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A Hidden Hierarchy for GD_4

Paolo CASATI

Dipartimento di Matematica, Università di Milano–Bicocca
Via degli Arcimboldi 8, I-20133 Milano, Italy
E-mail: casati@mat.unimi.it

Abstract

A reduction process to construct hidden hierarchies corresponding to the Gelfand–Dickey ones is outlined in a specific example, not yet treated in the literature.

1 Introduction

In a series of recent papers (see [4, 5] and references therein) integrable hierarchies associated with second order ("Schrödinger") or third order linear differential operators with energy-dependent potentials, usually called *hidden hierarchies* have been presented and studied. In the setting of Gelfand Dickey hierarchies (GD) they turn out to be the "hidden" hierarchies generalizing the well known KdV and Boussinesq hierarchies. Since these last are the two first GD hierarchies, it would be interesting to consider also some examples of higher dimension. But to the best of our knowledge no hidden hierarchies corresponding to Gelfand–Dickey hierarchies of dimension bigger then three has been until now examined.

On the other hand in [2, 3] it has been shown how to recover from a system of ordinary partial differential equations (called Central System (CS)) defined on the Sato Grassmanian, the GD hierarchies (as well the fractional ones) performing suitable reduction processes. It is therefore natural to wonder if there exist "Generalized Central Systems" (in what follows called Hidden Central Systems) which give rise to these new hidden hierarchies by performing the same reduction processes. It can be shown that these systems actually exist, but in these notes we can neither explain the general theory, which lies behind this results, nor recover the already known hidden hierarchies. The aim of this paper will be only to show through an example how this method allows to construct new hidden hierarchies corresponding to Gelfand–Dickey hierarchies of higher dimension, possibly not yet been considered in the literature. More precisely we will perform the reduction process leading to the most simple hidden hierarchy associated with the GD_4 hierarchy. I want to thank Marco Pedroni, Boris Konopelchenko, Franco Magri and Gregorio Falqui for many useful discussions and suggestions on this subject.

2 The Hidden Central System

The aim of this section is to define, in the general setting of the Sato Grassmanian, a particular set of systems of ordinary differential equations, which through a suitable reduction process give rise to hierarchies of soliton equations, which contain among others those studied in the papers [4, 5].

Let us first briefly define the Sato Grassmanian. Let $\mathcal{L}$ be the set of all Laurent series in a variable z, which are truncated from above, i. e.the space

$$\mathcal{L} := \left\{ \sum_{-\infty}^{N(l)} l_i z^{-i} | \quad N(l) \in \mathbb{Z} \right\}. \tag{2.1}$$

(In what follows the coefficients l_i are thought as depending on an infinite numbers of variables, later identified with the times of our dynamical systems.)

The space $\mathcal{L}$ has a natural decomposition in the direct sum

$$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_- \tag{2.2}$$

of a positive part $\mathcal{L}_+$: the subspace of all polynomials in z, and of a negative one $\mathcal{L}_-$: the subspace of all Laurent series with only negative terms. Let us denote by $\pi_\pm : \mathcal{L} :\to \mathcal{L}_\pm$ the projection associated with the decomposition (2.2).

Definition 2.1. *The Sato Grassmanian $\mathcal{S}$ is the set of all hyperplanes $\mathcal{W}$ in $\mathcal{L}$ such that the restriction $\pi_{+|\mathcal{W}}$ on $\mathcal{W}$ of the projection $\pi_+ : \mathcal{L} \to \mathcal{L}_+$ along $\mathcal{L}_-$ has finite kernel and cokernel [1]. The big cell of the Sato Grassmanian $\mathcal{S}$ is the subset of all points $\mathcal{W}$ of $\mathcal{S}$, for which the restriction of π_+ is a bijection. (Therefore $\mathcal{W}$ belongs to the big cell if and only if $\dim(\ker(\pi_{+|\mathcal{W}})) = \dim(\mathrm{coker}(\pi_{+|\mathcal{W}})) = 0$). Any points of the Grassmanian, which does not satisfy this condition is said to belong to the singular set of the Grassmanian.*

The singular set can be further decomposed in the Birkhoff strata. It can be shown (and this will be the subject of a forthcoming paper) that with every Birkhoff stratum one can associate systems of ordinary differential equations. For example to the big cell, which can be thought as the "trivial" Birkhoff stratum corresponds the usual central system studied in [2, 3]. Unfortunately we cannot pursue here this general approach, so let us simply define a new particular Central System, called in the following the Hidden Central System (HCS), which actually corresponds to a Birkhoff stratum of codimension one in the Grassmanian [4].

To this aim let us consider the set $\mathcal{N} = \{0, 2, 3, \ldots\}$ and for any element n in $\mathcal{N}$ let us define a Laurent series in $\mathcal{L}$, called *current*, of the type

$$H^{(n)} = z^n + \sum_{i \in \mathcal{N}^c} H^n_{-i} z^i, \qquad n \in \mathcal{N}, \tag{2.3}$$

where $\mathcal{N}^c$ denotes the complementary set of $\mathcal{N}$ in $\mathbb{Z}$ i.e., the set $\{1, -1, -2, \ldots\}$. For example the first current will be

$$\begin{aligned} H^{(0)} &= H^0_1 z + 1 + \sum_{i>0} H^0_i z^{-i}, \\ H^{(2)} &= z^2 + H^2_1 z + \sum_{i>0} H^2_i z^{-i}, \\ H^{(3)} &= z^3 + H^3_1 z + \sum_{i>0} H^3_i z^{-i}. \end{aligned} \tag{2.4}$$

Notice that contrary to what happens in the case of the currents of the usual central system, here the current $H^{(1)}$ is missing and moreover each current (even $H^{(0)}$) has an element of the type az. The set of currents (2.3) span a subspace of $\mathcal{L}$ which will be denoted by $\mathcal{H}_+$. Then, exactly as in the case of usual central system the following proposition defines uniquely our Hidden Central System.

Proposition 2.2. *The invariance relation*

$$\left(\frac{\partial}{\partial t_j} + H^{(j)}\right)(\mathcal{H}_+) \subset \mathcal{H}_+, \qquad j \in \mathcal{N} \tag{2.5}$$

define uniquely systems of ordinary differential equations called the Hidden Central System.

Proof. We have to show that this relation completely defines the action of the vector fields $\frac{\partial}{\partial t_j}$ on the currents $H^{(n)}$. Since the currents $H^{(k)}$ form a basis of $\mathcal{H}_+$, the invariance property (2.5) entails the existence of coefficients c_l^{jk} independent of z such that

$$\frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} = \sum_{l=0}^{j+k} c_l^{jk} H^{(l)}. \tag{2.6}$$

These coefficients are easily determined by comparing the expansion in powers of z of both sides of equation (2.6). The final result are the following equations defining the hidden central system:

$$\begin{aligned}\frac{\partial H^{(k)}}{\partial t_j} + H^{(j)} H^{(k)} &= H^{(j+k)} + \sum_{l=1}^{k-2} H_l^j H^{(k-l)} + \sum_{l=1}^{j-2} H_l^k H^{(j-l)} + H_k^j \\ &+ H_j^k + H_{-1}^j H^{(k+1)} + H_{-1}^k H^{(j+1)} + H_{-1}^k H_{-1}^j H^{(2)} + H_{-1}^k H_1^j + H_{-1}^j H_1^k.\end{aligned} \tag{2.7}$$

■

Observe that in the equations (2.7) the time–derivative are defined (like the currents) only for $j \in \mathcal{N}$, therefore there is no time t_1.

3 The general reduction scheme

Although the equations (2.7) looks a little complicate, it can be easily shown that the general theory developed for the usual CS still holds in this case (and in fact in any Hidden case). The main output of this fact is that the flows of (2.7) pairwise commute. This allows us to compose different kinds of reduction processes in order to obtain new integrable systems.

First of all let us remark that the submanifold defined by $H^{(0)} = 1$ is left invariant by the flows of the hierarchy so that in the following we can restrict equation (2.7) to this submanifold. Furthermore, it is easy to see that for any n in $\mathcal{N}$ the submanifold $\mathcal{S}_n$ of $\mathcal{L}$ defined by the equations

$$H^{(n)} = z^n \qquad \text{and} \qquad z^n(\mathcal{H}_+) \subset \mathcal{H}_+ \tag{3.1}$$

is invariant with respect equations (2.7). Therefore we can restrict the HCS on $\mathcal{S}_n$ getting a new hierarchy called the HCS_n. Finally it is possible to show that if m is an element of of $\mathcal{N}$ and we denote by x the corresponding time t_m, then any other vector field of HCS induces on the space Q_m of all solutions of the m-th a flow so that we can project on Q_m our HCS getting again a different new hierarchy denoted by HKP_n. Explicitly that means that on Q_m the coefficients of $\{H^{(m+1)}, H^{(m+2)}, \dots\}$ are polynomials in the coefficients of $\left\{H^{(2)}, H^{(3)}, \ldots, H^{(m)}\right\}$ and their x-derivatives. (This procedure has been called in the literature the spatialization of the time t_m.) We can indeed use the equations (2.7) for $j = m$ as recursion relations in the form:

$$H^{(m+k)} = \frac{\partial H^{(k)}}{\partial x} + H^{(m)}H^{(k)} - \sum_{l=1}^{k-2} H_l^m H^{(k-l)} - \sum_{l=1}^{m-2} H_l^k H^{(m-l)} - H_k^m + H_m^k$$
$$- H_{-1}^m H^{(k+1)} - H_{-1}^k H^{(m+1)} - H_{-1}^k H_{-1}^m H^{(2)} - H_{-1}^k H_1^m - H_{-1}^m H_1^k. \tag{3.2}$$

As stated before this two reduction processes can be composed to define the Hidden Gelfand–Dickey hierarchies which we are looking for.

For example if we reduce first HCS on the submanifold HCS_2 and then we spatialize the time $t_3 = x$, by performing the two reduction briefly described above we obtain the first hidden KdV hierarchies studied in [4] and [5]. On the other hand, if we reduce the HCS on the submanifold HCS_3 and spatialize the time $t_2 = x$ we obtain the hidden Boussinesq equation presented in [4]. Here we want to show that this reduction process can be applied also in higher dimensions giving rise to possibly new hidden hierarchies. More precisely let us show how to construct an hidden hierarchy for the GD_4, namely the one corresponding to the restriction to the submanifold S_4 and to the spatialization of the time t_3. It should be noticed that in principle the first hidden hierarchy to be considered for the GD_4 hierarchy is that corresponding to the spatialization of the time t_2 and not the one which we are going to consider. But unfortunately this first hierarchy seems to be more difficult to be handled, for reasons apparently related to the fact that 2 and 4 are not coprime, a problem which arises also in the theory of the fractional hierarchies [3] and [6], and which deserves further investigations.

Let us now return to the case we want present here. The first step in this direction is to restrict the HCS to the submanifold S_4 i.e., to impose the constraints

$$H^{(4)} = z^4, \qquad \frac{\partial H^{(j)}}{\partial t_4} = 0. \tag{3.3}$$

Equations (3.3) allows us to write all the currents in terms of the currents $H^{(2)}$, $H^{(3)}$, $H^{(5)}$.Indeed we have:

$$H^{(4j)} = z^{4j}, \tag{3.4}$$

$$H^{(j)} = p_1^j\left(z^4\right)H^{(5)} + p_2^j\left(z^4\right)H^{(3)} + p_3^j\left(z^4\right)H^{(2)} + p_4^j\left(z^4\right) \quad j \neq 0 \bmod 4, \tag{3.5}$$

where the $p_i^j\left(z^4\right)$ are polynomials in z^4.

We should now spatialize with respect to the time t_3; as briefly explained before, this amounts to take into account equations (3.2). Since we have only three free currents left (namely $H^{(2)}$, $H^{(3)}$, $H^{(5)}$) only the equations with $k = 1, 2, 3$ must be considered. The first one allows us to write $H^{(5)}$ in terms of the currents $H^{(2)} = h = z^2 + h_{-1}z + \sum_{i<0} h_i z^{-i}$ and $H^{(3)} = k = z^3 + k_{-1}z + \sum_{i<0} k_i z^{-i}$ as

$$H^{(5)} = h_x + hk - h_1k - h_3 - k_2 - h_{-1}z^4 - k_{-1}k - k_{-1}k - k_{-1}h_{-1}h - k_{-1}h_1 - k_1h_{-1}. \tag{3.6}$$

The last two give the constraints

$$k_x + k^2 = H^{(6)} + 2k_1h + 2k_3 + 2k_1z^4 + k_1^2h + 2k_{-1}k_1,$$
$$H_x^{(5)} + kH^{(5)} = z^8 + H_1^5h + k_1z^4 + k_2k + h_3h + k_5 + H_3^5$$
$$+ H_{-1}^5z^4 + k_{-1}H^{(6)} + H_{-1}^5k_{-1}h + H_{-1}^5k_1 + k_{-1}H_1^5 \tag{3.7}$$

on the coefficients of h and k. $H^{(5)}$ and its coefficients can be computed by using (3.6), and $H^{(6)}$ is determined by (3.5), which in this explicit case reads

$$H^{(6)} = -h_{-1}H^{(5)} - h_1 k + \left(z^4 - h_2\right) h - h_4. \tag{3.8}$$

To solve equations (3.7) we must decompose them in the powers of z. Then it is possible to prove, after some long computations, that h_3 and k_4 can be written in the terms of h_{-1} h_1, h_2 and k_{-1}, k_1, k_2, k_3, while for $i \geq 6$ they can be written in the normal form

$$\begin{aligned} &2k_i - h_{i+1} = \cdots, \\ &2k_i + h_{i+1} = \cdots. \end{aligned} \tag{3.9}$$

So finally all the coefficients of h and k can be written in terms of the coefficients $h_{-1}, h_1, h_2, h_4;\ k_{-1}, k_1, k_2, k_3, k_5$. Therefore we finally obtain a hierarchy in 9 fields, in an infinite numbers of times, whose explicit form, even for the simplest ones, are too complicate to be written here. What is important here, and this can hold as a final remark, is that the reduction process above described allows us to obtain hidden hierarchies until now apparently not considered in the literature.

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