \delta$, traceless, and rational in u, u', x, t. The better one has only one minor drawback, which is the same apparent singularity than in the scalar case.

2 On the discrete matrix Lax pair of $\mathrm{q}-\mathrm{P6}$

The unique discrete P6 equation which, up to now, admits a Lax pair has been found by Jimbo and Sakai [11], as an output to the matrix discrete isomonodromy problem

$$Y(x,qt)=A(x,t)Y(x,t), \tag{2.1}$$
$$A=A_0(x)+A_1(x)t+A_2(x)t^2, \tag{2.2}$$

where x is the independent variable, t is the spectral parameter, and the matrix A defines four singular points in the t complex plane. One of these is $t=\infty$, and its residue A_2 is chosen by the authors as the diagonal matrix [11, Eq. (10)]

$$A_2=\mathrm{diag}(\kappa_1,\kappa_2). \tag{2.3}$$

When $\kappa_1=\kappa_2$, one singularity is lost, and the isomonodromy problem cannot yield a $\mathrm{q}-\mathrm{P6}$ equation, as can be checked from the denominator $\kappa_1-\kappa_2$ in the discrete Lax pair.

This restriction $\kappa_1\neq\kappa_2$ prevents the Lax pair to exist for a one-dimensional subset of the four parameters, and it must be removed before any attempt to generalize the monodromy to more than four singularities, as probably required for the candidate discrete Painlevé equations with more than four parameters (see diagram above).

3 On the (continuous) matrix Lax pair of P6

One must wonder whether such a restriction also exists in the continuous case. The sixth Painlevé equation P6 is defined as [5, 17]

$$\begin{aligned} E(u) \;\equiv\; & -u''+\frac{1}{2}\left[\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-x}\right]u'^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{u-x}\right]u' \\ & +\frac{u(u-1)(u-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{u^2}+\gamma\frac{x-1}{(u-1)^2}+\delta\frac{x(x-1)}{(u-x)^2}\right]=0, \end{aligned}$$

together with the notation, adapted to the symmetry of P6 under permutation of its four singularities,

$$(2\alpha,-2\beta,2\gamma,1-2\delta)=(\theta_\infty^2,\theta_0^2,\theta_1^2,\theta_x^2). \tag{3.1}$$

In the matrix isomonodromy problem with four Fuchsian singularities as defined, and beautifully solved, by Jimbo and Miwa [10], after the definition of the problem

$$\frac{\partial Y}{\partial t}=MY, \tag{3.2}$$
$$M=\frac{M_0(x)}{t}+\frac{M_1(x)}{t-1}+\frac{M_x(x)}{t-x},\quad M_\infty+M_0+M_1+M_x=0. \tag{3.3}$$

the simplifying assumption that the residue M_∞ at ∞ is a constant diagonal traceless matrix,

$$M_\infty=\mathrm{diag}(\Theta_\infty/2,-\Theta_\infty/2), \tag{3.4}$$

ipso facto lowers the number of singularities from four to three when Θ_∞ vanishes, thus preventing the isomonodromy problem to yield P6. Indeed, the denominators in the resulting Lax pair make it inexistent on the subset

$$(\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2) = (1, 1, 1, 1), \tag{3.5}$$

i.e. a fourth iterate of the Picard case [16] $(\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2) = (0, 0, 0, 0)$ under the elementary birational transformation [15, 2] of P6.

Some remedies have been proposed ([12, p. 50] and [13, p. 180]) to this drawback, but the resulting Lax pair is then not a continuous function of Θ_∞.

4 All the existing Lax pairs of P6

There currently exist essentially two Lax pairs for P6:

1. the scalar one of R. Fuchs [5, 6], better written by Garnier [7], with four Fuchsian singularities $t = \infty, 0, 1, x$ and one apparent singularity [3] $t = u$; its monodromy exponents at the Fuchsian singularities are $\pm\theta_j, j = \infty, 0, 1, x$, with θ_j^2 defined in (3.1).

2. the matrix one of Jimbo and Miwa [10] , with four Fuchsian singularities $t = \infty, 0, 1, x$ and the associated monodromy exponents $\pm\Theta_j, j = \infty, 0, 1, x$,

$$((\Theta_\infty - 1)^2, \Theta_0^2, \Theta_1^2, \Theta_x^2) = (\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2), \tag{4.1}$$

   which does not fully display the permutation symmetry.

The reason for this apparent dissymetry is the requirement of the authors that the unique zero of the element M_{12} in the t complex plane should be a solution $u = u_{12}$ of P6, which implies the existence of another Lax pair in which M_{21} has a unique zero located at $t = u_{21}$, with u_{21} solution of another P6 equation having the monodromy exponents $\pm\Theta_j$,

$$((\Theta_\infty + 1)^2, \Theta_0^2, \Theta_1^2, \Theta_x^2) = (\theta_\infty^2, \theta_0^2, \theta_1^2, \theta_x^2). \tag{4.2}$$

The transformation between u_{12} and u_{21} is birational, and its expression can be found from Ref. [10],

$$u_{21} = \frac{P_4(u'_{12}; u_{12}, x)}{P_4(u'_{12}; u_{12}, x)}, \tag{4.3}$$

in which the two symbols P_4 denote polynomials of degree four in u'_{12}. It is then easy to generate balanced expressions in which u_{12} and u_{21} become expressions which only differ by signs. Let us indeed denote $\mathrm{T_G}$ the birational transformation of Garnier [8, 9] (this is

[3] A singularity is said *apparent* iff the ratio of two independent solutions of the associated linear system is singlevalued around this singularity.

basically the square [1] of the elementary transformation [15]) and $\mathrm{S_a}$ the operator which reverses the sign of θ_∞. Then one has the diagram

$$\underbrace{\begin{pmatrix} 1-\theta_\infty \\ \theta_0 \\ \theta_1 \\ \theta_x \end{pmatrix}}_{u_{12}} \overset{\mathrm{T_G}}{\longrightarrow} \underbrace{\begin{pmatrix} \theta_\infty \\ 1-\theta_0 \\ \theta_1 \\ \theta_x \end{pmatrix} \overset{\mathrm{S_a}}{\longrightarrow} \begin{pmatrix} -\theta_\infty \\ 1-\theta_0 \\ \theta_1 \\ \theta_x \end{pmatrix}}_{u_\mathrm{b}} \overset{\mathrm{T_G}}{\longrightarrow} \underbrace{\begin{pmatrix} 1+\theta_\infty \\ \theta_0 \\ \theta_1 \\ \theta_x \end{pmatrix}}_{u_{21}}, \tag{4.4}$$

in which u_b (b like balanced) denotes a third solution of P6. As a function of u_b, both u_{12} and u_{21} just differ by signs and are equal to

$$\mathrm{H}(u_\mathrm{b})\frac{R_n^+(u_\mathrm{b})R_n^-(u_\mathrm{b})}{R_d^+(u_\mathrm{b})R_d^-(u_\mathrm{b})}, \tag{4.5}$$

in which $\mathrm{H}(u_\mathrm{b})$ is some homographic transform of u_b, and the four symbols $R(u_\mathrm{b})$ denote the lhs of Riccati equations for u_b (full details can be found in [2, Eqs. (8)–11]).

Another matrix Lax pair has also been found [14, Eqs. (116), (118)], with four Fuchsian singularities $t = \infty, 0, 1, x$ but its associated monodromy exponents $\pm\Theta_j, j = \infty, 0, 1, x$, are

$$(\Theta_\infty^2, \Theta_0^2, \Theta_1^2, \Theta_x^2) = (\text{constant}, \text{constant}, 0, 0), \tag{4.6}$$

so it is nongeneric and we will not consider it.

Let us first recall these two Lax pairs (Fuchs, Jimbo and Miwa).

4.1 Lax pair by scalar isomonodromy

This pair [5, 6], as more nicely written in Ref. [7], is characterized by the two homographic invariants (S, C),

$$\partial_t^2\psi + (S/2)\psi = 0, \tag{4.7}$$

$$\partial_x\psi + C\partial_t\psi - (1/2)C_t\psi = 0, \tag{4.8}$$

with the cross-derivative condition,

$$X \equiv S_x + C_{ttt} + CS_t + 2C_tS = 0, \tag{4.9}$$

where

$$-C = \frac{t(t-1)(u-x)}{(t-u)x(x-1)}, \tag{4.10}$$

$$-\frac{S}{2} = \frac{3/4}{(t-u)^2} + \frac{\beta_1 u' + \beta_0}{(t-u)t(t-1)} + \frac{[(\beta_1 u')^2 - \beta_0^2]\dfrac{u-x}{u(u-1)} + f_\mathrm{G}(u)}{t(t-1)(t-x)} + f_\mathrm{G}(t), \tag{4.11}$$

$$\beta_1 = -\frac{x(x-1)}{2(u-x)}, \quad \beta_0 = -u + \frac{1}{2}, \tag{4.12}$$

$$f_G(z) = \frac{a}{z^2} + \frac{b}{(z-1)^2} + \frac{c}{(z-x)^2} + \frac{d}{z(z-1)}, \tag{4.13}$$

$$(2\alpha, -2\beta, 2\gamma, 1-2\delta) = (4(a+b+c+d+1), 4a+1, 4b+1, 4c+1). \tag{4.14}$$

4.2 Lax pair by matrix isomonodromy

Let us introduce the Pauli matrices σ_k

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_j\sigma_k = \delta_{jk} + i\varepsilon_{jkl}\sigma_l, \quad (4.15)$$
$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Lax pair

$$\partial_x\psi = L\psi, \quad \partial_t\psi = M\psi, \quad (4.16)$$

$$Z \equiv [\partial_x - L, \partial_t - M] = 0. \quad (4.17)$$

obtained by matrix isomonodromy [10] can be easily rewritten [12] in traceless form and with entries (L_{jk}, M_{jk}) which are all algebraic (not only with algebraic logarithmic derivatives). This traceless, algebraic form is the following.

$$\begin{aligned}
L &= -\frac{M_x}{t-x} + L_\infty,\ M = \frac{M_0}{t} + \frac{M_1}{t-1} + \frac{M_x}{t-x}, && (4.18)\\
L_\infty &= -\frac{(\Theta_\infty - 1)(u-x)}{2x(x-1)}\sigma_3, && (4.19)\\
2M_0 &= z_0\sigma_3 - \frac{u}{x}\sigma^+ + (z_0^2 - \theta_0^2)\frac{x}{u}\sigma^-, && (4.20)\\
2M_1 &= z_1\sigma_3 + \frac{u-1}{x-1}\sigma^+ - (z_1^2 - \theta_1^2)\frac{x-1}{u-1}\sigma^-, && (4.21)\\
2M_x &= \left((\theta_0^2 - z_0^2)\frac{x}{u} - (\theta_1^2 - z_1^2)\frac{x-1}{u-1}\right)\sigma^- - \frac{u-x}{x(x-1)}\sigma^+ \\
&\quad -(\Theta_\infty + z_0 + z_1)\sigma_3, && (4.22)\\
z_0 &= \frac{1}{2\Theta_\infty x(u-1)(u-x)}\Big[\left(x(x-1)u' - (u-1)(u - \Theta_\infty(u-x))\right)^2 \\
&\quad -(\Theta_\infty^2 + \theta_0^2)x(u-1)(u-x) + \theta_1^2(x-1)u(u-x) - \theta_x^2 x(x-1)u(u-1)\Big],\\
z_1 &= \frac{-1}{2\Theta_\infty(x-1)u(u-x)}\Big[\left(x(x-1)u' - u(u-1-\Theta_\infty(u-x))\right)^2 \\
&\quad +(\Theta_\infty^2 + \theta_1^2)(x-1)u(u-x) - \theta_0^2 x(u-1)(u-x) - \theta_x^2 x(x-1)u(u-1)\Big],\\
Z/E &= \frac{x(x-1)}{2\Theta_\infty t(t-1)(t-x)u(u-1)(u-x)} \times \\
&\quad \Big[t(u-1)(x(x-1)u' + u(\Theta_\infty(u-x) - (u-1))) \\
&\quad - (t-1)u(x(x-1)u' + (u-1)(\Theta_\infty(u-x) - u))\Big]\sigma_3 \\
&\quad + \frac{x(x-1)}{2\Theta_\infty^2 t(t-1)(t-x)(u(u-1)(u-x))^2} \times
\end{aligned}$$

$$P_{1,1,5,5,2,1,1,1}(t,u',u,x,\Theta_\infty,\theta_0^2,\theta_1^2,\theta_x^2)\sigma^-, \qquad (4.23)$$
$$(2\alpha,-2\beta,2\gamma,1-2\delta)=((\Theta_\infty-1)^2,\theta_0^2,\theta_1^2,\theta_x^2),\ M_\infty=-M_0-M_1-M_x,$$

in which the symbol P denotes a polynomial of its eight arguments with degrees indicated by its eight indices.

The question is: from either one of these two Lax pairs, can one build a “better” Lax pair?

The matrix form seems preferable to the scalar one because the apparent singularity $t=u$, unavoidable in the scalar case as shown by Poincaré [18, pp. 217–220], can be suppressed in the matrix case [10]. There are however other criteria, which will be discussed in section 6.

One can think of two strategies to build such a “better” matrix Lax pair:

1. In the matrix isomonodromy problem, remove the assumption that the residue M_∞ at ∞ is diagonal, and perform again the whole resolution, i.e. eliminate all the deformation parameters (free functions of x in the definition of M) but one, and establish the differential equation for this single remaining deformation parameter. Finally, integrate this second order, probably second degree, equation using the appropriate tables [4]. This is easy in theory, but certainly difficult in practice.

2. Convert the scalar pair to matrix form. Such a converted pair has already been given by two of us [3], but without preserving the Fuchsian nature of the four singularities in t.

Let us define a shorthand notation for the various matrix Lax pairs of P6, based on their structure of singularities in the complex plane of the spectral parameter t. The *type* of a matrix Lax pair will be denoted by $(n_{\mathrm{F}},n_{\mathrm{nF}},n_{\mathrm{A}})$, in which the three integers represent the number of singularities of the respective types : Fuchsian, nonFuchsian, apparent. The type of the pair (4.18) is $(4,0,0)$.

5 Conversion of the scalar pair to matrix form

Consider the traceless Lax pair

$$\left(\partial_x-\begin{pmatrix}L_{11} & L_{12}\\ L_{21} & -L_{11}\end{pmatrix}\right)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=0, \qquad (5.1)$$
$$\left(\partial_t-\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & -M_{11}\end{pmatrix}\right)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=0. \qquad (5.2)$$

The elimination of one component, say ψ_1, defines the scalar system

$$\left(\partial_t^2-\frac{M_{21,t}}{M_{21}}\partial_t+M_{11,t}-M_{11}^2-M_{11}\frac{M_{21,t}}{M_{21}}-M_{12}M_{21}\right)\psi_2=0, \qquad (5.3)$$
$$\left(\partial_x-\frac{L_{21}}{M_{21}}\partial_t+L_{11}-M_{11}\frac{L_{21}}{M_{21}}\right)\psi_2=0. \qquad (5.4)$$

By identifying the scalar pair (5.3)–(5.4) and the pair of Garnier (4.7)–(4.8), one can compute the elements L_{jk} and M_{jk} as follows [3].

The coefficients of the scalar Lax pair (5.3)–(5.4) have simple poles at $M_{21} = 0$. Let us first convert the double pole $t = u$ of the coefficients of the pair of Garnier into a simple pole, by the transformation

$$\Psi = \sqrt{t-u}\psi, \tag{5.5}$$

$$\left(\partial_t^2 - \frac{1}{t-u}\partial_t + \left(\frac{S}{2} + \frac{3/4}{(t-u)^2}\right)\right)\Psi = 0, \tag{5.6}$$

$$\left(\partial_x + C\partial_t + \frac{u' - (C + (t-u)C_t)}{2(t-u)}\right)\Psi = 0, \tag{5.7}$$

with S given by (4.11) and C by (4.10).

The six coefficients (L_{jk}, M_{jk}) of the Lax pair are then obtained by identifying the singularities $M_{21} = 0$ in (5.3)–(5.4) and $t - u = 0$ in (5.6)–(5.7). This results in

$$\Psi = \psi_2, \tag{5.8}$$

and

$$M_{11} = \frac{\beta_1 u' + \beta_0}{t(t-1)}, \tag{5.9}$$

$$M_{12} = \frac{M_{11,t} - M_{11}^2}{t-u} - \frac{M_{11}}{(t-u)^2} - \frac{1}{t-u}\left(\frac{S}{2} + \frac{3/4}{(t-u)^2}\right), \tag{5.10}$$

$$M_{21} = t - u, \tag{5.11}$$

$$L_{21} = -(t-u)C, \tag{5.12}$$

$$L_{11} = -CM_{11} + \frac{u' - (C + (t-u)C_t)}{2(t-u)} \tag{5.13}$$

$$L_{12} = \frac{-L_{11,t} + M_{11,x} + L_{21}M_{12}}{t-u} \tag{5.14}$$

(formula (5.10) corrects a misprint in Eq. (58) of Ref. [3]).

The resulting matrix pair [3] has the type $(0, 4, 0)$, which is unfortunate since no singularity at all is Fuchsian.

With other choices of the formulae (5.5) and (5.8), one arrives at quite different matrix Lax pairs. The results are the following.

5.1 A matrix Lax pair of type $(4, 0, 1)$

With the choice

$$\Psi = \sqrt{\frac{t(t-1)(t-x)}{t-u}}\left(\frac{1}{2}\psi_1 + \psi_2\right), \tag{5.15}$$

the four singularities $t = \infty, 0, 1, x$ remain Fuchsian, but the apparent singularity $t = u$ remains, the resulting matrix Lax pair has the type $(4, 0, 1)$,

$$L = -\frac{M_x}{t-x} + \frac{L_u}{t-u} + L_\infty, \quad M = \frac{M_0}{t} + \frac{M_1}{t-1} + \frac{M_x}{t-x} + \frac{M_u}{t-u}, \tag{5.16}$$

$$L_u = \frac{1}{4}\left(\frac{x^2(x-1)^2}{2u(u-1)(u-x)^2}u'^2 - \frac{(u-x)^2}{2x(x-1)} + \frac{\theta_\infty^2\,(u(u-x) - x(u-1))}{2x(x-1)}\right.$$

$$+\frac{\theta_0^2}{2u}-\frac{\theta_1^2}{2(u-1)}-\frac{(1-\theta_x^2)\,(u(x-1)+x(u-1))}{2(u-x)^2}\Bigg)(\sigma_3-\frac{\sigma_2}{i})$$
$$-\frac{1}{2}u'\sigma_1, \tag{5.17}$$

$$L_\infty = \left(\frac{\theta_\infty^2}{8}\sigma_3-\sigma_1+(1-\frac{\theta_\infty^2}{8})\frac{\sigma_2}{i}\right)\frac{(u-x)}{x(x-1)}. \tag{5.18}$$

$$M_0 = -\frac{1}{4}\left(\frac{x^3(x-1)^2}{2u^2(u-x)^2}u'^2+\frac{2x^2(x-1)}{u(u-x)}u'-\frac{\theta_0^2}{2}\frac{x}{u^2}\right)(\sigma_3-\frac{\sigma_2}{i})+\frac{u^2-x^2}{2x}\sigma_3$$
$$-\left(\frac{x(x-1)}{2(u-x)}u'+u\right)\sigma_1+\frac{u^2+x^2}{2x}\frac{\sigma_2}{i}, \tag{5.19}$$

$$M_1 = \frac{1}{4}\left(\frac{x^2(x-1)^3}{2(u-1)^2(u-x)^2}u'^2+\frac{2x(x-1)^2}{(u-1)(u-x)}u'-\frac{\theta_1^2}{2}\frac{(x-1)}{(u-1)^2}\right)(\sigma_3-\frac{\sigma_2}{i})$$
$$+\frac{(2-u-x)(u-x)}{2(x-1)}\sigma_3+\left(\frac{x(x-1)}{2(u-x)}u'+(u-1)\right)\sigma_1$$
$$-\frac{(u-1)^2+(x-1)^2}{2(x-1)}\frac{\sigma_2}{i}, \tag{5.20}$$

$$M_x = \frac{(u-x)^2}{2x(x-1)}(\sigma_3+\frac{\sigma_2}{i})-\frac{(1-\theta_x^2)x(x-1)}{8(u-x)^2}(\sigma_3-\frac{\sigma_2}{i})-\frac{1}{2}\sigma_1, \tag{5.21}$$

$$M_u = \frac{1}{4}\left(\frac{(u(u-x)-x(u-1))x^2(x-1)^2}{2u^2(u-1)^2(u-x)^2}u'^2+\frac{2x(x-1)}{u(u-1)}u'\right.$$
$$\left.+\frac{\theta_\infty^2}{2}-\frac{\theta_0^2}{2}\frac{x}{u^2}+\frac{\theta_1^2}{2}\frac{(x-1)}{(u-1)^2}+\frac{(1-\theta_x^2)x(x-1)}{2(u-x)^2}\right)(\sigma_3-\frac{\sigma_2}{i})+\frac{1}{2}\sigma_1, \tag{5.22}$$

$$Z/E = \frac{1}{z_1}\left(z_2(\sigma_3-\frac{\sigma_2}{i})+z_3\sigma_1\right),$$
$$z_1 = -4t(t-1)(t-u)(u-1)u^2(u-x)^2,$$
$$z_2 = xu(x-1)\left(x(x-1)((t-x)u-x(t-1))u'+2u(t-x)(u-1)(u-x)\right),$$
$$z_3 = 2x(x-1)u^2(u-1)(u-x)(t-u),$$

and the monodromy exponents $\pm\Theta_j, j=\infty,0,1,x$ take symmetric values

$$-4\det(\operatorname{res} M)|_{t=(\infty,0,1,x,u)} = (\theta_\infty^2,\theta_0^2,\theta_1^2,\theta_x^2,1),\ M_\infty=-M_0-M_1-M_x. \tag{5.23}$$

5.2 An unrestricted matrix pair without apparent singularity

If one wants to suppress the apparent singularity $t=u$, one can choose

$$\Psi=\sqrt{t(t-1)(t-x)}\,(\psi_1+\psi_2), \tag{5.24}$$

but the singularity $t=\infty$ becomes nonFuchsian, and the resulting matrix Lax pair has the type $(3,1,0)$:

$$L = -\frac{M_x}{t-x}+L_\infty^{(0)}+L_\infty^{(1)}t,\ M=\frac{M_0}{t}+\frac{M_1}{t-1}+\frac{M_x}{t-x}+\frac{\theta_\infty^2-4}{4}\left(\sigma_3-\frac{\sigma_2}{i}\right), \tag{5.25}$$

$$L_\infty^{(0)} = -\frac{(u-x)}{x(x-1)}\sigma_1+\frac{1}{2}\left(\frac{x^2(x-1)^2}{2u(u-1)(u-x)^2}u'^2+\frac{2(u-x)}{x-1}-\frac{\theta_\infty^2(u-1)}{2(x-1)}\right.$$

$$+\frac{\theta_0^2}{2u}-\frac{\theta_1^2}{2(u-1)}-\frac{(1-\theta_x^2)\left(u(x-1)+x(u-x)\right)}{2(u-x)^2}\Bigg)(\sigma_3-\frac{\sigma_2}{i}) \tag{5.26}$$

$$L_\infty^{(1)} = \left(\frac{\theta_\infty^2-4}{4}\right)\frac{u-x}{x(x-1)}\left(\sigma_3-\frac{\sigma_2}{i}\right), \tag{5.27}$$

$$\begin{aligned} M_0 &= \left(-\frac{x(x-1)}{2(u-x)}u'-u\right)\sigma_1-\frac{u}{4x}(\sigma_3+\frac{\sigma_2}{i}) \\ &+\left(\frac{x^3(x-1)^2}{4u(u-x)^2}u'^2+\frac{x^2(x-1)}{(u-x)}u'+xu-\frac{\theta_0^2}{4}\frac{x}{u}\right)(\sigma_3-\frac{\sigma_2}{i}), \end{aligned} \tag{5.28}$$

$$\begin{aligned} M_1 &= \left(\frac{x(x-1)}{2(u-x)}u'+(u-1)\right)\sigma_1 \\ &+\Bigg(-\frac{x^2(x-1)^3}{4(u-1)(u-x)^2}u'^2-\frac{x(x-1)^2}{(u-x)}u'-(x-1)(u-1) \\ &\quad+\frac{\theta_1^2}{2}\frac{(x-1)}{2(u-1)}\Bigg)(\sigma_3-\frac{\sigma_2}{i})+\frac{(u-1)}{4(x-1)}(\sigma_3+\frac{\sigma_2}{i}), \end{aligned} \tag{5.29}$$

$$M_x = -\frac{1}{2}\sigma_1+\frac{(1-\theta_x^2)x(x-1)}{4(u-x)}(\sigma_3-\frac{\sigma_2}{i})-\frac{(u-x)}{4x(x-1)}(\sigma_3+\frac{\sigma_2}{i}), \tag{5.30}$$

$$\begin{aligned} Z/E &= \frac{1}{z_1}\left(z_2(\sigma_3-\frac{\sigma_2}{i})+z_3\sigma_1\right), \\ z_1 &= -2t(t-1)(u-1)u(u-x)^2, \\ z_2 &= x(x-1)\left(x(x-1)(t(u-x)-x(u-1))u'+2(t-x)u(u-1)(u-x)\right), \\ z_3 &= x(x-1)u(u-1)(u-x). \end{aligned}$$

The monodromy exponents $\pm\Theta_j, j=0,1,x$ at the three Fuchsian points are

$$-4\det(\operatorname{res} M)|_{t=(0,1,x)} = (\theta_0^2,\theta_1^2,\theta_x^2), \tag{5.31}$$

the fourth point $t=\infty$ is Fuchsian iff $\alpha=2$, in which case

$$-4\det(\operatorname{res} M)|_{t=\infty} = 9. \tag{5.32}$$

6 Discussion. What the optimal Lax pair should be

None of the above two new Lax pairs, with types $(4,0,1)$ and $(3,1,0)$, is yet optimal. To be optimal, a Lax pair should, in our opinion, obey the following criteria:

1. have exactly four Fuchsian singularities, located at $t=\infty,0,1,x$,
2. (matrix form only) have no apparent singularity at $t=u$,
3. exist for any value of $\alpha,\beta,\gamma,\delta$,
4. (matrix form only) be "balanced", i.e., in the basis $(\sigma_+,\sigma_-,\sigma_3)$, have components on (σ_+,σ_-) which just differ by signs,
5. (matrix form only) be traceless,
6. be rational in (u',u,x),

7. allow an easy confluence to the lower Painlevé equations.

The various kinds of Lax pairs are compared in Table 1 according to these criteria.

Table 1. Advantages and inconveniences of the various kinds of Lax pairs (the scalar one of Fuchs and the matrix ones). N/A means not applicable. The ideal situation would be that all entries be "yes". The last four columns refer, respectively, to (4.18), (5.9)–(5.14), (5.16), (5.25).

	Scalar Fuchs	(4,0,0)	(0,4,0)	(4,0,1)	(3,1,0)
four Fuchsian singularities	N/A	yes	no	yes	no
no apparent singularity	no	yes	yes	no	yes
arbitrary $\alpha, \beta, \gamma, \delta$	yes	no	yes	yes	yes
balanced	N/A	no	no	yes	yes
traceless	N/A	yes	yes	yes	yes
rational (u', u, x)	yes	yes	yes	yes	yes
easy confluence	yes	no	yes	yes	yes

7 Conclusion

Until one has found an optimal matrix Lax pair for P6, it could be wiser, in order to devise discrete Lax pairs for the discretizations of P6 which still lack one, to take inspiration from the scalar pair of Fuchs.

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