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On Integrable Discretization of the Inhomogeneous Volterra Lattice

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Abstract

An integrable discretization of an inhomogeneous Volterra lattice is introduced. Some aspects of dynamics of a soliton affected by a random force are discussed.

1 Introduction

One of the main tasks of a full discretization of nonlinear continuous evolution equations is reproducing their behavior on a discrete level. In the case of integrable models there exist several well elaborated methods of doing this. In particular one can explore the discrete analog of the zero-curvature condition. However most of studies of particular models have been restricted, so far, to homogeneous equations. Meantime taking time- and coordinate-dependent coefficients into account is the natural extension of the theory as it brings the nonlinear models closer to the reality. Moreover, as it has been shown already in the first studies, where inhomogeneous integrable differential-difference equations were introduced, such generalization can lead to qualitatively new phenomena (i.e. phenomena which disappear in the continuum limit and in underline homogeneous models). For example, it has been obtained in Ref. [1] that the linear force added to the Ablowitz–Ladik (AL) model [2] results in oscillatory behavior of solutions. This does not happen in the continuum limit, i.e. in the case of the nonlinear Schrödinger equation with the linear force [3]. Later in Ref. [4] it has been argued that the mentioned effect has direct analogy in the solid state physics — an electron in a crystal lattice — and that is why it has been called *Bloch oscillations* (the discreteness of the nonlinear model corresponds to the periodicity of a potential of the crystal lattice). The analogy between a soliton in a discrete model and an electron in a crystal lattice has been found to be indeed deep since it has been shown analytically [4] and numerically [5] that another phenomenon — the dynamical localization — observed for electrons [6] is observed for solitons in of the forced AL model, as well. (The mentioned phenomenon consists of spatial localization of a soliton by a linear force which periodically depends on time with definite values of the frequency.) Bloch oscillations are obtained also for a "dark" lattice soliton [7].

A remarkable fact is that the results found for integrable AL lattice qualitatively hold also for nonintegrable, but having more wide physical applications, discrete nonlinear Schrödinger equation, the so-called self-trapping, model [8].

Integrable discretization of the inhomogeneous AL model has been proposed in [9]. Naturally, in a fully discrete model there appear new features of the soliton motion. As an example we mention a set of periods of Bloch oscillations which depend on the discretization step, and in particular, in order to be observable, require the step to be comensurable with the period of oscillations in the continuum-time model.

Another inhomogeneous lattice, discussed in [10], is the forced Volterra model

$$\dot{v}_n + v_n(v_{n-1} - v_{n+1}) - 2\gamma(t)v_n + (p_n - p_{n+1})v_n = 0, \tag{1}$$

$$p_n v_n - p_{n+2} v_{n+1} = 0, \tag{2}$$

where $\gamma(t)$ is an arbitrary real function of time and the dot stands for the derivative with respect to time. A releant property of the model (1), (2) is that its continuum limit [10] is the inhomogeneous Koreteweg-de Vries equation

$$v_t - 6vv_x + v_{xxx} = f(t)$$

studied in [11] from the viewpoint of the effect of a stochastic force on soliton dynamics.

The aim of the present paper is to introduce an integrable discretization of (1), (2) and discuss some properties of its one-soliton solution.

It is to be mentioned that the discretization of the homogeneous ($p_n = 0,\ f(t) = h^{-1}$) Volterra model as well as to its generalizations, so-called Bogoyavlensky lattices, is well known [12].

2 Discretization of the forced Volterra model

System (1), (2) can be included in the inverse scattering scheme if the respective spectral parameter is allowed to depend on time, $\dot{\lambda} = \gamma(t)\lambda$. The last property makes the form of lattice (1), (2) to be inconvenient for the purposes of the discretization. In order to avoid this inconvenience we notice that the simple transformation $v_n(t) = v(t)u_n(t)$ where $v(t) = \exp(2\int_0^t \gamma(t')dt')$, leads to the following system

$$\dot{u}_n + v(t)u_n(u_{n-1} - u_{n+1}) + (p_n - p_{n+1})u_n = 0, \tag{3}$$

$$p_n u_n - p_{n+2} u_{n+1} = 0, \tag{4}$$

which has a UV-pair with the spectral parameter independent on time [10]. Due to this reason in the present paper we concentrate on discretization of model (3), (4) which is subject to the boundary conditions as follows

$$\lim_{n\to\pm\infty} p(n,t) = p(t), \qquad \lim_{n\to\pm\infty} u(n,t) = 1. \tag{5}$$

It is to be mentioned here that further simplification of (3) can be achieved by trivial renormalization of both time, $t \mapsto \int_0^t v(t')dt'$ and p_n, $p_n \mapsto v(t)q_n$, allowing one to eliminate the explicit dependence on $v(t)$. In practice, this however not always leads to simplification of the problem, what happens for example in the case of stochastic perturbations: in that case for computing average values one has to return to original parameters.

Assume now that the variable t is discrete with the discretization step h. Consider the UV-pair as follows

$$U_n = \begin{pmatrix} \lambda & u_n \\ -1 & 0 \end{pmatrix}, \tag{6}$$

$$V_n = \begin{pmatrix} f - \tilde{u}_n c_n & -u_n\left[\lambda c_{n+1} + p_{n+1}/\lambda\right] \\ \lambda c_n + p_n/\lambda & f - \tilde{u}_{n-1}c_{n-1} + p_n + c_n\lambda^2 \end{pmatrix}, \tag{7}$$

where λ is independent on time spectral parameter, $f \equiv f(t)$ is an arbitrary function of t and tilde is used for the shift by h, e.g. $\tilde{u}_n \equiv u_n(t+h)$. Then the discrete analog of the zero curvative conditions

$$\tilde{U}_n V_n = V_{n+1} U_n \tag{8}$$

results in the following system of equations

$$f[\tilde{u}_n - u_n] + c_n\tilde{u}_n[\tilde{u}_{n+1} - u_{n-1}] + \tilde{u}_n p_n - u_n p_{n+1} = 0, \tag{9}$$

$$\tilde{u}_{n-1}p_{n-1} - u_n p_{n+1} = 0, \tag{10}$$

$$c_{n+1}u_n - c_n\tilde{u}_n = 0. \tag{11}$$

It is a straightforward algebra to verify that in the limit $h \to 0$ (9)–(11) are reduced to (2), (3) (in the last equation v_n being expressed through u_n) if one chooses $f(t) \equiv h^{-1}$ and $c_n \sim v(t)$.

The discrete analog of boundary conditions (5) reads ($p \equiv p(t)$ and $c \equiv c(t)$ are arbitrary functions of time)

$$\lim_{n\to\pm\infty} p_n = p, \qquad \lim_{n\to\pm\infty} u_n = 1, \qquad \lim_{n\to\pm\infty} c_n = c. \tag{12}$$

System (9)–(11) can be rewritten explicitly in a form of a map $u_n \mapsto \tilde{u}_n$. To this end we notice that it follows from (10) and (11) that

$$c_n = \phi u_n p_n p_{n+1}, \tag{13}$$

where $\phi \equiv c/p^2$ is a function on time only. Then the map we are interested in takes the form

$$\tilde{u}_n = \frac{p_{n+2}}{p_n} u_{n+1} \tag{14}$$

and p_n is computed with help of the formula

$$\begin{aligned} &f(u_{n+1}p_{n+2} - u_n p_n) + u_{n+1}p_n p_{n+2}(\phi u_n u_{n+2} p_{n+3} + 1) \\ &\qquad - u_n p_n p_{n+1}(\phi u_{n-1}u_{n+1}p_{n+2} + 1) = 0. \end{aligned} \tag{15}$$

3 Soliton dynamics

Let us now concentrate on a one-soliton solution of (9)–(11). Recalling the well known results on the inverse scattering technique applied to the Volterra lattice [13] we introduce the transfer matrix $T \equiv T(t)$ through the relation $T_-(n,t) = T_+(n,t)T$, where $T_\pm(n,t)$ is a solution of the eigenvalue problem $T_\pm(n+1,t) = U_n T_\pm(n,t)$ subject to the boundary conditions $\lim_{n\to\pm\infty} T_\pm(n,t) = MZ^n$ with $Z = \mathrm{diag}\,(1/z, z)$, $M = I - z\sigma_1$, $\lambda = z + 1/z$ and σ_1 being the Pauli matrix. Then the dependence of T on the discrete time is given by

$$\tilde{T} = ATA^{-1}, \tag{16}$$

where

$$A = \operatorname{diag}\left(f + z^2 c + \frac{p}{1+z^{-2}},\, f + z^{-2} c + \frac{p}{1+z^2}\right).$$

Now we consider in details the case when the solution of discrete problem is reduced to one-soliton solution of the inhomogeneous Volterra lattice [10]. To this end we put $f \equiv h^{-1}$ (i.e. to be constant) and assume that the Cauchy problem is defined by "initial"conditions given at $t = 0$ (generalization to the case $t = t_0 > 0$ is straightforward). Then $t = mh$ where m is a nonegative integer, and we introduce the notation

$$\mu_m(z) = \frac{\left(1 + hc^{(m)} z^2\right)\left(1 + z^2\right) + hp^{(m)} z^2}{\left(1 + hc^{(m)} z^{-2}\right)\left(1 + z^2\right) + hp^{(m)}}, \tag{17}$$

where $c^{(m)} \equiv c(mh)$, and $p^{(m)} \equiv p(mh)$. The dependence of the soliton solution on time is then governed by the equation

$$b^{(m)} = \prod_{l=1}^{m} \mu_l(z_1) b^{(0)}. \tag{18}$$

Finally the soliton solution can be written down in the form $[v^{(m)} \equiv v(mh)]$

$$v_n = v^{(m)} \frac{\cosh \Theta_{n+1}^{(m)} \cosh \Theta_{n-2}^{(m)}}{\cosh \Theta_n^{(m)} \cosh \Theta_{n-1}^{(m)}}, \tag{19}$$

where

$$\Theta_n^{(m)} = -Kn - n_0 + F^{(m)}, \tag{20}$$

$$F^{(m)} = \frac{1}{2} \sum_{l=1}^{m} \ln \frac{2\left(1 + hc^{(l)} e^{-2K}\right)\cosh(2K) + hp^{(l)} e^{-K}}{2\left(1 + hc^{(l)} e^{2K}\right)\cosh(2K) + hp^{(l)} e^{K}}, \tag{21}$$

K is a positive constant defining the soliton width and amplitude, and n_0 is a constant which plays a part of the initial position of the soliton (in what follows it will be taken equal to zero).

As an example we consider application of integrable inhomogeneous discrete models to study of effect of random forces on the soliton dynamics, which in general, i.e. in nonintegrable cases, seems to be still an open question. Indeed, most of conventional method of the theory of stochastical processes are not applicable here. Moreover, passing to numerical simulations one meets new problems related to interplay of temporal scales associated with random forces, i.e correlation radii, and with the discretization step. In order to explain the last point let us assume that we are interested in soliton dynamics subject to delta-correlated Gaussian noise $p(t)$ (D is a constant):

$$\langle p(t) \rangle = 0, \qquad \langle p(t) p(t') \rangle = 2D\delta(t - t'). \tag{22}$$

To this end one has to discretize the random force. In order to generate respective sequence associated with the discretization of $p(t)$ one introduces [14]

$$p^{(m)} = \frac{1}{h} \int_{mh}^{(m+1)h} p(t)\, dt \tag{23}$$

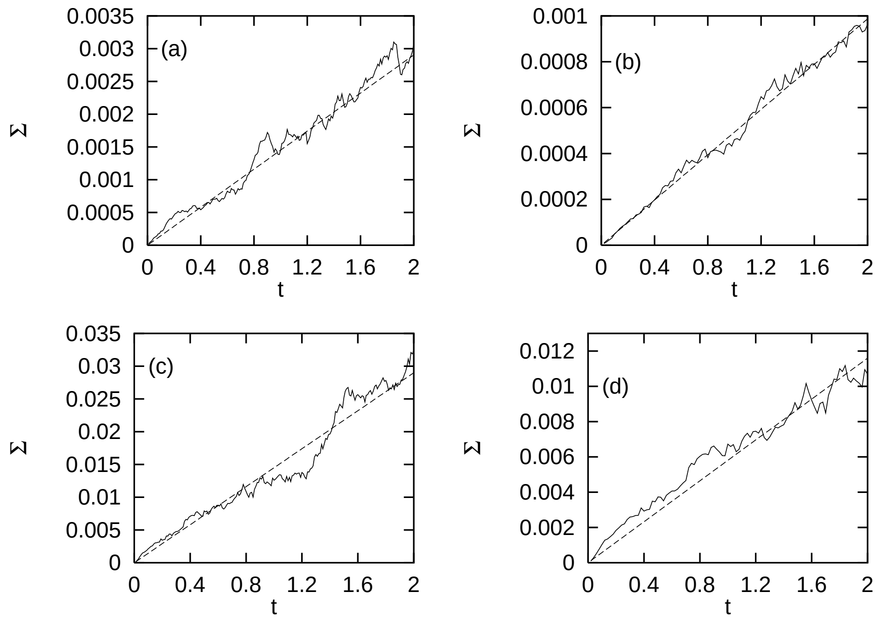

Figure 1. Dispersion of the trajectory of the soliton centre at (a) $h = 0.01$, $K = 2$; (b) $h = 0.02$, $K = 2$; (c) $h = 0.01$, $K = 1$; and (d) $h = 0.02$, $K = 1$. In all the cases $D = 0.1$. Dashed lines display the law $\mathcal{D}t$. In all the cases the mean values were computed over 40 realizations. Pseudo-random sequences were obtained using MAPLE generators of random numbers distributed according to the normal law.

and require

$$\langle p^{(m)} \rangle = 0, \qquad \langle p^{(m)} p^{(n)} \rangle = \sigma^2 \delta_{nm}, \tag{24}$$

where $\sigma^2 = 2D/h$ and δ_{nm} is the Kronecker delta.

However, as it is evident from the definition (23) strictly speaking the results of simulations with the discrete stochastic process defined by (24) must be interpreted as ones corresponding to a random force with the correlation radius τ less that the step of the discretization, $\tau < h$, rather than to a process with the zero correlation radius. This becomes especially important for long term simulations and for cases of strong fluctautions, i.e. large D. This is illustrated in Fig. 1, where we present typical results of numeriacal simulations of the dynamics of the center of the soliton at initial stages affected by the Gaussian noise. More specifically we present the dispersion Σ of the soliton center defined as $\Sigma = \langle (F^{(m)} - F_0^{(m)})^2 \rangle$ where $F_0^{(m)}$ is the position of the center of the soliton in the absence of the random force, i.e. $F_0^{(m)}$ is given by (21) with all $p^{(m)} = 0$. Comparing Figs. 1a and c with Figs. 1b and d, correspondingly, one conclude that in the moving frame the soliton displays the Brownian motion, i.e. $\Sigma = \mathcal{D}t$, as it must be in accordence with the continuum model. The diffusion coefficient $\mathcal{D}$ however essentially depends on the step of discretization. So in Figs. 1a and b one has $\mathcal{D} \approx 0.25\mathcal{D}_0$ and $\mathcal{D} \approx 0.095\mathcal{D}_0$, where $\mathcal{D}_0 = \left(\frac{\tanh K}{2K}\right)^2 \cdot D$ ($\approx$) is the diffusion coefficient of the continuum model. The relation $\mathcal{D}/\mathcal{D}_0$ is almost unity already at $h = 0.005$. Decreasing of the soliton amplitude and increasing of the soliton width results in more rapid convergence to the delta-correlated process. In Figs. 1c and d the same force is applied to the soliton with $K = 1$. The relation $\mathcal{D} = \mathcal{D}_0$ is achieved already at $h = 0.01$ while at $h = 0.02$ we have $\mathcal{D} \approx 0.4\mathcal{D}_0$.

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