Journal of Nonlinear Mathematical Physics 2001, V.8, Supplement, 69–78 Proceedings: NEEDS'99

Calogero–Moser Systems and Super Yang–Mills with Adjoint Matter

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Abstract

We review the construction of Lax pairs with spectral parameter for twisted and untwisted elliptic Calogero-Moser systems defined by a general simple Lie algebra $\mathcal{G}$, and the corresponding solution of $\mathcal{N}=2$ SUSY $\mathcal{G}$ Yang-Mills theories with a hypermultiplet in the adjoint representation of $\mathcal{G}$.

1 Introduction

A particularly important development in the theory of integrable models in recent years has been their unexpected close relation with $\mathcal{N}=2$ SUSY Yang–Mills theories. In this talk, we review aspects related to Calogero–Moser systems, which have played a central role.

2 $\mathcal{N}=2$ SUSY Yang–Mills with adjoint matter

In a general $\mathcal{N}=2$ SUSY Yang–Mills theory with gauge group $\mathcal{G}$ in 4 space-time dimensions, the $\mathcal{N}=2$ multiplet of the gauge field A_μ consists of $(A_\mu\ \lambda_{\alpha\pm}\ \phi)$, where $\lambda_{\alpha\pm}$ form a Dirac spinor and ϕ is a complex scalar, all transforming in the adjoint representation of $\mathcal{G}$. $\mathcal{N}=2$ SUSY also allows for additional matter hypermuliplets, of the form $(\psi_{\alpha+}\ H_\pm\ \psi_{\alpha-})$, where $\psi_{\alpha\pm}$ form a Dirac spinor, $H_\pm$ are complex scalars, transforming in a representation R of the gauge group $\mathcal{G}$. In this paper, we shall be mainly concerned with the theories with adjoint matter, that is, the theories where R is the adjoint representation of $\mathcal{G}$. In these theories, the complex gauge coupling $\tau=\frac{4\pi}{g^2}+i\frac{\theta}{2\pi}$ does not get renormalized. Nor does the mass of the hypermultiplet, which we denote by m. We also set $q=e^{2\pi i\tau}$.

The vacua are given by $F_{\mu\nu}=0$, $D_\mu\phi=0$, $[\phi,\phi^\dagger]=0$, so that the theory admits a moduli space of inequivalent vacua of dimension $r=\mathrm{rank}\,\mathcal{G}$. If we let h_j, $j=1,\ldots,r=\mathrm{rank}\,\mathcal{G}$ be a basis for the Cartan subalgebra of $\mathcal{G}$, we can set

$$\langle 0|\phi|0\rangle=\sum_{j=1}^{r}a_jh_j. \tag{2.1}$$

The complex parameters a_j, $j=1,\ldots,r$, are the *quantum moduli* or *order parameters* of the theory.

For generic vacua, the vacuum expectation value has distinct eigenvalues, so that the gauge group is spontaneously broken down to $U(1)^r$ by the Higgs mechanism. Since the $\mathcal{N}=2$ SUSY remains unbroken, the effective theory is a $\mathcal{N}=2$ SUSY theory of r interacting $U(1)$ multiplets $(A_{j\mu}\ \lambda_{j\pm}\ \phi_j)$. The $\mathcal{N}=2$ SUSY constrains the low energy Wilson effective action to be expressible in terms of a single function $\mathcal{F}(\phi,\tau)$ called the prepotential:

$$I = \mathrm{Im}\,(\tau_{ij})F^i_{\mu\nu}F^{j\mu\nu} + \mathrm{Re}\,(\tau_{ij})\,F^i_{\mu\nu}\tilde{F}^{j\mu\nu} + \mathrm{Im}\,(\partial_\mu\bar{\phi}^j\partial^\mu\phi_{Dj}) + \text{fermions}. \tag{2.2}$$

Here the effective couplings $\tau_{ij}(\phi)$ between the $U(1)$ gauge fields are given by $\tau_{ij}=\frac{\partial^2\mathcal{F}}{\partial\phi^i\phi^j}$, and $\phi_{Dj}=\frac{\partial\mathcal{F}}{\partial\phi^j}$. In the large ϕ regime, perturbative quantum field methods show that $\mathcal{F}$ is of the form

$$\mathcal{F} = \tau\sum_{j=1}^r \phi_j^2 + \sum_{\alpha\in\mathcal{R}(\mathcal{G})}\left\{\ln(\alpha\cdot\phi)^2 - \ln(\alpha\cdot\phi+m)^2\right\} + \sum_{n=1}^\infty q^n\mathcal{F}^{(n)}. \tag{2.3}$$

The terms on the right correspond respectively to the classical prepotential, the one-loop perturbative contributions, and the instanton contributions. The set $\mathcal{R}(\mathcal{G})$ denotes the roots of $\mathcal{G}$. The key Ansatz due to Seiberg and Witten [1] is that $\mathcal{F}$ can be determined in the following manner from a fibration of Riemann surfaces Γ over the moduli space of vacua, equipped with a meromorphic 1-form $d\lambda$

$$a_j = \frac{1}{2\pi i}\oint_{A_j} d\lambda, \qquad a_{Dj} = \frac{1}{2\pi i}\oint_{B_j} d\lambda, \qquad a_{Dj} = \frac{\partial\mathcal{F}}{\partial a_j}. \tag{2.4}$$

Here A_j and B_j are suitable homology cycles on the surface Γ. The Seiberg–Witten Ansatz gives rise to a symplectic structure $\omega=\delta\left(\sum_{j=1}^r d\lambda(z_j)\right)$ on (a subspace of) the Jacobian of Γ, with respect to which the vacuum moduli of the gauge theory are commuting Hamiltonians [2]. Thus to the 4-dimensional gauge theory corresponds an integrable model. The main problem is to determine which theory corresponds to which model. For the $SU(N)$ theory with adjoint matter, Donagi and Witten [2] have produced strong evidence that the corresponding integrable model is the $SU(N)$ Hitchin system [3]. Our goal here is to describe the results in [5, 6, 7, 8], which show that for general gauge group $\mathcal{G}$ given by a simple Lie algebra, the integrable model corresponding to the $\mathcal{N}=2$ SUSY $\mathcal{G}$ Yang–Mills theory with matter in the adjoint representation is the *twisted elliptic Calogero–Moser system associated to* $\mathcal{G}$. This is in accord with the Donagi–Witten proposal, since the spectral curves of the $SU(N)$ Hitchin system and the $SU(N)$ twisted elliptic Calogero–Moser system are the same.

3 Twisted elliptic Calogero–Moser systems

The original elliptic Calogero–Moser system is the N particle dynamical system with potential $m^2\sum_{i\neq j}\wp(x_i-x_j)$ [11]. Olshanetsky and Perelomov [12] recognized very early on that the system can actually be generalized to any simple Lie algebra, with potential $\sum_{\alpha\in\mathcal{R}(\mathcal{G})} m^2_{|\alpha|}\wp(\alpha\cdot x)$. It turns out that for the purpose of studying gauge theories, we need rather a new twisted version of these systems, first introduced in [5] and with Hamiltonian

$$H^{\text{twisted}}_{\mathcal{G}} = \frac{1}{2}\sum_{j=1}^r p_j^2 - \frac{1}{2}\sum_{\alpha\in\mathcal{R}(\mathcal{G})} m^2_{|\alpha|}\wp_{\nu(\alpha)}(\alpha\cdot x). \tag{3.1}$$

Here $m_{|\alpha|}$ are mass parameters depending only on the length of the root α, $\nu(\alpha) \in \{1,2,3\}$ is the relative ratio of the roots of $\mathcal{G}$, and $\wp_\nu(z)$ is the following twisted version of the Weierstrass $\wp$-function

$$\wp_\nu(z) = \sum_{k=0}^{\nu-1} \wp\left(z + 2\omega_a \frac{k}{\nu}\right), \tag{3.2}$$

where ω_a is a half-period of $\wp(z)$. The twisted system coincides of course with the usual system for $\mathcal{G}$ simply laced.

4 The $SU(N)$ Calogero–Moser system

We begin by discussing the basic case of $SU(N)$. The starting point is the Lax pair $L(z)$, $M(z)$ with spectral parameter obtained in 1980 by Krichever [13] for the elliptic Calogero–Moser system

$$L(z) = p_i\delta_{ij} - m(1-\delta_{ij})\Phi(x_i - x_j, z), \tag{4.1}$$

$$M(z) = m\delta_{ij}\sum_{k\neq i}\wp(x_i - x_k) + m(1-\delta_{ij})\Phi'(x_i - x_j, z). \tag{4.2}$$

The function $\Phi(x,z)$ is the Lamé function, defined by $\Phi(x,z) = \frac{\sigma(x-z)}{\sigma(x)\sigma(z)}e^{x\zeta(z)}$, where $\sigma(z)$, $\zeta(z)$ are the usual Weierstrass elliptic functions. Given a Lax pair with spectral parameter, the spectral curve Γ and the differential $d\lambda$ can be defined by $\Gamma = \{(k,z); \det(kI - L(z)) = 0\}$ and $d\lambda = kdz$. The main difficulty in establishing a correspondence between this model and the $SU(N)$ gauge theory is how to identify the vacua moduli in the context of Calogero–Moser systems. This is overcome by the following key fact [4], which provides a new parametrization of spectral curves for the $SU(N)$ Calogero–Moser systems

$$\det(\lambda I - L(z)) = \frac{\vartheta_1\left(\frac{1}{2\omega_1}\left(z - m\frac{d}{dk}\right)|\tau\right)}{\vartheta_1\left(\frac{z}{2\omega_1}|\tau\right)} H(k)\Big|_{k=\lambda+m\partial_z \ln\vartheta_1\left(\frac{z}{2\omega_1}|\tau\right)} \tag{4.3}$$

Here $H(k) = \prod_{j=1}^{N}(k-k_j)$ is a monic polynomial. Its zeroes k_j are manifestly integrals of motion for the Calogero–Moser system. On the other hand, they can be shown to be classical limits of the vacua moduli for the gauge theory. Once the classical vacua moduli have been identified, we can apply the methods of [18] to determine the logarithmic singularities of the prepotential from the Calogero–Moser system. We find agreement with the one-loop contributions in (2.3), which is the main criterion for the correct integrable model. A complete correspondence can now be summarized in the following table:

The function $F(y)$ in the table is defined by

$$F(y) = \sum_{n=1}^{\infty} q^{\frac{1}{2}n(n+1)}(-)^n\left[y^{-n}\eta_n(k,\beta) - y^{n+1}\eta_n(k-\beta m, -\beta)\right], \qquad \beta = -\frac{i\pi}{\omega_1}, \tag{4.4}$$

$$\eta_n(k,\beta) = \frac{H(k+\beta mn)H(k-\beta m)^n}{H(k)^{n+1}}. \tag{4.5}$$

$\mathcal{N}=2$ SUSY Gauge Theory	Calogero–Moser
Γ	$\det(kI - L(z)) = 0$
$d\lambda$	kdz
$\lim_{q\to 0} \frac{1}{2\pi i}\oint_{A_j} d\lambda$	$k_j - \frac{1}{2}m$
$a_j = \frac{1}{2\pi i}\oint_{A_j} d\lambda$	$a_j = \frac{1}{2\pi i}\oint_{A_j} k\, d\ln \frac{H(k)}{H(k-m)}\left(1+\sum_{n=1}^{\infty}\frac{q^n}{n!}\frac{\partial^n}{\partial y^n}F^n(y)_{\vert y=1}\right)$
Renormalization Group Equation Beta Function	$\frac{\partial \mathcal{F}}{\partial \tau} = TrL(z)^2$ Hamiltonian $\mathrm{Tr}\, L(z)^2$

Table 1. Gauge Theory and Calogero–Moser Correspondence for $SU(N)$

Note that the expressions of a_j in terms of k_j can be inverted recursively. It follows that the renormalization group equation in table 1 allows us to solve for instanton contributions to all orders. The decoupling limits as $m \to \infty$ (see below) also agree with all instanton contributions computed by other methods [18, 19]. We observe also that the integrals of motion k_j can be expressed explicitly in terms of Calogero–Moser dynamical variables p, x [8]. This requires some remarkable identities relating $\Phi(x, z)$ to free fermion determinants [10].

5 The case of $\mathcal{G}$ general simple Lie algebra

5.1 Scaling limits of Calogero–Moser systems

We begin by explaining why the Calogero–Moser systems have to be twisted. As the hypermultiplet becomes infinitely heavy, it decouples and the gauge theory should reduce to a pure $\mathcal{N}=2$ SUSY Yang–Mills theory without additional matter hypermultiplet. Now standard renormalization group arguments show that the gauge coupling τ and the mass should be related by

$$m = Mq^{-\frac{1}{2}\delta_{\mathcal{G}}^{\vee}}, \tag{5.1}$$

where M is fixed, and $\frac{1}{\delta_{\mathcal{G}}^{\vee}}$ is the quadratic Casimir of $\mathcal{G}$, or equivalently the dual Coxeter number of $\mathcal{G}$. It turns out that for $\mathcal{G}$ not simply laced, the finiteness of the scaling limits under (5.1) of the $\mathcal{G}$ elliptic Calogero–Moser system is not compatible with other requirements such as the retention of simple roots. The point of (3.2) is the following improvement in the asymptotics for the $\wp$-function, $\wp_\nu(x) = \frac{\nu^2}{2}\sum_{n=-\infty}^{\infty}\frac{1}{\mathrm{ch}\,\nu(x-2n\omega_2)-1}$, thanks to which

$$H_{\mathcal{G}}^{\mathrm{twisted}} \to H^{\mathrm{Toda}}_{\left(\mathcal{G}^{(1)}\right)^{\vee}} \equiv \frac{1}{2}p^2 - M^2 \sum_{\alpha\in\mathcal{R}*\left(\left(\mathcal{G}^{(1)}\right)^{\vee}\right)} e^{-\alpha\cdot x} \tag{5.2}$$

when $x = X + 2\omega_2 \frac{1}{h_{\mathcal{G}}^{\vee}}\rho$, $\rho = \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+(\mathcal{G})}\alpha$ is the Weyl vector, and $q \to 0$, $m \to \infty$ as in (5.1). Since the Toda system defined by the affine algebra $\left(\mathcal{G}^{(1)}\right)^{\vee}$ is the integrable model corresponding to the pure Yang–Mills theory [14], the above limit is a crucial consistency requirement. The usual elliptic Calogero–Moser systems also have a finite scaling limit

$$H_{\mathcal{G}} \to H^{\rm Toda}_{\mathcal{G}^{(1)}} \equiv \frac{1}{2}p^2 - M^2 \sum_{\alpha\in\mathcal{R}*(\mathcal{G}^{(1)})} e^{-\alpha\cdot x} \tag{5.3}$$

but under the scaling law $x = X + 2\omega_2 \frac{1}{h_{\mathcal{G}}}\rho^\vee$, $\rho^\vee = \frac{1}{2}\sum_{\alpha^\vee\in\mathcal{R}_+(\mathcal{G})^\vee} \alpha^\vee$ is the level vector, and $q \to 0$, $m = Mq^{-\frac{1}{2}\delta_{\mathcal{G}}} \to \infty$ with M fixed and $\frac{1}{\delta_{\mathcal{G}}}$ equal to the Coxeter number $h_{\mathcal{G}}$ [6].

5.2 Lax pairs for $\mathcal{G}$ general simple algebra

Lax pairs with spectral parameter were known only for $SU(N)$ (Olshanetsky and Perelomov [12] had constructed Lax pairs for the classical Lie algebras in the untwisted case of Calogero–Moser systems, but these were without spectral parameter). Our general Ansatz for Lax pairs with spectral parameter for both twisted and untwisted Calogero–Moser systems is as follows.

Let $\mathcal{L}$ be a representation of $\mathcal{G}$ of dimension N, of weights λ_I, $1 \le I \le N$. We fix a Cartan subalgebra $\mathcal{H}$ for $GL(N, \mathbf{C})$ which contains the Cartan subalgebra $\mathcal{H}_{\mathcal{G}}$ of $\mathcal{G}$. A basis for $\mathcal{H}$ consists then of a basis h_i, $1 \le i \le r$, of Cartan generators for $\mathcal{G}$, together with a complementary set $\tilde h_i$, $r+1 \le i \le N$, of generators of $GL(N, \mathbf{C})$ satisfying $[h_i, h_j] = [h_i, \tilde h_j] = [\tilde h_i, \tilde h_j] = 0$. By adding to the $\tilde h_i$ suitable linear combinations of the h_i, we may assume that the h_i and $\tilde h_i$ are mutually orthogonal with respect to the Cartan–Killing form $\mathrm{tr}\,[\mathrm{Ad}_{h_i}, \mathrm{Ad}_{\tilde h_j}] = 0$.

Let $u_I \in \mathbf{C}^N$ be the weights of the fundamental representation of $GL(N, \mathbf{C})$. Project orthogonally the u_I's onto the λ_I's as

$$su_I = \lambda_I + v_I, \qquad \lambda_I \perp v_J. \tag{5.4}$$

The coefficient s is easily determined to be $s^2 = \frac{1}{r}\sum_{I=1}^{N} \lambda_I^2 = I_2(\Lambda)$, where $I_2(\Lambda)$ is by definition the second Dynkin index of the representation Λ of $\mathcal{G}$. We note that $\alpha_{IJ} = \lambda_I - \lambda_J$ is a weight of $\mathcal{L}\otimes\mathcal{L}^*$ associated to the root $u_I - u_J$ of $GL(N, \mathbf{C})$. The centralizer of $\mathcal{H}_{\mathcal{G}}$ in $GL(N, \mathbf{C})$ may be larger that the Cartan subalgebra $\mathcal{H}$ of $GL(N, \mathbf{C})$. We denote it by $\mathcal{H}\oplus GL_0$. For all simple Lie algebras in their lowest dimensional faithful representation, we have $GL_0 = 0$, except in the cases of F_4 and E_8, where the dimension of GL_0 is 2 and 56 respectively.

The Lax pairs for both untwisted and twisted Calogero–Moser systems will be of the form

$$L = P + X, \qquad M = D + Y, \tag{5.5}$$

where the matrices P, X, D and Y are given by

$$X = \sum_{I\ne J} C_{IJ}\Phi_{IJ}(\alpha_{IJ}, z)E_{IJ}, \qquad Y = \sum_{I\ne j} C_{IJ}\Phi'_{IJ}(\alpha_{IJ}, z)E_{IJ}, \tag{5.6}$$

$$P = p\cdot h, \qquad D = d\cdot(h\oplus\tilde h) + \Delta. \tag{5.7}$$

Here Δ is an element of GL_0, and $E_{IJ} = u_I u_J^T$ are the usual generators of $GL(N, \mathbf{C})$. The functions $\Phi_{IJ}(x,z)$ and the coefficients C_{IJ} are yet to be determined. It is convenient to introduce the following notation

$$\Phi_{IJ} = \Phi_{IJ}(\alpha_{IJ}\cdot x), \qquad \Phi'_{IJ} = \Phi'_{IJ}(\alpha_{IJ}\cdot x), \tag{5.8}$$

$$\wp'_{IJ} = \Phi_{IJ}(\alpha_{IJ}\cdot x, z)\Phi'_{JI}(-\alpha_{IJ}\cdot x, z) - \Phi'_{IJ}(\alpha_{IJ}\cdot x, z)\Phi'_{JI}(-\alpha_{IJ}\cdot x, z). \tag{5.9}$$

Then the Lax equation $\dot{L}(z) = [L(z), M(z)]$ implies the (twisted or untwisted) Calogero–Moser system if and only if the following three identities are satisfied

$$\sum_{I\neq J} C_{IJ}C_{JI}\wp'_{IJ}\alpha_{IJ} = s^2 \sum_{\alpha\in\mathcal{R}(\mathcal{G})} m^2_{|\alpha|}\wp_{\nu(\alpha)}(\alpha\cdot x), \tag{5.10}$$

$$\sum_{I\neq J} C_{IJ}C_{JI}\wp'_{IJ}(v_I - v_J) = 0, \tag{5.11}$$

$$\sum_{K\neq I,J} C_{IK}C_{KJ}(\Phi_{IK}\Phi'_{KJ} - \Phi'_{IK}\Phi_{KJ}) = sC_{IJ}\Phi_{IJ} d\cdot(u_I - u_J) + \sum_{K\neq I,J} \Delta_{IJ}C_{KJ}\Phi_{KJ} - \sum_{K\neq I,J} C_{IK}\Phi_{IK}\Delta_{KJ}. \tag{5.12}$$

Theorem. *A representation Λ, functions Φ_{IJ}, and coefficients C_{IJ} with a spectral parameter z satisfying (5.10), (5.11), (5.12) can be found for all twisted and untwisted elliptic Calogero–Moser systems associated with a simple Lie algebra $\mathcal{G}$, except possibly in the case of twisted G_2. In the case of E_8, we have to assume the existence of a ± 1 cocycle.*

Some important features of the Lax pairs we obtain in this manner for the untwisted Calogero–Moser systems are the following.

- In the case of the *untwisted* Calogero–Moser systems, we can choose $\Phi_{IJ}(x,z) = \Phi(x,z)$, $\wp_{IJ}(x) = \wp(x)$ for all $\mathcal{G}$.
- $\Delta = 0$ for all $\mathcal{G}$, except for E_8.
- For A_n, the Lax pair (5.5) corresponds to the choice of the fundamental representation for $\mathcal{L}$. A different Lax pair can be found by taking $\mathcal{L}$ to be the antisymmetric representation.
- For the BC_n system, the Lax pair is obtained by imbedding B_n in $GL(N, \mathbf{C})$ with $N = 2n+1$. When $z = \omega_a$ (half-period), the Lax pair obtained this way reduces to the Lax pair obtained by Olshanetsky and Perelomov [12].
- For the B_n and D_n systems, additional Lax pairs with spectral parameter can be found by taking $\mathcal{L}$ to be the spinor representation.
- For G_2, a first Lax pair with spectral parameter can be obtained by the above construction with $\mathcal{L}$ chosen to be the **7** of G_2. A second Lax pair with spectral parameter can be obtained by restricting the **8** of B_3 to the $\mathbf{7}\oplus\mathbf{1}$ of G_2.
- For F_4, a Lax pair can be obtained by taking $\mathcal{L}$ to be the $\mathbf{26}\oplus\mathbf{1}$ of F_4, viewed as the restriction of the **27** of E_6 to its F_4 subalgebra.
- For E_6, $\mathcal{L}$ is the **27** representation.

- For E_7, $\mathcal{L}$ is the **56** representation.
- For E_8, a Lax pair with spectral parameter can be constructed with $\mathcal{L}$ given by the **248** representation, if coefficients $c_{IJ} = \pm 1$ exist with the following cocycle conditions

$$c(\lambda, \lambda-\delta)c(\lambda-\delta, \mu) = c(\lambda, \mu+\delta)c(\mu+\delta, \mu) \qquad \text{when} \quad \begin{array}{l} \delta \cdot \lambda = -\delta \cdot \mu = 1, \\ \lambda \cdot \mu = 0, \end{array} \tag{5.13}$$

$$c(\lambda, \mu)c(\lambda-\delta, \mu) = c(\lambda, \lambda-\delta) \qquad \text{when} \ \delta\cdot\lambda = \lambda\cdot\mu = 1, \ \delta\cdot\mu = 0, \tag{5.14}$$

$$c(\lambda, \mu)c(\lambda, \lambda-\mu) = -c(\lambda-\mu, -\mu) \qquad \text{when} \ \lambda\cdot\mu = 1. \tag{5.15}$$

The matrix Δ in the Lax pair is then the 8×8 matrix given by

$$\begin{aligned} \Delta_{ab} = & \sum_{\substack{\delta\cdot\beta_a=1 \\ \delta\cdot\beta_b=1}} \frac{m_2}{2}\big(c(\beta_a, \delta)c(\delta, \beta_b) + c(\beta_a, \beta_a - \delta)c(\beta_a - \delta, \beta_b)\big)\wp(\delta \cdot x) \\ & - \sum_{\substack{\delta\cdot\beta_a=1 \\ \delta\cdot\beta_b=-1}} \frac{m_2}{2}\big(c(\beta_a, \delta)c(\delta, \beta_b) + c(\beta_a, \beta_a - \delta)c(\beta_a - \delta, \beta_b)\big)\wp(\delta \cdot x), \\ \Delta_{aa} = & \sum_{\beta_a\cdot\delta=1} m_2\wp(\delta \cdot x) + 2m_2\wp(\beta_a \cdot x), \end{aligned} \tag{5.16}$$

where β_a, $1 \le a \le 8$, is a maximal set of 8 mutually orthogonal roots.

Explicit expressions for the constants C_{IJ} and the functions $d(x)$, and thus for the Lax pair are particularly simple when the representation Λ consists of only a single Weyl orbit of weights. This is the case when Λ is either

- the defining representation of A_n, C_n or D_n;
- any rank p totally anti-symmetric representation of A_n;
- an irreducible fundamental spinor representation of B_n or D_n;
- the **27** of E_6; the **56** of E_7.

For the twisted elliptic Calogero–Moser systems, we have:

- For B_n, the Lax pair is of dimension $N = 2n$, admits two independent couplings m_1 and m_2, and

$$\Phi_{IJ}(x, z) = \begin{cases} \Phi(x, z), & \text{if } I - J \neq 0, \pm n; \\ \Phi_2\left(\dfrac{1}{2}x, z\right), & \text{if } I - J = \pm n. \end{cases} \tag{5.17}$$

Here a new function $\Phi_2(x, z)$ is defined by

$$\Phi_2\left(\frac{1}{2}x, z\right) = \frac{\Phi\left(\frac{1}{2}x, z\right)\Phi\left(\frac{1}{2}x + \omega_1, z\right)}{\Phi(\omega_1, z)}. \tag{5.18}$$

- For C_n, the Lax pair is of dimension $N = 2n + 2$, admits one independent coupling m_2, and

$$\Phi_{IJ}(x,z) = \Phi_2(x+\omega_{IJ}, z), \tag{5.19}$$

where ω_{IJ} are given by

$$\omega_{IJ} = \begin{cases} 0, & \text{if } I \neq J = 1, 2, \cdots, 2n+1; \\ \omega_2, & \text{if } 1 \leq I \leq 2n,\ J = 2n+2; \\ -\omega_2, & \text{if } 1 \leq J \leq 2n,\ I = 2n+2. \end{cases} \tag{5.20}$$

- For F_4, the Lax pair is of dimension $N = 24$, two independent couplings m_1 and m_2,

$$\Phi_{\lambda\mu}(x,z) = \begin{cases} \Phi(x,z), & \text{if } \lambda \cdot \mu = 0; \\ \Phi_1(x,z), & \text{if } \lambda \cdot \mu = \frac{1}{2}; \\ \Phi_2\left(\frac{1}{2}x, z\right), & \text{if } \lambda \cdot \mu = -1. \end{cases} \tag{5.21}$$

where the function $\Phi_1(x,z)$ is defined by

$$\Phi_1(x,z) = \Phi(x,z) - e^{\pi i \zeta(z) + \eta_1 z} \Phi(x+\omega_1, z) \tag{5.22}$$

Here it is more convenient to label the entries of the Lax pair directly by the weights $\lambda = \lambda_I$ and $\mu = \lambda_J$ instead of I and J.

- For G_2, candidate Lax pairs can be defined in the **6** and **8** representations of G_2, but it is still unknown whether elliptic functions $\Phi_{IJ}(x,z)$ exist which satisfy the required identities.

We observe that other Lax pairs have subsequently been proposed in [15]. However, they do not have finite limits under the scaling law (5.1), and hence do not appear as relevant to $\mathcal{N} = 2$ super Yang–Mills.

5.3 Spectral curves for super Yang–Mills

We obtain now the Seiberg–Witten curves for the $\mathcal{L} = 2$ SUSY $\mathcal{G}$ gauge theory with adjoint matter by setting $\Gamma = \{(k,z) \in \mathbf{C} \times \Sigma; \det(kI - L(z)) = 0\}$, $d\lambda = kdz$, where $L(z), M(z)$ is the Lax pair of the *twisted* elliptic Calogero–Moser system associated to the Lie algebra $\mathcal{G}$ [7].

This prescription satisfies all the available consistency requirements. In particular, the spectral curve Γ is invariant under the Weyl group of $\mathcal{G}$, the position and residues of the poles of $d\lambda$ are independent of the moduli, and the residues are linear in the hypermultiplet mass m. The logarithmic singularities of the prepotential are more difficult to check in general, but they can be seen to be verified in the case of D_r, at least in the trigonometric limit, which is dependable when $r \leq 5$. More precisely, in the trigonometric limit $\tau \to i\infty$, change from variables z, k to new variables Z, A by

$$\frac{1}{Z} = \frac{1}{2}\coth\frac{z}{2}, \qquad 0 = A^2 + mA + 2k\frac{m}{Z} - k^2. \tag{5.23}$$

Then the equation $R(k,z) = \det(kI - L(z))$ for the spectral curve can be expressed in terms of a polynomial $H(A)$ just as in the case of $SU(N)$

$$R(k,z) = \frac{m^2 + mA - 2k\frac{m}{Z}}{m^2 + mA} H(A) + \frac{mA + 2k\frac{m}{Z}}{m^2 + 2mA} H(A+m). \tag{5.24}$$

The methods of [4] can now be applied as in the $SU(N)$ case to show that the prepotential obtained from our construction has the desired logarithmic singularities.

Finally, we note that Seiberg–Witten curves for many models have also been obtained from M Theory [16] as well as geometric engineering [17].

Acknowledgements

This work was supported in part by National Science Foundation grants NSF-PHY-9531023 and DMS-9800783.

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