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Nonlinear Gauge Transformation for a Quantum System Obeying an Exclusion-Inclusion Principle

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Abstract

We introduce a nonlinear and noncanonical gauge transformation which allows the reduction of a complex nonlinearity, contained in a Schrödinger equation, into a real one. This Schrödinger equation describes a canonical system, whose kinetics is governed by a generalized Exclusion-Inclusion Principle. The transformation can be easily generalized and used in order to reduce complex nonlinearities into real ones for a wide class of nonlinear Schrödinger equations. We show also that, for one dimensional system and in the case of solitary waves, the above transformation coincides with the one already adopted to study the Doebner–Goldin equation.

Let us consider the kinetics of N particles in a one-dimensional discrete space, which is an one-dimensional Markovian chain. The generic site is labeled by the index i ($i = 0, \pm 1, \pm 2, \ldots$); the position at the ith site is $x_i = i\Delta x$, where Δx is a constant.

We call $\rho_i(t)$ the occupational probability of the ith site. Let us assume that only transitions to the nearest neighbors are allowed and define the transition probability $\pi_i^{\pm}(t)$ from the site i to the site $i \pm 1$ in the following way:

$$\pi_i^{\pm}(t) = \frac{\alpha_i^{\pm}(t)}{\Delta x^2}\rho_i(t)[1+\kappa\rho_{i\pm1}(t)]. \tag{1}$$

The factor $1 + \kappa\rho_{i\pm1}(t)$ means that the transition probability depends on the particle population $\rho_{i\pm1}(t)$ of the arrival site. If $\kappa > 0$ the $\pi_i^{\pm}(t)$ introduces an inclusion effect. In fact, the population at the arrival site $i \pm 1$ stimulates the transition and $\pi_i^{\pm}(t)$ increases linearly with $\rho_{i\pm1}(t)$. In the case $\kappa < 0$ the $\pi_i^{\pm}(t)$ takes into account an exclusion effect. If the arrival site is empty $\rho_i(t) = 0$, the $\pi_i^{\pm}(t)$ depends only on the population of the starting point. If the arrival site is populated $0 < \rho_i(t) \le \rho_{\max}$ the transition is inhibited.

The Pauli master equation can be written as follows:

$$\frac{d\rho_i(t)}{dt} = \pi_{i-1}^{+}(t) + \pi_{i+1}^{-}(t) - \pi_i^{+}(t) - \pi_i^{-}(t). \tag{2}$$

If we define:

$$j_i(t) = \left[\pi_i^+(t) - \pi_{i+1}^-(t)\right] \Delta x, \tag{3}$$

which represents a discrete current, the master equation can be written as:

$$\frac{d\rho_i(t)}{dt} + \frac{\Delta j_i(t)}{\Delta x} = 0, \tag{4}$$

where

$$\Delta j_i(t) = j_{i+1}(t) - j_i(t). \tag{5}$$

Equation (4) represents a forward continuity equation in a discrete one-dimensional space. Analogously we can obtain a backward continuity equation. The half of the sum of these equations, in the limit $\Delta x \to 0$ gives the following continuity equation:

$$\frac{\partial\rho(t,x)}{\partial t} + \frac{\partial j(t)}{\partial x} = 0, \tag{6}$$

where $j(t,x)$ assumes the form:

$$j(t,x) = u(t,x)[1+\kappa\rho(t,x)]\rho(t,x). \tag{7}$$

We note that the current velocity $u(t,x)$ depends on the nature of the particle interaction while the factor $1+\kappa\rho(t,x)$ takes into account the generalized Exclusion-Inclusion Principle (EIP) [1].

In Ref. [2, 3, 4] was recently considered by us the nonlinear canonical model defined by the Lagrangian density:

$$\mathcal{L} = i\frac{\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}\left|\boldsymbol{\nabla}\psi\right|^2 - U_{\rm EIP}[\psi,\psi^*] - V\psi^*\psi, \tag{8}$$

with

$$U_{\rm EIP}[\psi,\psi^*] = -\kappa\frac{\hbar^2}{8m}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right)^2, \tag{9}$$

in order to study, in a many-body 3-dimensional system, the effect of collective interactions due to the generalized EIP.

The evolution equation for the field ψ can be obtained from a variational principle $\delta\mathcal{A}/\delta\psi^* = 0$, where the action is given by $\mathcal{A} = \int \mathcal{L} d^3x dt$, obtaining the following nonlinear Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + W(\rho,\boldsymbol{j}_\psi)\psi + i\mathcal{W}(\rho,\boldsymbol{j}_\psi)\psi + V\psi, \tag{10}$$

where real and imaginary parts of the nonlinearity, introduced by the EIP, are given respectively by:

$$W(\rho,\boldsymbol{j}_\psi) = \kappa\frac{m}{\rho}\left(\frac{\boldsymbol{j}_\psi}{1+\kappa\rho}\right)^2, \tag{11}$$

$$\mathcal{W}(\rho,\boldsymbol{j}_\psi) = -\kappa\frac{\hbar}{2\rho}\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{j}_\psi\rho}{1+\kappa\rho}\right). \tag{12}$$

From (10) we can see that the probability density $\rho = |\psi|^2$ is conserved because satisfies the following continuity equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j}_\psi = 0, \tag{13}$$

where the quantum current $\boldsymbol{j}_\psi$ is:

$$\boldsymbol{j}_\psi = (1+\kappa\rho)\boldsymbol{j}_0, \tag{14}$$

being

$$\boldsymbol{j}_0 = \frac{-i\hbar}{2m}\left(\psi^* \boldsymbol{\nabla}\psi - \psi \boldsymbol{\nabla}\psi^*\right), \tag{15}$$

the standard quantum current to which $\boldsymbol{j}_\psi$ reduces when the EIP is switched off.

From Eq. (14) we see that the action of EIP on the system is given by the factor $1+\kappa\rho$, which means enhancement ($\kappa > 0$) or inhibition ($\kappa < 0$) in the expression of the current. Introducing the hydrodynamical variable:

$$\psi(\boldsymbol{x}, t) = \rho^{1/2}(\boldsymbol{x}, t) \exp\left[-\frac{i}{\hbar} S(\boldsymbol{x}, t)\right], \tag{16}$$

the expression of the current $\boldsymbol{j}_\psi$ takes the form:

$$\boldsymbol{j}_\psi = \frac{\boldsymbol{\nabla} S}{m} \rho(1+\kappa\rho). \tag{17}$$

We note that $\boldsymbol{j}_\psi$ is nonlinear in ρ as a consequence of the presence of the complex nonlinearity $W + i\mathcal{W}$ in the evolution equation of the field ψ.

In the present work we introduce a unitary gauge transformation $\mathcal{U}^\dagger = \mathcal{U}^{-1}$ for the field ψ:

$$\psi(\boldsymbol{x}, t) \to \phi(\boldsymbol{x}, t) = \mathcal{U}(\rho, S)\psi(\boldsymbol{x}, t), \tag{18}$$

and the expression of $\boldsymbol{j}_\psi$ can be linearized by means of the phase transformation:

$$\boldsymbol{\nabla} S \to \boldsymbol{\nabla}\sigma = (1+\kappa\rho)\boldsymbol{\nabla} S, \tag{19}$$

which can be performed only if $\boldsymbol{\nabla} \times (\rho \boldsymbol{\nabla} S) = 0$. It is easy to see that $\mathcal{U}$ is:

$$\mathcal{U}(\rho, S) = \exp\left(i\frac{\kappa}{\hbar}\int_\gamma \rho \boldsymbol{\nabla} S \cdot d\boldsymbol{l}\right). \tag{20}$$

Let us note that the definition of $\mathcal{U}$ uses a path integral over the path γ_i from the initial fixed point P_0 to a point P, joined by the curve C_i: $\gamma_i \equiv (P_0 \xrightarrow{C_i} P)$.

We consider two different paths γ_1 and γ_2, both joining the starting point P_0 to the final point P. If we indicate with $\overline{\gamma}_1$ the path opposed to γ_1 it is easy to see that:

$$\int_{\gamma_2} \rho \boldsymbol{\nabla} S \cdot d\boldsymbol{l} = \int_{\gamma_1} \rho \boldsymbol{\nabla} S \cdot d\boldsymbol{l} + \oint_{\overline{\gamma}_1 \cup \gamma_2} \rho \boldsymbol{\nabla} S \cdot d\boldsymbol{l} = \int_{\gamma_1} \rho \boldsymbol{\nabla} S \cdot d\boldsymbol{l} + \text{const.} \tag{21}$$

The relation (21) means that $\mathcal{U}$, introduced in (20), transforms S in σ and defines the new phase, modulo an arbitrary additive constant.

$$S \to \sigma = \int_\gamma \rho \boldsymbol{\nabla} S \cdot d\boldsymbol{l} + \text{const.} \tag{22}$$

The transformation (20) is well defined if the phase appears in the evolution equation only in derivative form. The current $\boldsymbol{j}_\phi$, associated to the new field ψ, takes the standard form of the linear quantum mechanics:

$$\boldsymbol{j}_\phi = \frac{\boldsymbol{\nabla}\sigma}{m}\rho, \tag{23}$$

while the continuity equation (14) is written as:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j}_\phi = 0. \tag{24}$$

The evolution equation for the field ϕ is again nonlinear:

$$i\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\phi + \kappa m\frac{\boldsymbol{j}_\phi^2}{\rho(1+\kappa\rho)}\phi - \kappa\frac{\hbar^2}{4m}\left[\Delta\rho - \frac{(\boldsymbol{\nabla}\rho)^2}{\rho}\right]\phi + V\phi, \tag{25}$$

but now the nonlinearity is a real quantity.

Note that the fields ϕ and ψ describe the same physical system because $|\phi|^2 = |\psi|^2 = \rho$ and the transformation $\mathcal{U}$, introduced in (20), has the property to make real the nonlinearity that appears in the evolution equation of the system. The price that we pay is that the new system, described by ϕ, is noncanonical because of the nonlinearity of the transformation.

We analyze the gauge transformation in the case of a solitary wave $\rho(\boldsymbol{x},t) \equiv \rho(\boldsymbol{\xi})$ with $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{u}t$.

The continuity equation (14) becomes:

$$\boldsymbol{u}\cdot\frac{\partial\rho}{\partial\boldsymbol{\xi}} = \frac{1}{m}\frac{\partial}{\partial\boldsymbol{\xi}}\cdot\left[\frac{\partial S}{\partial\boldsymbol{\xi}}\rho(1+\kappa\rho)\right], \tag{26}$$

and after integration and neglecting the integration constant, we obtain:

$$\frac{\partial S}{\partial\boldsymbol{\xi}} = \frac{m\boldsymbol{u}}{1+\kappa\rho}. \tag{27}$$

Taking into account Eq. (27), we can obtain the following expression for the transformation $\mathcal{U}$:

$$\mathcal{U}(\rho, S) = \exp\left\{\frac{i}{\hbar}\left[m\boldsymbol{u}\cdot\boldsymbol{\xi} - S(\boldsymbol{\xi})\right]\right\}. \tag{28}$$

The transformation $\mathcal{U}$, given by (28) for the solitary wave, is a particular case of the transformation introduced by Doebner and Goldin in Ref. [5]

$$\psi(\boldsymbol{x},t) \to \phi(\boldsymbol{x},t) = \rho^{1/2}(\boldsymbol{x},t)\exp\left[i\left(\frac{\gamma(t)}{2}\log\rho(\boldsymbol{x},t) + \frac{\lambda(t)}{\hbar}S(\boldsymbol{x},t) + \theta(\boldsymbol{x},t)\right)\right]. \tag{29}$$

Finally, let us generalize the method above introduced (to transform into a real quantity the nonlinearity introduced by EIP) to other physical systems.

The quantum current of a nonlinear system, canonical or not, can be written in the form:

$$\boldsymbol{j}_\psi = \frac{\boldsymbol{\nabla}S}{m}\rho - \boldsymbol{F}(\rho, S), \tag{30}$$

where $\boldsymbol{F}(\rho, S)$ is an arbitrary vectorial function.

The nonlinearity that appears in the evolution equation of the field ψ has an imaginary part given by:

$$\mathcal{W}(\rho, S) = \frac{1}{2\hbar\rho}\boldsymbol{\nabla}\cdot\boldsymbol{F}(\rho, S), \tag{31}$$

and the number of particle $N = \int \rho d^3x$ is conserved. It is not difficult to see that the unitary transformation $\mathcal{U}$: $\psi \to \mathcal{U}\phi$, defined as:

$$\mathcal{U}(\rho, S) = \exp\left[-i\frac{m}{\hbar}\int\limits_{\gamma}\frac{\boldsymbol{F}(\rho, S)}{\rho}d\boldsymbol{l}\right], \tag{32}$$

with $\boldsymbol{\nabla}\times(\boldsymbol{F}/\rho) = 0$, gives an evolution equation for the new field ϕ containing a real nonlinearity. The transformation (32) can be used to transform different nonlinear Schrödinger equations [5, 6, 7] to find their solutions.

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