Journal of Nonlinear Mathematical Physics 2001, V.8, Supplement, 128–132 Proceedings: NEEDS'99

Asymptotic Solutions of the Whitham Equations

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Abstract

We extend a previous result, namely we show that the solution of the Whitham equations is asymptotically self-similar for generic monotone polynomial initial data with smooth perturbation.

1 Introduction

It is known [1, 2, 3] that the evolution of the smooth initial data

$$u(x,t=0)=u_0(x), \qquad t,x\in\mathbb{R}, \tag{1}$$

according to the Korteweg de Vries (KdV) equation

$$u_t+6uu_x+\epsilon^2u_{xxx}=0, \tag{2}$$

is described locally as $\epsilon\to 0$ by the solution of an initial value problem for the Whitham equations. Lax Levermore [2] and Venakides [3] studied the Cauchy problem for (2) as $\epsilon\to 0$ for certain particular classes of initial data in the frame of the zero-dispersion asymptotics for the solution of the inverse scattering problem of KdV. According to their results, to the solution $u(x,t,\epsilon)$ as $\epsilon\to 0$ of KdV it corresponds a decomposition of the (x,t) plane into a number of domains D_g, $g=0,1,\ldots$. In the domain D_g the principal term of the asymptotics is given by the so called g-phase solution [4] of the KdV equation with the wave parameters depending on the functions $u_1(x,t)>\cdots>u_{2g+1}(x,t)$ which satisfy the g-phase Whitham equations

$$\frac{\partial u_i}{\partial t}+\lambda_i(\vec{u})\frac{\partial u_i}{\partial x}=0, \qquad i=1,\ldots,2g+1, \qquad g\geq 0. \tag{3}$$

Here $\vec{u}=(u_1,\ldots,u_{2g+1})$, $u_1>u_2>\cdots>u_{2g+1}$, and the $\lambda_i(\vec{u})$, $i=1,\ldots,2g+1$, depends on complete hyperelliptic integrals of genus g.

The equations (3) were found by Whitham [5] in the single phase case $g=1$ and more generally by Flaschka, Forest and McLaughlin [6] in the multiphase case.

The hyperbolic nature of the equations has been shown by Levermore [7]. Dubrovin and Novikov [8] found the geometric Hamiltonian structure of the equations (3). Based on this structure, Tsarev [9] showed that equations (3) can be integrated by a generalization of the method of characteristics. This result was put into an algebro-geometric setting by Krichever [10]. In this frame he build the so called self-similar solutions of the Whitham equations, namely solutions which are time-free.

The investigation of the initial value problem of the Whitham equations began with Gurevich and Pitaevskii [1]. In the case $g \leq 1$ they solved the initial value problem of system (3) numerically for cubic initial data The Cauchy problem for the Whitham equations was widely investigated by Tian in the case $g \leq 1$. He proved that, for monotone decreasing initial data $x = f(u)|_{t=0}$ satisfying the condition $f'''(u) < 0$ except at one point, the solution of the Whitham equations exists for all times $t > 0$ and the phase is just zero or one [12]. In [13] we considered monotone decreasing initial data of the form $x = f_a(u) + f_s(u)$, where $f_a(u)$ is an analytic function and $f_s(u)$ is a smooth function rapidly decreasing at infinity. It was shown that, if such initial data satisfies the condition $f^{(2m+1)}(u) < 0$ except for some number of isolated points, then the solution of the Whitham equations has genus $g \leq m$ for all x and $t \geq 0$. In [14] it was shown that for polynomial initial data of degree $2N+1$ the solution of the Whitham equations is asymptotically close to the solution of the Whitham equations with initial data $x = -u^{2N+1}$. Here we extend this result considering a smooth perturbation of the polynomial initial data. Namely we consider monotone decreasing initial data of the form $x = P_N(x) + f_s(x)$, where $P_N(x)$ is a polynomial of degree $2N+1$ and $f_s(u)$ is a smooth function rapidly decreasing at infinity. Under certain conditions on such initial data the solution of the Whitham equations is asymptotically close to the solution of the Whitham equations with initial data $x = -u^{2N+1}$.

2 Preliminaries on the theory of the Whitham equations

In the following we give a brief review concerning the theory of the Whitham equations and their solution in the case $g \leq 1$. The one phase Whitham equations are a system of 3 quasi-linear hyperbolic PDE's defined by the expression [5, 6]

$$\frac{\partial u_i}{\partial t} + \lambda_i(u_1, u_2, u_3)\frac{\partial u_i}{\partial x} = 0, \qquad u_1 > u_2 > u_3, \qquad i = 1, 2, 3, \tag{4}$$

where

$$\begin{aligned} \lambda_i(u_1, u_2, u_3) &= 2(u_1 + u_2 + u_3) + 4\,\frac{\prod_{j\neq i}(u_i - u_j)}{\alpha_0 + u_i}, \qquad j, i = 1, 2, 3, \\ \alpha_0 &= -u_1 - (u_3 - u_1)\frac{E(s)}{K(s)}, \end{aligned} \tag{5}$$

and $K(s)$ and $E(s)$ are the complete elliptic integrals of the first and second kind respectively of modulus $s = \frac{u_3 - u_2}{u_3 - u_1}$. The zero-phase Whitham equation coincides with the so called Burgers equation

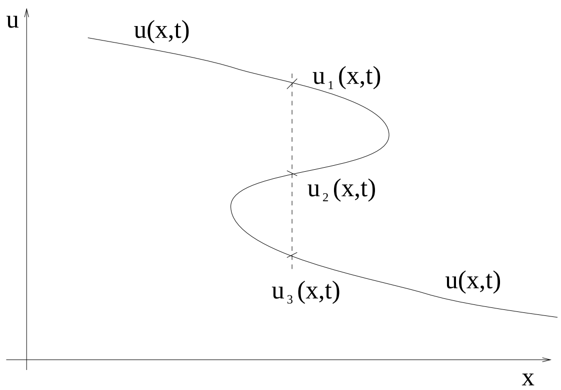
$$u_t + 6uu_x = 0. \tag{6}$$

For monotone decreasing initial data $x = f(u)|_{t=0}$, the solution of the Burgers equation (6) is obtained by the method of characteristics and is given by the expression

$$x = 6tu + f(u). \tag{7}$$

This solution is globally well defined only for $t < t_c$, where $t_c = \frac{1}{6}\min_{u\in\mathbb{R}}[-f'(u)]$ is the time of gradient catastrophe of the solution. The breaking is caused by an inflection point in the initial data. At later time the solution of the Whitham equations is obtained gluing together C^1-smoothly solutions of different genera.

We consider the case $g \leq 1$.



On the picture the evolution of $u_1(x,t) > u_2(x,t) > u_3(x,t)$ is ruled by the one-phase Whitham equations (4), while $u(x,t)$ satisfies the Burgers equation. The quantities u_1, u_2, u_3 match the Burgers solution at the boundaries of the multi-valued region, namely:

a) trailing edge

$$\begin{cases} u_1 = \text{solution of the Burgers equation outside the multi-valued region,} \\ u_2 = u_3; \end{cases} \tag{8}$$

b) leading edge

$$\begin{cases} u_1 = u_2, \\ u_3 = \text{solution of the Burgers equation outside the multi-valued region.} \end{cases} \tag{9}$$

The solution of the one phase equations which matches the solution of the zero phase equation at the boundaries of the multi-valued region can be written in the form [12]

$$x = \lambda_i(u_1, u_2, u_3)t + w_i(u_1, u_2, u_3), \qquad i = 1, 2, 3, \tag{10}$$

where the $\lambda_i = \lambda_i(u_1, u_2, u_3)$ have been defined in (4) and $w_i = w_i(u_1, u_2, u_3)$ is given by the expression

$$w_i = \left(\frac{1}{2}\lambda_i - u_1 - u_2 - u_3\right)\frac{\partial q}{\partial u_i} + q, \qquad i = 1, 2, 3. \tag{11}$$

The function $q = q(u_1, u_2, u_3)$ reads [12]

$$q(u_1, u_2, u_3) = \frac{1}{2\sqrt{2}\pi}\int_{-1}^{1}\int_{-1}^{1}\frac{f\left(\frac{1+\mu}{2}\frac{1+\nu}{2}u_1 + \frac{1+\mu}{2}\frac{1-\nu}{2}u_2 + \frac{1-\mu}{2}u_3\right)}{\sqrt{(1-\mu)(1-\nu^2)}}d\mu d\nu, \tag{12}$$

where f is the initial data.

The next theorem provides conditions for the existence of a global solution of the Whitham equations.

Theorem 2.1 [13]. *Suppose that the monotone decreasing initial data $x = f(u)$ is of the form $f(u) = P_N(u) + f_s(u)$ where $P_N(u)$ is a polynomial of degree $2N + 1$ and $f_s(u)$ is a smooth function rapidly decreasing at infinity. If the initial data satisfies the condition*

$$f^{(2m+1)}(u) < 0 \qquad \forall\, u \in \mathbb{R}, \quad m < N, \tag{13}$$

except for some number of isolated points, then the solution of the Whitham equations exists for all x and $t > 0$ and it has genus at most equal to m.

The following theorem characterizes the solution of the Whitham equations for all times bigger than a certain time $T > 0$.

Theorem 2.2 [11]. *Let us assume that the solution of the Cauchy problem for the Whitham equations with monotone decreasing initial data $x = f(u)$ exists for any $x, t \geq 0$. Suppose that the function $f(u)$ defined on the whole real axis satisfies the conditions*

$$\begin{aligned} &\lim_{u\to-\infty} f''(u) = +\infty, \qquad f''(u) < 0 \quad \text{for} \quad u \to +\infty, \\ &f'''(u) < 0 \quad \text{for} \quad u > u_+ \quad \text{and} \quad u < u_-, \end{aligned} \tag{14}$$

where $u_+ \geq u_-$ are some real numbers. Then there is a time $T \geq 0$ such that for all $t > T$ the solution of the Whitham equations is of genus one inside the interval $x^-(t) < x < x^+(t)$, where $x^-(t) < x^+(t)$ are two real functions of t. For $x = x^\pm(t)$ boundary conditions (8) and (9) are satisfied. It is of genus zero outside this interval.

3 Self-similar asymptotic solutions

When the initial data is of the form $f(u) = -u^k$, $k = 3, 5, 7, \ldots$, one obtains the distinguished self-similar solutions [15, 10]. Their genus is at most equal to one.

These solutions have the form $u_i(x,t) = t^{\frac{1}{k-1}} U_i\left(t^{-\frac{k}{k-1}} x\right)$, $i = 1, 2, 3$. Indeed, introducing the new variables

$$\begin{aligned} &X = t^{-\frac{k}{k-1}} x, \qquad u(x,t) = t^{\frac{1}{k-1}} U\left(t^{-\frac{k}{k-1}} x\right), \\ &u_i(x,t) = t^{\frac{1}{k-1}} U_i\left(t^{-\frac{k}{k-1}} x\right), \qquad i = 1, 2, 3 \end{aligned} \tag{15}$$

the system (10) becomes time free:

$$X = \lambda_i(\vec{U}) + \left[\frac{1}{2}\lambda_i(\vec{U}) - U_1 - U_2 - U_3\right] \frac{\partial}{\partial U_i} q_k(\vec{U}) + q_k(\vec{U}), \tag{16}$$

where $\vec{U} = (U_1, U_2, U_3)$ and $q_k = q_k(\vec{U})$ is the function defined in (12) for the initial data $X = -U^k$. The characteristic equation (7) becomes

$$X = 6U - U^k. \tag{17}$$

We have the following corollary [12].

Corollary 3.1. *For $x = -u^k$, $k = 3, 5, 7, \ldots$, the Whitham equations have a global self-similar one-phase solution $u_1 > u_2 > u_3$:*

$$u_i(x,t) = t^{\frac{1}{k-1}} U_i\left(t^{-\frac{k}{k-1}} x\right), \qquad i = 1, 2, 3 \tag{18}$$

within a cusp in the $x-t$ plane: $x_-(k)\,t^{\frac{k}{k-1}} < x < x_+(k)\,t^{\frac{k}{k-1}}$, where $x_-(k) < x_+(k)$ are two real constants and $t > 0$. On the boundary of the cusp the one-phase solution is attached C^1-smoothly to the solution $u(x,t)$ of the zero-phase equation.

Let us consider monotone decreasing initial data of the form

$$x = f(u) = -u^{2N+1} - \left(c_0 + c_1 u + \cdots + c_{2N} u^{2N}\right) + f_s(u), \tag{19}$$

where $f_s(u)$ is a smooth function rapidly decreasing at infinity. Suppose that such initial data satisfies (13). It follows from Theorem 2.1 that the solution of the Whitham equations for such initial data exists for all x and $t \geq 0$ and it has a number of interacting oscillatory phases less or equal than $m < N$.

If the monotone decreasing initial data (19) satisfies (13) then it also satisfies the hypothesis of Theorem 2.2. Hence there exists a time $T > 0$ such that for all times $t > T$ the solution of the Whitham equations for such initial data has genus $g \leq 1$.

From the above considerations and doing the rescaling (16) for $k = 2N+1$ to the initial data (19) we obtain the next theorem.

Theorem 3.2. *The solution of the Whitham equations (4) with initial data (19) is asymptotically close for $t \to +\infty$ to the self-similar solution (16)–(17) with initial data $x = -u^{2N+1}$.*

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