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Nonlinear Renormalization Group Flow and Optimization

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Abstract

Renormalization group flow equations for scalar $\lambda\Phi^4$ are generated using smooth smearing functions. Numerical results for the critical exponent ν in $d = 3$ are calculated by polynomial truncation of the blocked potential. It is shown that the convergence of ν with the order of truncation can be improved by fine tuning of the smoothness parameter.

1 Introduction

The Renormalization Group (RG) is a powerful tool for investigating nonperturbative physical phenomena [1]. Through the continuous elimination of degrees of freedom, RG provides a systematic resummation of the perturbative series and yields information about the nonperturbative nature of the system. However, the power of RG relies on the existence of efficient analytic and computational methods since the full RG equation cannot be solved exactly. The goal in this contribution is to show how the approximate RG can be optimized to improve the nonperturbative calculations.

For a field theoretical system a continuous RG transformation can be realized by introducing a smearing function $\rho_k(x)$ which governs the coarse-graining procedure [2]. The scale k acts as an effective IR cutoff that separates the low- and high-momentum modes. Using this smearing function an averaged blocked field can be defined as

$$\phi_k(x) = \int_y \rho_k(x-y)\phi(y). \tag{1.1}$$

The modified propagator becomes $\Delta_k(p) = \tilde{\rho}_k(p)/p^2$, where $\rho_k(p) + \tilde{\rho}_k(p) = 1$. By varying the blocked action $\widetilde{S}_k[\Phi]$ infinitesimally with k, we obtain the following RG equation [3]:

$$k\frac{\partial \widetilde{S}_k}{\partial k} = -\frac{1}{2}\,\mathrm{Tr}\left[\tilde{\rho}_k^{-1}\left(k\frac{\partial \tilde{\rho}_k}{\partial k}\right)\left(1+\frac{\tilde{\rho}_k}{p^2}\frac{\delta^2 \widetilde{S}_k}{\delta\Phi^2}\right)^{-1}\right]. \tag{1.2}$$

A reliable method for approximating $\widetilde{S}$ is the derivative expansion [4]:

$$\widetilde{S}_k[\Phi] = \int_x \left\{\frac{Z_k(\Phi)}{2}(\partial_\mu\Phi)^2 + U_k(\Phi) + O\left(\partial^4\right)\right\}, \tag{1.3}$$

where $Z_k(\Phi)$ and $U_k(\Phi)$ are, respectively, the wave function renormalization and the blocked potential. At leading order where $Z_k(\Phi) = 1$, the low-energy effective action is described solely in terms of $U_k(\Phi)$ [5]. This local potential approximation (LPA) results in the following flow equation:

$$\begin{aligned} k\frac{\partial U_k(\Phi)}{\partial k} &= \frac{1}{2}\int_p \left(k\frac{\partial \tilde{\rho}_k(p)}{\partial k}\right) \frac{U_k''(\Phi)}{p^2 + \tilde{\rho}_k(p)U_k''(\Phi)} \\ &= \frac{S_d k^d}{2}\int_{t(z=0)}^{t(z=\infty)} \frac{dt\; z^d(t)\bar{U}_k''(\Phi)}{z^2(t) + (1-t)\bar{U}_k''(\Phi)}, \end{aligned} \tag{1.4}$$

where $t = 1 - \tilde{\rho}_k(p)$, $S_d = 2/(4\pi)^{d/2}\Gamma(d/2)$, $z(t) = p(t)/k$ and $\bar{U}_k''(\Phi) = U_k''(\Phi)/k^2$.

Clearly, the form of the RG equation for $U_k(\Phi)$ depends on the choice of $\rho_k(p)$. However, according to the universality principle the shape of $\rho_k(p)$ should not influence the physics of finite length scales; so long as the effective action contains all marginal or relevant operators the resulting RG flow must be scheme independent. This indeed holds at leading order in the derivative expansion. On the other hand, universal properties of the system may vary when they are determined in the vicinity of the fixed point of the truncated equation given in Eq. (1.4).

2 Smooth cutoff functions

The smooth smearing functions considered here are

$$\begin{aligned} &\text{(i)} \qquad \rho_{k,\varepsilon}(p) = \frac{1}{2}\left[1 + \tanh\left(\frac{k^2 - p^2}{pk\varepsilon}\right)\right], \\ &\text{(ii)} \qquad \rho_{k,b}(p) = e^{-a(p/k)^b}, \\ &\text{(iii)} \qquad \rho_{k,m}(p) = \frac{1}{1 + (p/k)^m}. \end{aligned} \tag{2.1}$$

They approach $\Theta(k-p)$ as the smoothness parameter $\sigma(\varepsilon, b, m)$ tends to zero. For the general case of a smooth cutoff, it is difficult to obtain a non-truncated solution, and we circumvent the problem with polynomial truncation at order ϕ^{2M} [6]:

$$U_k(\Phi) = \sum_{\ell=1}^{M} \frac{g_k^{(2\ell)}}{(2\ell)!}\Phi^{2\ell}, \qquad g_k^{(2\ell)} = U_k^{(2\ell)}(0) = \left.\frac{\partial^{2\ell} U_k}{\partial \Phi^{2\ell}}\right|_{\Phi=0}. \tag{2.2}$$

The remaining task is to solve a system of M integro-differential equations for the coupling constants $g^{(2\ell)*}$, $\ell = 1, \ldots, M$.

Eq. (1.4) in dimensionless form reads:

$$\left[k\frac{\partial}{\partial k} - \frac{1}{2}(d-2)\bar{\Phi}\frac{\partial}{\partial \bar{\Phi}} + d\right]\bar{U}_k(\bar{\Phi}) = -\int_0^1 dt\, \frac{z^d(t)\bar{U}_k''(\bar{\Phi})}{z^2(t) + (1-t)\bar{U}_k''(\bar{\Phi})}, \tag{2.3}$$

where $\bar{U}_k(\bar{\Phi}) = \zeta^2 k^{-d} U_k(\Phi)$, $\bar{\Phi} = \zeta k^{-(d-2)/2}\Phi$, $\zeta = \sqrt{2/S_d} = \sqrt{(4\pi)^{d/2}\Gamma(d/2)}$, and $\eta = 0$. To characterize the critical behavior of the theory, we must first identify all the fixed points around which RG is linearlized. For a general smooth $\rho_{k,\sigma}(p)$, the location of the Wilson–Fisher fixed point will depend on σ.

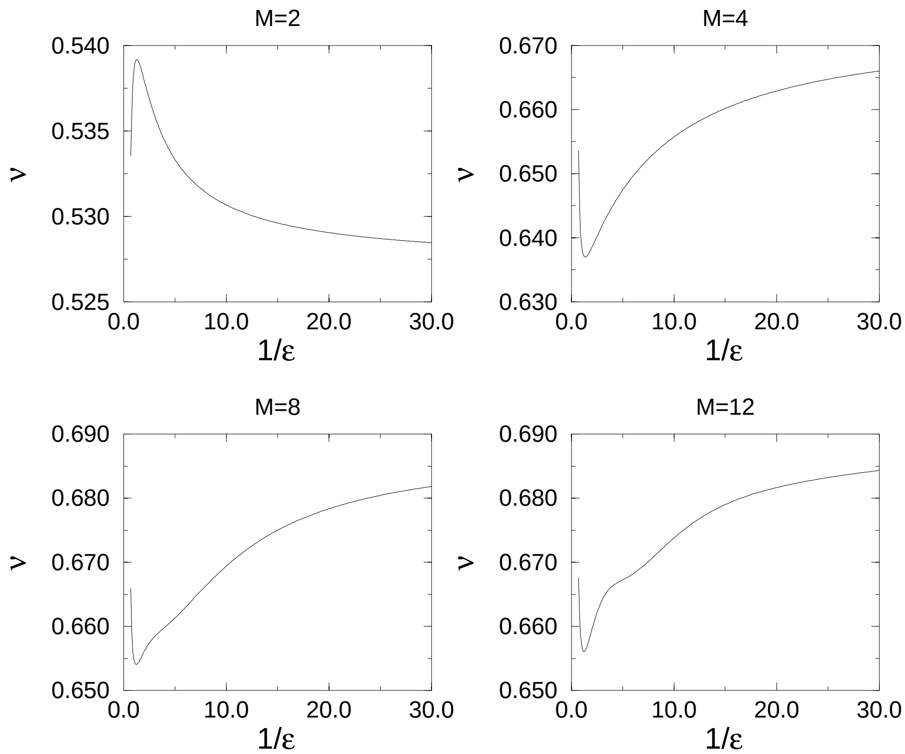

Figure 1. Critical exponent ν as a function of $1/\varepsilon$ using the hyperbolic tangent smearing function. Results for four different levels of polynomial truncation are shown.

Around the fixed point(s), the theory exhibits scale invariance, i.e. $\partial_k \bar{U}_k^* = 0$, and the RG flow can be linearized by writing $\bar{U}_k(\bar{\Phi}) = \bar{U}^*(\bar{\Phi}) + \bar{v}(\bar{\Phi})e^{-\lambda \ln k}$. We first determine the fixed-point solution analytically in the sharp cutoff limit, which is then used as an initial guess in our numerical root-finding subroutine for the smooth case. The smoothness of the cutoff function is increased in small steps by decreasing the parameters. In this manner we are able to track the solution associated with a physical fixed point from the sharp cutoff limit to an arbitrary smoothness.

In Fig. 1 we plot the dependence of ν in $d = 3$ as a function of ε^{-1}. We see that ν varies by 2–5 % over the range shown. The variation would be greater had we taken $\varepsilon^{-1} \to 0$. The exponents in this limit are mean-field like, e.g., $\nu = 0.5$. It is also interesting to note that at each order in M there is a dip, or an extremum whose position changes only slightly with M. The behavior remains qualitatively the same for the exponential and power-law smearing functions.

3 Scheme dependence and optimization

Even though non-truncated solutions for the fixed-point potential $\bar{U}^*(\bar{\Phi})$ can be obtained in special cases [4], this is not possible in general. Any approximation inevitably brings in scheme-dependent effects. Our results are seen to depend both on the order of truncation M as well as the smoothness parameter σ.

At order M we obtain M fixed-point solutions parameterized by the critical coupling constants $\left(g^{(2)*}, \ldots, g^{(2M)*}\right)$, each having its own eigenvectors and eigenvalues which in

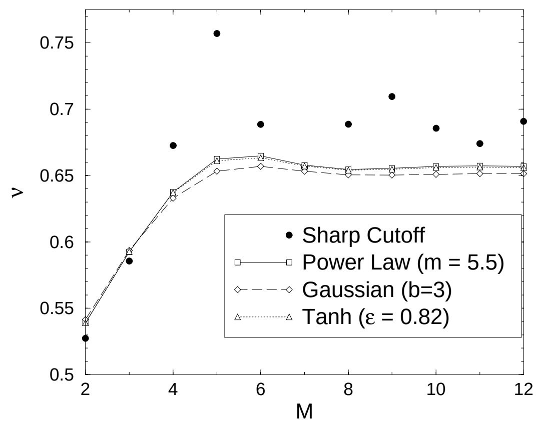

Figure 2. Critical exponent ν as a function of the level of polynomial truncation for all three optimized smearing functions and the sharp cutoff.

turn can be used for calculating the critical exponents. However, only one of the solutions corresponds to the Wilson–Fisher (WF) fixed point which is characterized by one relevant operator, with the rest being irrelevant. Computational artifacts are induced with a polynomial truncation of $\bar{U}_k(\bar{\Phi})$ [7, 4]. For example, as M is increased, numerous unphysical fixed points are also generated. The salient feature of the WF solution is that it is stable against an increase in M. Unphysical solutions will be sensitive to the higher order equations and their eigenvalues will fluctuate in an unpredictable manner. Another prominent artifact of the polynomial truncation is the oscillation of the critical exponents with M when a sharp cutoff is used.

Why can we not reproduce the exact critical exponent(s) by keeping only the renormalizable coupling constants? The main reason lies in that fact the that fixed point of the truncated solution is applicable only up to the operators which have been neglected. At the point which we call a "fixed point", the would-be ignored set of irrelevant operators does not vanish but continues to evolve. On the other hand, in the exact RG approach, the fixed-point condition implies a complicated cancellation between different operators.

From Fig. 1, we see that ν clearly depends on σ. That is, because of the polynomial truncation, physical results now depend on M as well as on σ. How can such dependence be reconciled with universality? The key to the problem here again lies in the role played by the irrelevant operators in the approximate solution. If we solve the RG equation for the full theory and come down to the IR regime, the dependence on the initial condition for the irrelevant operators must die out according to the universality hypothesis. The explicit verification of this scenario by reaching the IR fixed point in principle requires an infinite number of iterations. This is not what was followed in this work. Instead we move close to the UV fixed point and a linearized RG prescription is utilized to deduce the critical exponents and the corresponding scaling laws. Using the derivative expansion and the polynomial truncation schemes gives the advantage of a readily accessible (approximate) fixed-point solution; however, we are no longer certain what the omitted irrelevant operators do. The comparison with the exact RG can offer a direction of improvement in our approximation.

We wish to choose the cutoff to be as smooth as possible in order to minimize the generation of non-local, higher-derivative irrelevant interactions, as well as the order of truncation M. For the three smearing functions discussed, there is an "optimal" value of σ for which ν converges most rapidly. As illustrated in Fig. 2, the optimal value for the exponential function is found to be $b = 3$. Although there remains small oscillations, we see a dramatic improvement in the convergence. In fact, the variation beyond $M = 7$ is only 1–2 %.

In general, as σ increases and more fast modes are being included in the loop integrations, more cancellations between the effects of the irrelevant operators take place and the theory moves closer to the *true* Wilson–Fisher fixed point found in the exact approach. Nevertheless, when σ becomes too large and the resulting smearing function is too sharp, non-local effects begins to set in and the theory drifts away from the true RG trajectory. At $\sigma = \infty$ where the smearing function becomes a step function $\Theta(k-p)$, non-local effects become maximal and LPA is no longer adequate. One must then include derivative operators to all orders in the blocked action $\widetilde{S}_k[\Phi]$ in order to arrive at a scheme-independent result.

4 Summary and discussions

In the present work we have demonstrated how one can improve the convergence of critical exponents calculated using polynomial truncations of the RG flow equation for the blocked potential $U_k(\Phi)$. We find that there exists an optimal smoothness value which gives the most rapid convergence as a function of M. This is due to the maximal cancellation of the effects generated by the irrelevant operators in the blocked action. Our optimal smearing functions yield $\nu = 0.65(5)$ for $M \geq 7$ with a variation of 1–2 %. The convergence behavior of the polynomial truncation scheme is intimately related to the non-truncated solution of $\bar{U}_k(\bar{\Phi})$.

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