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Nonlinear Models in Quantum Optics through Quantum Algebras

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Abstract

The $su_q(2)$ algebra is shown to provide a natural dynamical algebra for some nonlinear models in Quantum Optics. Applications to the computation of eigenvalues and eigenvectors for the Hamiltonian describing second harmonics generation are proposed.

1 Introduction

The trilinear boson Hamiltonian

$$\mathcal{H} = \omega_a\, a^\dagger\, a + \omega_b\, b^\dagger\, b + \omega_c\, c^\dagger\, c + \lambda\,(a^\dagger\, b^\dagger\, c + c^\dagger\, b\, a), \tag{1.1}$$

with a, b, c being boson operators for three different modes of the radiation field, describes (under certain conditions) nonlinear quantum optical processes as frequency conversion and Raman and Brillouin scattering [1]. Through a Jordan–Schwinger transformation, this model is algebraically equivalent to the resonant interaction of a sample of two-level atoms with a single mode of the quantized radiation field (the so called Dicke model, see [2] and references therein):

$$\mathcal{H} = H_0 + H_D = \omega\, a^\dagger\, a + \omega_{at}\, S_z + g(a^\dagger\, S_- + a\, S_+). \tag{1.2}$$

Here, $S_i = \sum_{k=1}^{N} S_i^{(k)}$, where N is the number of atoms in the sample and $S_i^{(k)}$ are the pseudospin operators of the k-th atom.

Exact general solutions for the Dicke Hamiltonian are very difficult to obtain. Bethe ansatz techniques have been used [3] but they reduce the problem to an algebraic equation that does not lead to a useful form for the general eigenvalues and eigenfunctions of $\mathcal{H}$. In [2, 4] a perturbative approach based on an approximate $su(2)$ dynamical symetry of $\mathcal{H}$ was introduced. This approach is based on a block-diagonal form of the Dicke interaction Hamiltonian $H_D = (a^\dagger\, S_- + a\, S_+)$ whose blocks are of the form

$$H_D^{(s)} = \begin{pmatrix} 0 & A_l & 0 & \dots & 0 \\ A_l & 0 & A_{l-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & A_{-l+2} & 0 & A_{-l+1} \\ 0 & \dots & 0 & A_{-l+1} & 0 \end{pmatrix}. \tag{1.3}$$

These operators can be obtained provided we consider that the initial state of the system is a given eigenstate of the excitation number operator $\hat{s} = \hat{n}_a + S_z + N/2$, where $\hat{n}_a$ is the photon number operator and N is the number of atoms in the sample. The dimension of the matrix (1.3) is $(2\,l+1)$ and l is related to the number of atoms through $2\,l = N$. In this basis, the matrix elements of $H_D^{(s)}$ read

$$A_m = \sqrt{(l+m)(l-m+1)(2s-l-m+1)}, \qquad m = -l{+}1, -l{+}2, \ldots, l. \tag{1.4}$$

Here $2\,s$ (with $s \geq l$) is just the (constant) eigenvalue of the excitation number operator in that subspace.

Under certain dynamical conditions (essentially, in either a strong or weak field regimes) the matrix (1.3) can be written as the $(2\,l+1)$ dimensional irreducible representation of the $2\,J_x = J_+ + J_-$ generator of $su(2)$ plus some additional smaller terms. In this way, a perturbative approach to the spectrum and dynamical properties of the Dicke model can be developed [2].

In this contribution we show that the $su_q(2)$ quantum algebra can be also used to define a Hamiltonian of the type $H_D^{(s)}$ whose eigenvalues and eigenvectors can be analytically found [5]. The spectrum of the $su_q(2)$ model is essentially anharmonic, and we show that this q-deformed Hamiltonian can efficiently serve as the zero-th order operator in order to work out the corresponding perturbation theory for the regimes with the strongest nonlinear properties and for which the $su(2)$ perturbation theory is not useful. In particular, we shall analyse the case $s = l$ that corresponds, in the three-boson language, to second harmonics generation.

2 A tridiagonal Hamiltonian defined on $su_q(2)$

The quantum algebra $su_q(2)$ [6]–[8] is generated by $J_z\,, J_\pm$ and has the following commutation rules

$$[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = [2\,J_z], \tag{2.1}$$

where the following symbol is introduced

$$[x] := \frac{q^x - q^{-x}}{q - q^{-1}}, \qquad q = e^z. \tag{2.2}$$

Note that we can always recover 'classical' (undeformed) $su(2)$ results when $q \to 1$ (or, equivalently, in the limit $z \to 0$).

Representation theory of $su_q(2)$ is a smooth deformation of the $su(2)$ one and it can be constructed by introducing the 'bare' basis of eigenvectors of J_z,

$$2\,J_z|l,m\rangle = 2\,m\,|l,m\rangle. \tag{2.3}$$

Therefore, $q^{2J_z}|l,m\rangle = q^{2m}|l,m\rangle$. In this basis, the $(2\,l+1)$-dimensional irreducible representation of $su_q(2)$ is given by (2.3) and

$$J_\pm|l,m\rangle = \sqrt{[l \mp m][l \pm m + 1]}\,|l,m+1\rangle. \tag{2.4}$$

Let us consider the following operator defined on $su_q(2)$:

$$H_q = q^{J_z/2}\,(J_+ + J_-)\,q^{J_z/2}\,. \tag{2.5}$$

We stress that the $q \to 1$ limit of H is just the $2\,J_x$ generator used in [2]. The $(2\,l+1)$-dimensional representation for H takes the tridiagonal form (1.3) with

$$A_m(q) = q^{m-1/2}\sqrt{[l+m][l-m+1]}\,. \qquad (2.6)$$

By making use of the algebraic properties of $su_q(2)$ (in particular, by exploiting the coproduct map in order to construct tensor product representations) it can be proven that the spectrum of this operator for a given l is given by the q-numbers $[2m]$, with $m = -l, \ldots, l$ (see Fig. 1, in which the anharmonicity of H_q is clearly shown). Moreover, the corresponding normalized eigenvectors can be also explicitly found as well as the q-Clebsch–Gordan coefficients in both the 'bare' and 'dressed basis' [5].

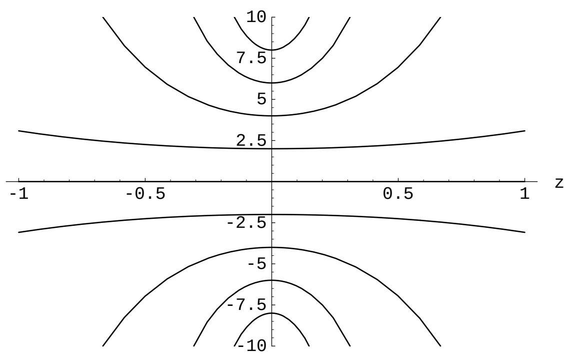

Figure 1. Spectrum of H_q in the $l = 4$ representation as a function of $z = \log q$.

3 Second harmonics generation through $su_q(2)$

Let us now consider the second harmonics generation (SHG) analogue of the Dicke Hamiltonian, that is obtained when $s = l$. This is a strongly nonlinear regime of H_D, as it can be appreciated in Fig. 2, where we have assumed A_m to be a continuous function of m and we have plotted it together with the $2\,J_x$ matrix elements of $su(2)$ (the symmetric curve).

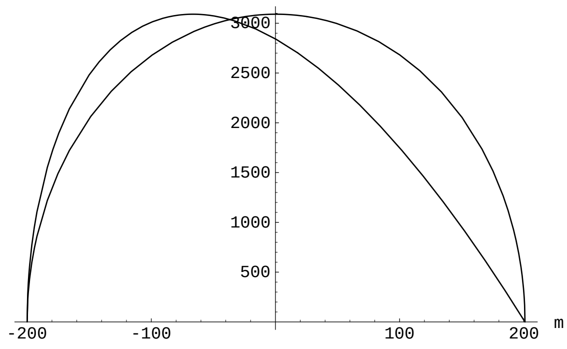

Figure 2. Matrix elements of $H_D^{(s)}$ and of a $su(2)$ model for $l = 200$ and $s = l$.

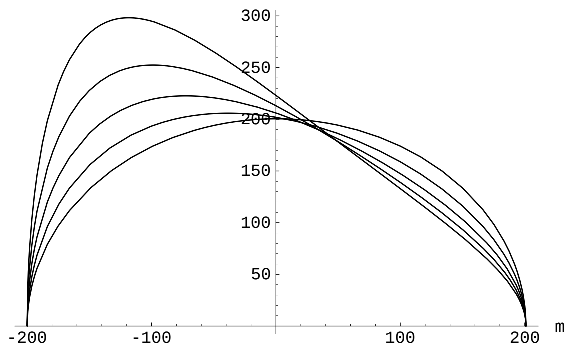

Figure 3. Matrix elements of H_q for $z = 0, -1.10^{-3}, -2.10^{-3}, -3.10^{-3}, -4.10^{-3}$ and $l = 200$.

However, if we plot the function $A_m(q)$ corresponding to the H_q Hamiltonian, it becomes clear that we could approach the SHG model by fitting an appropriate deformation parameter (Fig. 3). Therefore, let us try to approximate analytically the Dicke/SHG operator $H_D^{(s)}$ through a Hamiltonian of the type

$$T_0 = \Omega\, H_q. \tag{3.1}$$

Here we have two free parameters (q and Ω) in order to get the closest T_0 to $H_D^{(s)}$. The simplest way to do this is to choose both parameters in such a way that the matrix elements A_m of $H_D^{(s)}$ and the matrix elements $\tilde{A}_m(q) = \Omega\, A_m(q)$ of the Hamiltonian T_0 coincide in their maxima. This choice gives rise (for $s = l$) to the following relations defining q and Ω in terms of the number of atoms N:

$$\alpha = N \log q = \frac{3}{2} \log \frac{\sqrt{5}-1}{2} \approx -0.7218\,, \tag{3.2}$$

$$\Omega = \frac{4(N+1)^{3/2}}{\sqrt{27}[N+1]}\,. \tag{3.3}$$

In this way, both the maxima of A_m and $\tilde{A}_m(q)$ (considered as functions of m) occur in the point $m_0 = -(l-1)/3$ (see Fig. 4).

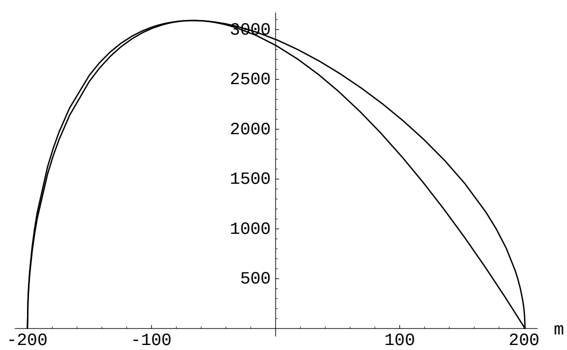

Figure 4. Fitting between $\Omega\, H_q$ with $z = -1.8\, 10^{-3}$ and SHG hamiltonian ($l = 200$).

In the neighbourhood of m_0, it can be proven that the Dicke/SHG matrix elements can be approximated as [5]:

$$A_m \approx \Omega\, A_m(q)\, \phi(m), \qquad \phi(m) = 1+\phi_1\Delta-\phi_2\Delta^2+\phi_3\Delta^3\,, \qquad \Delta = m-m_0\,. \tag{3.4}$$

If we substitute $\Delta = m - m_0 = J_z + (l-1)/3$ we can rewrite (3.4) in the matrix form:

$$H_D^{(s)} \approx \Omega\,[J_+\phi(J_z - m_0) + \phi(J_z - m_0)J_-] = 2\,\Omega\,\{H_q, f(J_z)\}\,. \tag{3.5}$$

The (degree three) polynomial $f(J_z)$ can be explicitly obtained. This formula leads to a straightforward estimation of the ground state energy of $H_D^{(s)}$ as follows:

$$\langle\underline{-l,l}|H_D^{(s)}|\underline{-l,l}\rangle \approx -\Omega\,[2l]\sum_{k=0}^{3} f_k\langle\underline{-l,l}|(J_z)^k|\underline{-l,l}\rangle, \tag{3.6}$$

where $|\underline{-l,l}\rangle$ is the ground state for the H_q hamiltonian in the dressed basis. This method turns out to be valid for arbitrary eigenstates, and no higher order expansion is needed to obtain the correct $N\to\infty$ properties (large photon numbers). As a conclusion,$su_q(2)$ dynamical symmetry provides a powerful algebraic approach to the dynamics of trilinear quantum optical Hamiltonians, thus improving in an essential way the already known approaches to this problem [9].

Acknowledgements

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References

[1] Mandel L and Wolf E, Optical Coherence and Quantum Optics, Cambridge University Press, Cambridge, 1995.

[2] Chumakov S M and Kozierowski M, *Quant. Semiclas. Optics*, 1996, V.8, 775.

[3] Jurco B, *J. Math. Phys.*, 1989, V.30, 1739.

[4] Kozierowski M, Mamedov A A and Chumakov S M, *Phys. Rev. A*, 1990, V.42, 1762.

[5] Ballesteros A and Chumakov S M, *J. Phys. A: Math. Gen.*, 1999, V.32, 6261.

[6] Kulish P and Reshetikhin N, *Zap. Nauch. Seminarov LOMI*, 1981, V.101, 101; *J. Soviet. Math.*, 1983, V.23, 2435.

[7] Drinfel'd V G, in Quantum Groups, Proceedings of the International Congress of Mathematics, Berkeley, MRSI, 1986.

[8] Jimbo M, *Lett. Math. Phys.*, 1985, V.10, 63.

[9] Chumakov S M, Klimov A B and Sánchez-Mondragón J J, *Phys. Rev. A*, 1994, V.49, 4972; Karassiov V P, *Phys. Lett. A*, 1998, V.238, 19.