Journal of Nonlinear Mathematical Physics 2001, V.8, Supplement, 156–160 Proceedings: NEEDS'99

Singular Solution of the Liouville Equation under Perturbation

L A KALYAKIN

Institute of Mathematics RAS, Ufa, Russia

Abstract

The Cauchy problem for the Liouville equation with a small perturbation is considered. We are interested in the asymptotics of the perturbed solution under the assumption that one has singularity. The main goal is to study both the asymptotic approximation of the singular lines and the asymptotic approximation of the solution everywhere outside the narrow neighbourhood of the singular lines.

The Cauchy problem for the Liouville equation with a small perturbation

$$\partial_t^2 u - \partial_x^2 u + 8 \exp u = \varepsilon \mathbf{F}[u], \qquad 0 < \varepsilon \ll 1, \tag{0.1}$$

$$u|_{t=0} = \psi_0(x;\varepsilon), \qquad \partial_t u|_{t=0} = \psi_1(x;\varepsilon), \qquad x \in R \tag{0.2}$$

is considered. We are interesting for asymptotics of the perturbed solution $u(x,t;\varepsilon)$ as $\varepsilon \to 0$.

Perturbation theory for integrable equations remains a very attractive task. As a rule a perturbation of smooth solutions such as a single soliton were usually considered. We intend here to discuss a perturbation of a singular solution under assumption that the perturbed solution has singularities as well. A simple well known instance of this kind is a chock wave under weak perturbation as given by the Hopf equation with a small perturbation term $u_t + uu_x = \varepsilon f(u)$, $0 < \varepsilon \ll 1$. In this paper we consider a more complicated problem namely a perturbation of the singular solution of Liouville equation. We deal with the singularities studied by Pogrebkov and Polivanov [1].

1 Unperturbed equation

Unperturbed equation (as $\varepsilon = 0$) $\partial_t^2 u - \partial_x^2 u + 8 \exp u = 0$ was solved by Liouville [2] and a formula of the general solution is well known

$$u(x,t) = \ln \frac{r'_+(s^+) r'_-(s^-)}{r^2(s^+,s^-)}, \qquad r = r_+(s^+) + r_-(s^-), \qquad s^\pm = x \pm t. \tag{1.1}$$

Initial equations (0.2) give two ODE's for the $r_\pm$ which may be linearized.

The singularities of the solution occur due to zero of the denominate $r(x+t, x-t) \equiv r_+(x+t) + r_-(x-t) = 0$ under condition $r'_\pm > 0$. We consider just this case. The singular solution is unique under matching condition imposed on the singular lines Γ [3]. That is continuity (zero jump across the singular lines) of two expressions

$$\left[1/2\left(u_t^2 + u_x^2\right) + 2\exp u - 2u_{xx}\right]_\Gamma = 0, \qquad [-u_t\phi_x + 2u_{tx}]_\Gamma = 0. \tag{1.2}$$

2 Perturbed problem ($\varepsilon \neq 0$)

As regards the perturbation operator $F[u]$ we assume that one has no higher order derivatives. We desire the singularities do not disperse so that singular lines only deform slowly under perturbation. Note that an existence theorem for the perturbed problem is not proved up to now. We only give a formal asymptotic solution.

The main goals are both an asymptotic approximation of the singular lines and an asymptotic approximation of the solution everywhere out of narrow neighborhood of the singular lines.

Definition. *The formal asymptotic solution (FAS) of order N is a function $U_N(x,t;\varepsilon)$ satisfied to both the equation (0.1) and the initial condition (0.2) up to order $\mathcal{O}\left(\varepsilon^N\right)$ everywhere in $\{x \in R,\ 0 \le t \le T = \mathrm{const}\}$ out of narrow strips of order $\mathcal{O}\left(\varepsilon^N\right)$. In the strips there are smooth lines on which the matching conditions (1.2) are satisfied.*

Remark. A direct asymptotic expansion as given by $u \approx \sum_n \varepsilon^n \overset{n}{u}(x,t)$, $\varepsilon \to 0$ does not provide an approach to both the singular lines and the solution inside of stripes (near limit singular lines) whose width has the order of $\mathcal{O}(\varepsilon)$. This assertion can be verified on a simple example when $F \equiv 0$ and exact solution is taken in the explicit form (1.1) with the smooth functions $r_\pm(s^\pm;\varepsilon)$ depending on the parameter ε. Asymptotic expansion of such solution as $\varepsilon \to 0$ has coefficients with the increasing order singularities $\mathcal{O}\left(r_0^{-n}\right)$ at the limit singular lines $r_0^{-n} \equiv r_+(s^+;0) + r_-(s^-;0) = 0$.

Ansatz. A FAS (for any N) is taken as a finite peace of the asymptotic series

$$u(x,t;\varepsilon) \approx \sum_{n=0}^{\infty} \varepsilon^n \overset{n}{u}(x,t;\varepsilon), \qquad \varepsilon \to 0. \tag{2.1}$$

The leading order term is here taken as an exact solution of the unperturbed equation

$$\overset{0}{u} = \ln \frac{r'_+(s^+;\varepsilon) r'_-(s^-;\varepsilon)}{r^2(s^+,s^-;\varepsilon)}, \qquad (r = r_+ + r_-). \tag{2.2}$$

We permit dependence on a small parameter in the functions $r_\pm(s_\pm;\varepsilon)$. For these functions an asymptotics is constructed in the form

$$r_\pm(s^\pm;\varepsilon) \approx \sum_{n=0}^{\infty} \varepsilon^n \overset{n}{r}_\pm(s^\pm), \qquad \varepsilon \to 0. \tag{2.3}$$

The formulas (2.1)–(2.3) are usually named Bogolubov–Krylov ansatz. This approach provides an asymptotic approximation of the singular lines up to any order from the equation $r_+(s^\pm;\varepsilon) + r_-(s^\pm;\varepsilon) = 0$. Thus the matter is reduced to define the coefficients $\overset{n}{u}$, $\overset{n}{r}_\pm$.

3 Linearized problem for the correction

Corrections $\overset{n}{u}$ $(n \ge 1)$ are determined from the linear equations

$$\partial_t^2 \overset{n}{u} - \partial_x^2 \overset{n}{u} + 8 \frac{r'_+ r'_-}{r^2} \overset{n}{u} = \overset{n}{f}(x,t;\varepsilon), \qquad (x,t) \in R^2$$

with the corresponding initial data. The right sides are here calculated from previous steps, for example $\overset{1}{f} = \mathbf{F}\left[\overset{0}{u}\right]$, $\overset{2}{f} = \delta\mathbf{F}\left[\overset{0}{u}\right]\overset{1}{u} - 4r'_+r'_-\left(\overset{1}{u}\right)^2/r^2$.

General solution of the homogeneous linear equation is given by the formula

$$u(x,t;\varepsilon) = j'_+/r'_+ + j'_-/r'_- - 2(j_+ + j_-)/r.$$

Here $j_\pm = \overset{n}{j}_\pm (s^\pm;\varepsilon)$ are arbitrary functions. In context of the Cauchy problem they are determined by the initial functions and an expression similar to the D'Alembert formula takes place. Take into account the singularities of order $\mathcal{O}(1/r)$.

To solve the nonhomogeneous linear equation the similar formula may be used with the functions $j_\pm(x,t)$, depending on two variables, which are defined from ODE's in the explicit form. So a solution of the linear equation is determined by the integral

$$u(x,t) = \int_{s^-}^{s^+}\int_{s^-}^{\sigma^+} K(s^+,s^-,\sigma^+,\sigma^-) f(\sigma^+,\sigma^-)\, d\sigma^-\, d\sigma^+$$

taken over the characteristic triangle. The kernel K is here expressed by

$$K(s^+,s^-,\sigma^+,\sigma^-) = \frac{1}{2r(s_+,s_-)r(\sigma_+,\sigma_-)}\Big\{ r_+(s^+)r_-(s^-) + r_+(\sigma^+)r_-(\sigma^-) + \frac{1}{2}\left[r_+(s^+) - r_-(s^-)\right]\left[r_+(\sigma^+) - r_-(\sigma^-)\right]\Big\}.$$

Let us denote by $\overset{n}{u}_1 (x,t;\varepsilon)$ a part of the correction corresponding to a solution of the nonhomogeneous linear equation. One depends on both the $r_\pm(s^\pm;\varepsilon)$ and the $\overset{m}{j}_\pm (s^\pm;\varepsilon)$ $(0 < m < n)$ and one is determined by means of the quite well specific operator $\overset{n}{u}_1 = \overset{n}{\mathbf{U}} [r_\pm, \overset{1}{j}_\pm, \ldots, \overset{n-1}{j}_\pm]$. It is convenient to identify the singular part in these representation

$$\overset{n}{\mathbf{U}} = \overset{n}{\mathbf{V}}\left[r_\pm, \overset{1}{j}_\pm, \ldots, \overset{n-1}{j}_\pm\right] - (2/r)\,\overset{n}{\mathbf{W}}\left[r_\pm, \overset{1}{j}_\pm, \ldots, \overset{n-1}{j}_\pm\right]. \tag{3.1}$$

Singularities play here a role of secular terms. If we take a FAS in the form of the direct expansion, then we find that the singularities became stronger on each step. One can guess that this effect is due to the poor approximation of the perturbed singular lines. Hence the singularities must to be eliminated from the corrections. Just this elimination gives us a good approximation of the perturbed singular lines[1]. Of course a representation (3.1) with smooth functions $\overset{n}{\mathbf{V}}, \overset{n}{\mathbf{W}}$ is possible just only under some restrictions on the perturbation operator $F[u]$. In fact, the more general condition on the $F[u]$ is a representation (3.1) with smooth functions $\overset{n}{\mathbf{V}}$, $\overset{n}{\mathbf{W}}$ on each step n.

4 Identification of corrections

The system of linear equations are based on the leading term whose parameters $r_\pm$ are determined by initial data from equation:

$$\ln\frac{r'_+r'_-}{r^2} + \sum_{n=1}^{\infty}\varepsilon^n\left[\overset{n}{j}{}'_+/r'_+ + \overset{n}{j}{}'_-/r'_- - \frac{2}{r}\left(\overset{n}{j}_+ + \overset{n}{j}_-\right) + \overset{n}{u}_1\right] = \psi_0(x;\varepsilon), \tag{4.1}$$

[1]One can think these ideas are suitable for another problems with singularities under perturbations.

$$\begin{aligned}\frac{r''_+}{r'_+} - \frac{r''_-}{r'_-} - 2\frac{r'_+ - r'_-}{r} &+ \sum_{n=1}^{\infty} \varepsilon^n \Bigg[\left(\overset{n}{j}{}'_+ / r'_+\right)' - \left(\overset{n}{j}{}'_- / r'_-\right)' - \frac{2}{r}\left(\overset{n}{j}{}'_+ - \overset{n}{j}{}'_-\right) \\ &+ \frac{2}{r^2}\left(\overset{n}{j}_+ + \overset{n}{j}_-\right)(r'_+ - r'_-) + \partial_t \overset{n}{u}_1\Bigg] = \psi_1(x;\varepsilon).\end{aligned} \tag{4.2}$$

Note that there are unknown functions $\overset{n}{j}_\pm$ apart from $r_\pm$.

A naive approach to these equations with a small parameter is as follows. Functions $r_\pm$ are identified with the leading term $r_\pm = \overset{0}{r}_\pm (s^\pm)$ which is defined from the nonlinear equations as $\varepsilon = 0$. After that all functions $\overset{n}{j}_\pm (s^\pm)$ are determined from the recurrent system of linear equations. In this way a direct asymptotic expansion is just obtained, whose coefficients have the singularities of increasing order $\mathcal{O}\left(r_0^{-n}\right)$ at the limit singular lines $r_0 \equiv r_+(s^+;0) + r_-(s^-;0) = 0$. Deformation of the singular lines is not determined in this way.

In our approach the functions $r_\pm$ are taken in the form of asymptotic series (2.3). Additional ambiguities in the coefficients $\overset{n}{r}_\pm (s^\pm)$, $n \geq 1$ are used to solve the (4.1), (4.2) under additional requirements. We desire to eliminate the singularities of order $\mathcal{O}\left(r^{-1}\right)$ from the corrections $\overset{n}{u}$ on each step. Elimination of these terms give rise to the algebraic equations

$$\overset{n}{j}_+ (s^+;\varepsilon) + \overset{n}{j}_- (s^-;\varepsilon) + \overset{n}{\mathbf{W}} \left[r_\pm, \overset{1}{j}_\pm, \ldots, \overset{n-1}{j}_\pm\right] = 0 \qquad \text{as} \quad r(s^+, s^-;\varepsilon) = 0. \tag{4.3}$$

Moreover an additional condition is used

$$\overset{n}{r}_+ (x) + \overset{n}{r}_- (x) = 0, \qquad \forall\, x \in R \quad (n \geq 1). \tag{4.4}$$

So we obtain the systems of equations (4.1)–(4.4) for the four functions $\overset{n}{r}_\pm$, $\overset{n}{j}_\pm$ on each step $n = 1, 2, \ldots$. However these equations contain a small parameter throw $r_\pm(x;\varepsilon)$. Hence an asymptotic expansion have to be constructed for the functions $\overset{n}{j}_\pm$ like (2.3) $\overset{n}{j}_\pm (x;\varepsilon) \approx \sum\limits_m \varepsilon^m \overset{n,m}{j}_\pm (x)$. Coefficients can be here obtained from the recurrent system of algebraic equations

$$\overset{n,m}{j}_+ (s^+) + \overset{n,m}{j}_- (s^-) = \overset{n,m}{W} \qquad (n \geq 1,\ 0 < m < n) \quad \text{as} \quad \overset{0}{r} (s^+, s^-) = 0. \tag{4.5}$$

So all functions $\overset{n}{r}_\pm$, $\overset{n,0}{j}_\pm (x)$ $(n \geq 1)$ and $\overset{n,m}{j}_\pm$ $(1 \leq m < n)$ are defined step-by-step from the algebraic and differential equations. The main objects are differential equations, which can be represented in the form

$$\overset{n}{y}{}'_+ / \overset{0}{r}{}'_+ (x) + \overset{n}{y}{}'_- / \overset{0}{r}{}'_- (x) - 2\frac{\overset{n}{y}_+ + \overset{n}{y}_-}{r(x,x)} = \overset{n}{\Psi}_0 (x),$$

$$\left(\overset{n}{y}{}'_+ / \overset{0}{r}{}'_+ (x) - \overset{n}{y}{}'_- / \overset{0}{r}{}'_- (x)\right)' - 2\frac{\overset{n}{y}{}'_+ - \overset{n}{y}{}'_-}{r(x,x)} + 2\frac{\left(\overset{n}{y}_+ + \overset{n}{y}_-\right)\left(\overset{0}{r}{}'_+ (x) - \overset{0}{r}{}'_- (x)\right)}{r^2(x,x)} = \overset{n}{\Psi}_1 (x)$$

for the combinations $\overset{n}{y}_{\pm}(x) = \overset{n}{r}_{\pm}(x) + \overset{n,0}{j}_{\pm}(x)$. If we take into account $r(x,x) = \overset{0}{r}_{+}(x) + \overset{0}{r}_{-}(x)$, then a solution is obtained in the explicit form

$$y_{\pm} = \frac{1}{2}\int (\Psi_0 \pm g)\, \overset{0}{r}{}_{\pm}'\, dx + \frac{1}{2}\int \overset{0}{r}{}_{\pm}' \int \left(\Psi_0 r'/r + \left(\overset{0}{r}_{+} - \overset{0}{r}_{-}\right)' g/r\right) dx\, dx,$$

$$g(x) = r(x,x)\int \big(\Psi_1(x)/r(x,x) + \Psi_0(x)\,(r_+(x) - r_-(x))'/r^2(x,x)\big)\, dx.$$

After that the algebraic equations (4.4)–(4.5) are solved. There are some arbitrariness in the solution $\overset{n}{r}_{\pm}$, $\overset{n,m}{j}_{\pm}$, which however do not effect the obtained approximation for the $u(x,t;\varepsilon)$.

5 Conclusion

The main result of the paper is the given above manner of determination of the FAS (2.1)–(2.3) in the occurrence of singularities.

Acknowledgements

This research has been supported by the Russian Foundation for Fundamental Research under Grants 99-01-00139, 00-15-96038 and INTAS under Grant 99–1068.

References

[1] Jeorjadze J P, Pogrebkov A K and Polivanov M C, *Teor. Mat. Fiz.*, 1979, V.40, N 2, 221–234 (in Russian).

[2] Liouville J, *J. Math. Pure et Appl.*, 1853, V.18, 71–74.

[3] Pogrebkov A K, *Doklady Akad. Nauk USSR*, 1979, V.244, 873–876 (in Russian).