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Inhomogeneous Burgers Lattices

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Abstract

We study statistical properties of inhomogeneous Burgers lattices which are solved by the discrete Cole–Hopf transformation. Using exact solutions we investigate effect of various kinds of noise on the dynamics of solutions.

1 Introduction

During the last few years differential-difference nonlinear equations (lattices) became a subject of great interest in the nonlinear physics. The first (to the best of authors knowledge) integrable discretization of a dissipative system, the Burgers equation, has been reported in Ref. [1], where using the representation in a Lax form [2] a hierarchy of nonlinear matrix lattices which are linearized by the discrete Cole–Hopf transformation has been introduced. The mentioned hierarchy includes, as a particular case, equations with n-dependent coefficients. Further development of a theory of the discretized Burgers equation has received in the paper [3].

On the other hand in the continuum limit the forced Burgers equation is a general model describing the evolution of nonlinear diffusive systems under the influence of an external driver. In particular, the case of stochastic forcing is relevant in physical applications as a model for the time evolution of the profile of a growing interface [4]. In this context several studies have been reported for the case of additive noise [5] and multiplicative noise [6].

In the present paper we study inhomogeneous Burgers lattices, i.e. differential difference equations the homogeneous counterpart of which in the continuous limit is reduced to the conventional Burgers equation

$$\partial_\tau w(x,\tau) = \partial_x^2 w(x,\tau) + 2w(x,\tau)\partial_x w(x,\tau) \tag{1}$$

or to one of its modifications including integrable inhomogeneous terms.

2 Burgers lattices

Let us consider a differential-difference equation

$$\frac{du_n}{dt} = u_{n+1} + u_{n-1} - 2u_n + f_n(t), \tag{2}$$

where

$$f_n(t) = \sum_{k=1}^{N} a_n^{(k)}(t) u_{n+k} \tag{3}$$

and $a_n^{(k)}$ are arbitrary functions on n and t. Then the relation

$$v_n = \frac{u_{n+1}}{u_n} \tag{4}$$

can be interpreted as a discrete Cole–Hopf transformation [1]. The so introduced function v_n solves the differential difference equation

$$\frac{dv_n}{dt} = v_n(v_{n+1} - v_n) + \left(1 - \frac{v_n}{v_{n-1}}\right) + F_n(t), \tag{5}$$

where

$$F_n(t) = \sum_{k=1}^{N} \left[a_{n+1}^{(k)} v_{n+k} - a_n^{(k)} v_n\right] v_{n+k-1} \dots v_n. \tag{6}$$

In what follows equation (5) is refered to as an inhomogeneous Burgers lattice. The name becomes clear from the continuum limit of (5) which is obtained as the limit $\epsilon \to 0$ with help of the substitution $v_n = 1 + \epsilon w\left(\epsilon n, \epsilon^2 t\right)$.

In order to construct the whole hierarchy of inhomogeneous nonlinear lattices, which can be linearized by means of Cole–Hopf transformation, we use the method of Ref. [1] and introduce the operators

$$\mathbf{J}\phi(n) = [\phi(n+1) - \phi(n)]v_n, \tag{7}$$

$$\mathbf{L}\phi(n) = \phi(n+1)v_n, \qquad \tilde{\mathbf{L}}\phi(n) = \frac{\phi(n-1)}{v_{n-1}}. \tag{8}$$

Then the hierarchy of the nonlinear lattices is given by the formula

$$\frac{dv_n}{dt} = \mathbf{J} \sum_{k=0}^{N} \left(\mathbf{L}^k \alpha_n^{(k)}(t) + \tilde{\mathbf{L}}^k \alpha_n^{(k)}(t)\right), \tag{9}$$

where $\alpha_n^{(k)}(t)$ and $\tilde{\alpha}_n^{(k)}(t)$ are arbitrary functions of n and t.

The proof of the possibility to linearize (9) by means of the Cole–Hopf transformation (4) is similar to one given in [1]. It is based on the properties

$$\mathbf{L}^k \alpha_n^{(k)}(t) = \alpha_{n+k}^{(k)}(t) \frac{u_{n+k}}{u_n}, \qquad \tilde{\mathbf{L}}^k \tilde{\alpha}_n^{(k)}(t) = \tilde{\alpha}_{n+k}^{(k)}(t) \frac{u_{n-k}}{u_n}. \tag{10}$$

Thus the functions $u(n)$ solve the equation

$$\frac{du(n)}{dt} = \sum_k \left[\alpha_{n+k}^{(k)}(t) u_{n+k} + \tilde{\alpha}_{n-k}^{(k)}(t) u_{n-k}\right]. \tag{11}$$

It is interesting to mention that it follows from the above analysis that in the homogeneous case, $\alpha_n^{(k)}(t) = \tilde{\alpha}_n^{(k)}(t) = \delta_{k,1}$, we arrive at the discrete Burgers equation in the form

$$\frac{dv_n}{dt} = 1 - \frac{v_n}{v_{n-1}}. \tag{12}$$

By the substitution $\tilde{v}_n = 1/v_n$ it is reduced to[1]

$$\frac{d\tilde{v}_n}{dt} = 2\tilde{v}_n(\tilde{v}_{n+1} - \tilde{v}_n). \tag{13}$$

The respective continuum limit is provided by the ansatz $v_n = 1 + \epsilon w\left(\epsilon(n-t), \epsilon^2 t\right)$.

Let us consider a shock wave solution of (2)

$$u_n = \exp(\lambda_1 t + k_1 n) + \exp(\lambda_2 t + k_2 n), \tag{14}$$

where $\lambda_j = 4\sin^2(k_j/2)$, and can be written down in the form

$$v_n = V + \Delta \tanh[(\beta(n - \mathrm{v}t)], \tag{15}$$

where $V = (1/2)(\exp k_1 + \exp k_2)$, $\Delta = (1/2)(\exp k_1 - \exp k_2)$, $\mathrm{v} = (\lambda_2 - \lambda_1)/(k_1 - k_2)$, and $\beta = (1/2)(k_1 - k_2)$.

3 Discrete shock waves affected by multiplicative noise

Consider an inhomogeneous Burgers lattice in the form

$$\frac{dv_n}{dt} = v_n(v_{n+1} - v_n) + \left(1 - \frac{v_n}{v_{n-1}}\right) + f(t)(A_{n+1} - A_n)v_n, \tag{16}$$

where $f(t)$ is a random function. This model is linearized by the discrete Cole–Hopf transformation with the function u_n solving the equation

$$\frac{du_n}{dt} = u_{n+1} + u_{n-1} - 2u_n + f(t)A_n u_n. \tag{17}$$

In the present section we concentrate on a particular case $A_n = n$ which corresponds to the Burgers lattice with the multiplicative noise. The respective shock wave solution reads

$$v_n = (V + \Delta \tanh\{\beta[n - X(t)]\})\, e^{F(t)}, \tag{18}$$

where $X(t) = \frac{2}{\beta}[\gamma_1(t) - \gamma_2(t)]$,

$$\gamma_j(t) = \int_0^t \cosh[F(t') + k_j]\, dt', \qquad \text{and} \qquad F(t) = \int_0^t f(t')\, dt'. \tag{19}$$

Due to the random noise, the shock solution (4) experiences fluctuations. It is therefore of interest to study statistical properties related to the motion of the shock (such as, for example, mean displcement, mean velocity, etc.) and also some related correlation functions. To this end we choose $f(t)$ be a Gaussian random process characterized by the zero average $\langle f(t)\rangle = 0$ and by the two-time correlator

$$\langle f(t)f(t')\rangle = B(t - t')\frac{D}{2\tau}\exp\left(-\frac{|t - t'|}{\tau}\right) \tag{20}$$

which describes a weakly correlated noise. In (20) D is the noise intensity and τ is the correlation time of the noise. In the limit $\tau \to 0$ (D fixed) the right hand side of (20) reduces to the δ-function (white noise).

We now turn our attention to the statistical properties of the shock motion, and to their long time behaviour. We first observe that due to the multiplicative noise the mean value of the background exhibits an exponential growth given by

$$\langle v(t)\rangle = (V \pm \Delta)\exp\left(\frac{1}{2}\langle F^2(t)\rangle\right) \tag{21}$$

with

$$\langle F^2(t)\rangle = D\left[t + \tau\left(e^{-t/\tau} - 1\right)\right]. \tag{22}$$

The exponential growth of the background induces a similar growth also in the dynamical variables characterising the shock motion.

We start with the mean displacement which assymptotics at $t \to \infty$ is given by (the higher order turms are dropped)

$$\langle X(t)\rangle = \frac{2A(k_1,k_2)}{\beta D}\exp\left[\frac{1}{2}D(t-\tau)\right], \tag{23}$$

where $A(k_1,k_2) = \cosh k_1 - \cosh k_2$. Similarly we compute the mean velocity and mean acceleration of the shock wave

$$\langle \dot{X}(t)\rangle = \frac{A(k_1,k_2)}{\beta}\exp\left[\frac{1}{2}D(t-\tau)\right], \tag{24}$$

$$\langle \ddot{X}(t)\rangle = \frac{DA(k_1,k_2)}{2\beta}\exp\left[\frac{1}{2}D(t-\tau)\right]. \tag{25}$$

Thus all the mean value exhibit an exponential grows and in particular we observe a stochastic acceleration of the shock. The accelerated motion takes place along the positive or negative direction of the lattice according to the sign of $A(k_1,k_2)$. The mentioned exponential behaviour results in anomalous diffusion of the shock which in the assymptotic region ($t \to \infty$) is characterised by the mean square displacement

$$\langle X^2(t)\rangle = \cosh(k_1+k_2)[\cosh(k_1-k_2)-1]\frac{2}{D}\exp\left[2D(t-\tau)\right]. \tag{26}$$

Let us now consider the two time correlation between the shock displacement and the noise given by

$$\langle X(t_1)f(t_2)\rangle = \frac{B(k_1,k_2)}{\beta}\int_0^{t_1}\langle F(t')f(t_2)\rangle\exp\left[\frac{1}{2}\langle F^2(t')\rangle\right]dt' \tag{27}$$

with $B(k_1,k_2) = \sinh k_1 - \sinh k_2$.

In order to evaluate the correlator in the integrand of (27) we have to distinguish the the cases $t_2 > t_1$ and $t_2 < t_1$. In the former case we get

$$\langle F(t')f(t_2)\rangle = \frac{D}{2}e^{-t_2/\tau}\left(e^{t'/\tau} - 1\right) \tag{28}$$

which when substituted back to (27) yields (for $t_2 \to \infty$ and t_1 fixed) the assymptotic behaviour

$$\langle X(t_1)f(t_2)\rangle = \frac{\Delta B(k_1,k_2)}{2\beta}e^{-t_2/\tau}e^{-D\tau/2}\mathcal{A}(t_1,D,\tau), \tag{29}$$

where $\mathcal{A}(t_1, D, \tau)$ is a constant for t_1 fixed. Relation (29) describes a correlation assymptotically vanishing as t_2 grows. Moreover as $\tau \to 0$ a correlation is identically zero (as it is expected from the causality principle).

In the case $t_1 > t_2$ instead, we find in the limit $t_1 \to \infty$ (t_2 fixed) an assymptotically growing correlator

$$\langle X(t_1) f(t_2) \rangle = \frac{B(k_1, k_2)}{\beta} (2 + D\tau) \exp\left[\frac{1}{2} D(t_1 - \tau)\right]. \tag{30}$$

4 Shock wave affected by nonlinear multiplicative noise

Let us now consider one more model with a sorce allowing linearization by the Cole–Hopf transformation. It reads

$$\frac{dv_n}{dt} = (1 + f(t)) v_n (v_{n+1} - v_n) + \left(1 - \frac{v_n}{v_{n-1}}\right). \tag{31}$$

The respective linear equation reads

$$\frac{du_n}{dt} = u_{n+1} + u_{n-1} - 2u_n + f(t) u_{n+1}. \tag{32}$$

The physical meaning of the introduced model becomes clear from its continuum limit, assuming that $f(t) = \epsilon F(\tau)$:

$$\partial_\tau u = 2u \partial_x u + \partial_x^2 u + F(\tau) \partial_x u. \tag{33}$$

This continuum model has been studied in [6].

Equation (31) admits the shock solution

$$v_n(t) = V + \Delta \tanh[\beta(n - X(t))] \tag{34}$$

with

$$X(t) = vt - \tilde{X}(t), \qquad \tilde{X}(t) = \gamma F(t). \tag{35}$$

In the above relations we are useing the same notations as in (15), and γ is a constant given by $\gamma = \Delta/\beta$.

We choose $f(t)$ to be a Gaussian white noise, characterised by zero mean value and by the two-point correlator

$$\langle f(t) f(t') \rangle = \frac{D}{2} \delta(t - t'). \tag{36}$$

It then follows form (34), (35) that the multiplicative noise induces a random shift of the shock (centre of mass) position.

The random shift is a Brownian motion with average $\langle \tilde{X}(t) \rangle = 0$. From (35) it then follows that the shock motion is characterised by the mean displacement $\langle X(t) \rangle = vt$, by a constant mean velocity $\langle \dot{X}(t) \rangle$, by the two-point correlator

$$\langle X(t_1), X(t_2) \rangle = v^2 t_1 t_2 + \gamma^2 D t_m$$

with $t_m = \min\{t_1, t_2\}$.

We now turn our attention to the statistical properties of the shock solution (33). The white noise $f(t)$ induces an exponential growth of the statistical average. We obtain

$$\langle v_n(t)\rangle = V+\Delta\left[1+\sum_{\alpha=1}^{\infty}(-1)^{\alpha}\exp\left[-2\alpha\beta(n-vt)+\frac{1}{2}\alpha^2\Delta^2\langle F^2\rangle\right]\right] \tag{37}$$

with $\langle F^2(t)\rangle = Dt$. It is also of interest to look at the two-time correlator between the shock and the noise. Taking into account the zero mean value of the noise form (33) we get

$$\langle v_n(t_1), f(t_2)\rangle = \Delta\sum_{\alpha=1}^{\infty}(-1)^{\alpha}C(t_1,t_2)\exp[-2\alpha\beta(n-vt_1)] \tag{38}$$

with $C(t_1,t_2) = \langle\exp[-\alpha\Delta F(t_1)], f(t_2)\rangle$. In the case $t_2 > t_1$ we obtain $C(t_1,t_2) = 0$, which implies that the solution at a given time t_1 is not influenced by the value of the noise at a future time t_2, in agreement with the causality principle. For $t_2 > t_1$ we get instead

$$C(t_1,t_2) = \frac{\alpha}{2}\Delta D\exp\left(\frac{1}{2}\alpha^2 Dt_1\right). \tag{39}$$

As expected, the correlator exhibits in this case the same exponential growth as the statistical average (37).

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