Journal of Nonlinear Mathematical Physics 2001, V.8, Supplement, 178–182 Proceedings: NEEDS'99

Construction of Variable Mass Sine-Gordon and Other Novel Inhomogeneous Quantum Integrable Models

Anjan KUNDU

Saha Institute of Nuclear Physics, Theory Group
1/AF Bidhan Nagar, Calcutta 700 064, India
E-mail: anjan@tnp.saha.ernet.in

Abstract

The inhomogeneity of the media or the external forces usually destroy the integrability of a system. We propose a systematic construction of a class of quantum models, which retains their exact integrability inspite of their explicit inhomogeneity. Such models include variable mass sine-Gordon model, cylindrical NLS, spin chains with impurity, inhomogeneous Toda chain, the Ablowitz–Ladik model etc.

1 Introduction

The physical systems often encounter the inhomogeneity in the form of impurities, defects or the density fluctuations in the media or it can enter as variable magnetic fields or other external forces. Such inhomogeneities can depend on space as well as time variables and can appear as explicit space-time dependent coefficients in the Hamiltonian. They usually destroy the integrability of a system making its analytic study almost impossible [1]. However there are several examples of classical models like deformed MKDV [2] or NLS [3], spherical or cylindrical symmetric NLS [4], inhomogeneous Ablowitz–Ladik (AL) model [5] etc., where the exact integrability could be retained along with their Lax operators, in spite of the presence of space-time dependent coefficients in their evolution equations. Nevertheless, in case of quantum models there seems to be no systematic attempts to explore such possibilities, except certain construction for the impurity chains [6].

We propose here a scheme for introducing inhomogeneities in the known quantum integrable models, which retains their integrability and allows explicit construction of their R-matrix as well as the Lax operators. This systematic scheme is based on a novel quadratic algebra derived from the quantum Yang–Baxter equation, where its Casimir operators play the role of the inhomogeneity parameters and the proper realization of other generators construct the Lax operator of the model. The R-matrix simply corresponds to that of the standard homogeneous model. Applying this procedure one first constructs quantum models with inhomogeneity on discrete lattices, which preserves the exact integrability and then taking the continuum limit builds the corresponding field models. Thus one obtains a series of new quantum integrable inhomogeneous models like a variable mass quantum sine-Gordon model, cylindrical quantum NLS, inhomogeneous Toda and Ablowitz–Ladik chains. It also provides a different way of introducing integrable impurities in the spin chain models.

2 The generating scheme

We start with the quadratic algebra [7]

$$[S^3, S^{\pm}] = \pm S^{\pm}, \qquad [S^+, S^-] = \left(M^+ \sin\left(2\alpha S^3\right) + M^- \cos\left(2\alpha S^3\right)\right) \frac{1}{\sin\alpha}, \\ [M^{\pm}, \cdot] = 0, \tag{1}$$

and the quantum L-operator

$$L_t(\xi) = \begin{pmatrix} \xi c_1^+ e^{i\alpha S^3} + \xi^{-1} c_1^- e^{-i\alpha S^3} & 2\sin\alpha S^- \\ 2\sin\alpha S^+ & \xi c_2^+ e^{-i\alpha S^3} + \xi^{-1} c_2^- e^{i\alpha S^3} \end{pmatrix}, \qquad \xi = e^{i\alpha\lambda}, \tag{2}$$

where $M^{\pm} = \pm\sqrt{\pm 1}\left(c_1^+ c_2^- \pm c_1^- c_2^+\right)$ are the Casimir operators of algebra (1). Taking the well known trigonometric $R(\lambda)$-matrix solution [8] along with (2) the quadratic algebra (1) can be shown to be equivalent to the quantum Yang Baxter relation [8] $R(\lambda-\mu)L(\lambda)\otimes L(\mu) = (I \otimes L(\mu)) \otimes (L(\lambda) \otimes I)R(\lambda-\mu)$. Therefore the associated L-operator (2) may serve as the generating Lax operator for the quantum integrable models belonging to the relativistic or the anisotropic class of models, the parameter $q = e^{i\alpha}$ playing the role of the deformation parameter [9]. The integrable inhomogeneities are introduced in fact through different representations of the Casimir operators $c_a^{\pm}$ by choosing their eigenvalues as position and time dependent functions.

For definiteness we consider here α to be real. At the undeformed limit $q \to 1$ or equivalently at $\alpha \to 0$ all the entries in the above scheme, i.e. algebra (1), L-operator (2) and the trigonometric R-matrix are reduced to their corresponding rational forms. The reduced algebra is simplified but still represents a quadratic algebra:

$$[s^+, s^-] = 2m^+ s^3 + m^-, \qquad [s^3, s^{\pm}] = \pm s^{\pm} \tag{3}$$

with $m^+ = c_1^0 c_2^0$, $m^- = c_1^1 c_2^0 + c_1^0 c_2^1$ as the new central elements. Note that both (1) and (3) are Hopf algebras with explicit coproduct structure, counit, antipode etc. [9].

Due to $\alpha \to 0$ and $\xi \to 1 + \alpha\lambda$ the L-operator takes the form

$$L_r(\lambda) = \begin{pmatrix} c_1^0\left(\lambda + s^3\right) + c_1^1 & s^- \\ s^+ & c_2^0\left(\lambda - s^3\right) - c_2^1 \end{pmatrix}, \tag{4}$$

with spectral parameter λ and the quantum R-matrix is reduced to its well known rational form [8]. Remarkably, our scheme with these reduced entries belonging to the rational class becomes suitable for generating quantum integrable nonrelativistic models with inhomogeneity.

3 Inhomogeneous quantum integrable models

3.1 Variable mass sine-Gordon model

Since this is a relativistic model we have to use the objects belonging to the trigonometric class. Through canonical operators u, p a representation of (1) may be given by

$$S^3 = u, \qquad S^+ = e^{-ip} g(u), \qquad S^- = g(u) e^{ip}, \tag{5}$$

where the operator function

$$g(u) = \big(1 - M^+ \sin\alpha u \cdot \sin\alpha(u+1)\big)^{1/2} \frac{1}{\sin\alpha}. \tag{6}$$

By choosing the eigenvalues of the Casimirs as $M_j^+ = -(\Delta m_j)^2$, $M_j^- = 0$ and inserting (5) in (2) one gets a quantum integrable lattice model involving bosonic operators and the inhomogeneity parameter m_j. Comparing with the well known result [10] we may conclude that the model thus constructed is a generalization of the exact lattice version of the quantum sine-Gordon model. For going to the continuum limit we may scale p by lattice constant Δ and take the limit $\Delta \to 0$. As a result one derives from (2) the Lax operator of the sine-Gordon field model

$$\mathcal{L} = im\left(u_t\sigma^3 + \left(k_1 \cos u\sigma^1 + k_0 \sin u\sigma^2\right)\right), \tag{7}$$

where the mass parameter $m = m(x,t)$ now is not a constant as in the standard case, but an arbitrary function of x, t. The variable mass also enters the Hamiltonian of this novel sine-Gordon model as

$$\mathcal{H} = \int dx\left[m(x,t)(u_t)^2 + (1/m(x,t))(u_x)^2 + 8(m_0 - m(x,t)\cos(2\alpha u))\right], \tag{8}$$

which is integrable both at classical and the quantum level for the arbitrary mass function $m(x,t)$. Note that if the mass is independent of time and depends only on the space coordinate: $m = m(x)$, one can formally convert the evolution equation into the standard sine-Gordon through a coordinate change: $x \to X = \int^x m(y)dy$ and can find its exact soliton solution as

$$u = 2\tan^{-1}\left[\exp\left(\gamma \int^x m(y)dy + vt\right)\right], \tag{9}$$

which exhibit intriguing structure depending on the choice of the mass-function $m(x)$. Such variable mass sine-Gordon equations may arise in physical situations [1] and therefore the related exact results become important.

3.2 Inhomogeneous NLS model

Nonlinear Schrödinger equation belongs to the nonrelativistic class. Therefore we should use the rational R-matrix and the rational L-operator (4) with suitable realizations of algebra (3). A simple such realization may be given by considering site-dependent values for central elements in (4) and in the generalized HPT

$$\begin{aligned} &s^3 = s - N, \qquad s^+ = g_0(N)\psi, \qquad s^- = \psi^\dagger g_0(N), \\ &g_0^2(N) = m^- + m^+(2s - N), \qquad N = \psi^\dagger\psi. \end{aligned} \tag{10}$$

This exactly integrable quantum discrete model is an inhomogeneous generalization of the known lattice NLS [10]. In the continuum limit one may introduce the inhomogeneity by choosing the eigenvalues of the central elements as

$$c_1^1 = \frac{1}{\Delta} + f, \qquad c_2^1 = -\left(\frac{1}{\Delta} - f\right), \qquad c_1^0 = -c_2^0 = g$$

with f and g being space-time dependent arbitrary functions. The Lax operator of the field model would be given formally by that of the NLS model, where the constant spectral parameter should be replaced by $\tilde{\lambda} = g\lambda + f$ and the field variables by $\psi/\sqrt{g}$. Particular choice of these functions as $f = \frac{4x}{t}$, $g = \frac{1}{t}$ would yield integrable cylindrical NLS [4] like equation at the quantum level.

3.3 Inhomogeneous Toda chain

Another interesting realization of algebra (3) may be given by

$$s^3 = -ip, \qquad s^{\pm} = \alpha e^{\mp u} \tag{11}$$

with $m^{\pm} = 0$, which leads to the construction of quantum Toda chain model. A consistent choice of the Casimir eigenvalues like $c_2^0 = c_2^1 = 0$ together with c_1^0 and c_1^1 taken as space-time dependent coefficients $c_j^0(t)$ and $c_j^1(t)$, would now result a novel quantum integrable inhomogeneous Toda chain given by the Hamiltonian

$$H = \sum_j \left(p_j + \frac{c_j^1}{c_j^0} \right)^2 + \frac{1}{c_j^0 c_{j+1}^0} e^{u_j - u_{j+1}}. \tag{12}$$

For $c_j^1 = 0$ the evolution equation can be written down in an interesting compact form

$$\frac{\partial^2}{\partial t^2} u_j = e^{u_j^+ - u_{j+1}^-} - e^{u_{j-1}^+ - u_j^-}, \tag{13}$$

where $u_j^{\pm}(t) = u_j(t) \pm \phi_j(t)$, with $\phi_j(t)$ being an arbitrary function inducing inhomogeneity in the system. Note that, when the c's are time dependent coefficients such inhomogeneities can not be removed through gauge transformation or variable change.

Using similar procedure one may construct impurity spin chains in a different way by seeking various spin operator realizations of the Lax operators (2) or (4) at the impurity sides. An inhomogenegious version of the quantum Ablowitz–Ladik model can also be constructed generalizing the result of [11] obtained through twisting transformation in trigonometric case.

4 Concluding remarks

Thus we have prescribed a systematic scheme for constructing a novel series of inhomogeneous quantum integrable models belonging to the lattice as well as the field models of both relativistic ($q \neq 1$) and nonrelativistic ($q = 1$) class along with their corresponding classical counterparts. The scheme is based on an algebraic approach, where the generators through different realizations construct nonlinear functions of field operators and the Casimir operators with space-time dependent eigenvalues introduce inhomogeneity into the system. In our scheme one also obtains automatically the Lax operators and the R-matrices of the models constructed.

Acknowledgements

I thank the Humboldt Foundation, Germany and the organizers of NEEDS'99 for financial support.

References

[1] Malomed et al, *Phys. Rev. B*, 1990, V.41, 11271; *Phys. Lett. A*, 1990, V.144, 351;
Sanchez S, Bishop A R and Dominguez-Adame F, e-print patt-sol/9401005;
Woafo P, *J. Phys. Soc. Japan*, 1998, V.67, 3734;
Sen D and Lal S, e-print cond-mat/9811330.

[2] Burtsev S P, Zakharov V E and Mikhailov A V, *Teor. Mat. Fiz.*, 1987, V.70, 323.

[3] Chen H H and Liu C S, *Phys. Rev. Lett.*, 1976, V.37, 693;
Horvatthy P A and Yera J C, The Variable Coefficient NLS Equation and the Conformal Properties of Non-Relativistic Space Time, Preprint 18/3/99.

[4] Radha R and Lakshmanan M, *Chaos Solitons and Fractals*, 1994, V.4, 181.

[5] Bruschi M, Levi D and Ragnisco O, *Nuovo Cimento A*, 1979, V.53, 21;
Konotop V V, Chubaikalo O A and Vazquez L, *Phys. Rev. E*, 1993, V.48, 563.

[6] Schmitteckert P, Schwab P and Eckern U, *Erophys. Lett.*, 1995, V.30, 543;
Eckle H P, Punnoose A and Römer R, *Erophys. Lett.*, 1997, V.39, 293;
Bedürftig G, Essler F and Frahm H, *Nucl. Phys. B*, 1997, V.489, 697.

[7] Kundu A, *Phys. Rev. Lett.*, 1999, V.82, 3936.

[8] Kulish P and Sklyanin E K, *Lect. Notes in Phys.*, Vol.151, Editors J Hietarinta et al, Springer, 1982, 61.

[9] Kundu A, Algebraic Construction of Quantum Integrable Models Including Inhomogeneous Models, Proc. of the Annual Conf. of Math. Rev., Tarun, Poland, 1999.

[10] Izergin A G and Korepin V E, *Nucl. Phys. B*, 1982, V.205, 401.

[11] Kundu A and Basumallick A, *Mod Phys. Lett. A*, 1992, V.7, 61.