Journal of Nonlinear Mathematical Physics 2001, V.8, Supplement, 254–260 Proceedings: NEEDS'99

Universal Lax Pair for Generalised Calogero–Moser Models

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Abstract

In this talk we introduce generalised Calogero–Moser models and demonstrate their integrability by constructing universal Lax pair operators. These include models based on non-crystallographic root systems, that is the root systems of the finite reflection groups, H_3, H_4, and the dihedral group $I_2(m)$, besides the well-known ones based on crystallographic root systems, namely those associated with Lie algebras. Universal Lax pair operators for all of the generalised Calogero–Moser models and for any choices of the potentials are linear combinations of the reflection operators. The equivalence of the Lax pair with the equations of motion is proved by decomposing the root system into a sum of two-dimensional sub-root systems, A_2, B_2, G_2, and $I_2(m)$. The root type and the minimal type Lax pairs, derived in our previous papers, are given as the simplest representations. The spectral parameter dependence plays an important role in the Lax pair operators, which bear a strong resemblance to the Dunkl operators.

1 Introduction

This talk is based on a series of papers on Calogero–Moser models [1, 2, 3], in particular [4]. Generalized Calogero–Moser models are integrable many-particle dynamical systems based on finite reflection groups, which include the dihedral groups $I_2(m)$ and H_3 and H_4 together with the Weyl groups of the root systems associated with Lie algebras, called crystallographic root systems. Integrability of classical Calogero–Moser models based on the crystallographic root systems [5, 6, 7] is shown in terms of Lax pairs. The root and the minimal type Lax pairs derived in [1] provide a universal framework for these Calogero–Moser models, including those based on exceptional root systems and with the twisted potentials. On the other hand, a theory of classical integrability for the models based on non-crystallographic root systems has been virtually non-existent. This is in sharp contrast with the quantum counterpart. Dunkl operators, which are useful for solving quantum Calogero–Moser models, were first explicitly constructed for the models based on the dihedral groups [8].

In this talk we present a Lax pair in an operator form for generalized Calogero–Moser models, which applies universally to the models based on non-crystallographic root systems as well as those based on crystallographic ones. In this Lax pair the reflection operators play a central role and the spectral parameter dependence is also essential. When suitable representation spaces are chosen, the universal Lax pair reproduces the root type and the minimal type Lax pairs for the models based on the crystallographic root systems [4]. For the applications of Calogero–Moser models to the theories based on Lie algebras, for example, the Toda models [9, 2] and the supersymmetric gauge theories [7], the minimal type Lax pairs are relevant.

For the general background of this paper and the physical applications of the Calogero–Moser models with various potentials to lower-dimensional physics, we refer to our previous papers [1] and references therein.

2 Generalized Calogero–Moser models

A generalized Calogero–Moser model is a Hamiltonian system associated with a root system. A root system Δ of rank r is a set of vectors in $\mathbf{R}^r$ which is invariant under reflections in the hyperplane perpendicular to each vector in Δ. In other words,

$$\Delta \ni s_\alpha(\beta) = \beta - 2(\alpha\cdot\beta/|\alpha|^2)\alpha, \qquad \forall\, \alpha, \beta \in \Delta. \tag{2.1}$$

Dual roots are defined by $\alpha^\vee = 2\alpha/|\alpha|^2$, in terms of which

$$s_\alpha(\beta) = \beta - (\alpha^\vee\!\cdot\beta)\alpha. \tag{2.2}$$

The root systems for finite reflection groups may be divided into two types: crystallographic and non-crystallographic root systems. Crystallographic root systems satisfy the additional condition

$$\alpha^\vee\!\cdot\beta \in \mathbf{Z}, \qquad \forall\, \alpha, \beta \in \Delta. \tag{2.3}$$

These root systems are associated with simple Lie algebras: A_r, B_r, C_r, D_r, E_6, E_7, E_8, F_4, and G_2 and BC_r. The remaining non-crystallographic root systems [10] are H_3, H_4, and the dihedral group of order $2m$, $\{I_2(m),\ m \geq 4\}$.

The dynamical variables are the coordinates $\{q^j\}$ and their canonically conjugate momenta $\{p_j\}$, with the Poisson brackets

$$\{q^j, p_k\} = \delta^j_k, \qquad \{q^j, q^k\} = \{p_j, p_k\} = 0, \qquad j, k = 1, \ldots, r. \tag{2.4}$$

These will be denoted by vectors in $\mathbf{R}^r$, $q = (q^1, \ldots, q^r)$, $p = (p_1, \ldots, p_r)$. The Hamiltonian for the generalized Calogero–Moser model is

$$\mathcal{H} = \frac{1}{2}p^2 + \sum_{\alpha\in\Delta} \frac{g_{|\alpha|}^2}{|\alpha|^2} V_{|\alpha|}(\alpha\cdot q), \tag{2.5}$$

in which the real coupling constants $g_{|\alpha|}$ and potential functions $V_{|\alpha|}$ are defined on orbits of the corresponding finite reflection group. This then ensures that the Hamiltonian is invariant under reflections of the phase space variables about a hyperplane perpendicular to any root

$$q \to s_\alpha(q), \qquad p \to s_\alpha(p), \qquad \forall \alpha \in \Delta. \tag{2.6}$$

The Lax pair that we will construct will apply for the following potentials:

1. *Untwisted elliptic potential* for all of the crystallographic root systems.

$$V_{|\alpha|}(\alpha\cdot q) = \wp(\alpha\cdot q|\{2\omega_1, 2\omega_3\}), \quad \text{for all roots}, \tag{2.7}$$

in which $\wp$ is the Weierstrass $\wp$ function with a pair of primitive periods $\{2\omega_1, 2\omega_3\}$.

2. *Twisted elliptic potential* for all of the non-simply laced root systems.

$$V_{|\alpha|}(\alpha\cdot q)=\begin{cases}\wp(\alpha\cdot q|\{2\omega_1,2\omega_3\}), & \text{for long roots,}\\ \wp(\alpha\cdot q|\{\omega_1,2\omega_3\}), & \text{for short roots,}\end{cases}\qquad \text{except for}\quad G_2, \tag{2.8}$$

$$V_{|\alpha|}(\alpha\cdot q)=\begin{cases}\wp(\alpha\cdot q|\{2\omega_1,2\omega_3\}), & \text{for long roots,}\\ \wp\left(\alpha\cdot q|\left\{\dfrac{2\omega_1}{3},2\omega_3\right\}\right), & \text{for short roots,}\end{cases}\qquad \text{for}\quad G_2. \tag{2.9}$$

3. *Trigonometric and hyperbolic potentials* for all crystallographic systems.

$$V_{|\alpha|}(\alpha\cdot q)=\begin{cases}a^2/\sin^2 a(\alpha\cdot q),\\ a^2/\sinh^2 a(\alpha\cdot q),\end{cases}\qquad \text{for all roots,}\quad a\ \text{is real const.} \tag{2.10}$$

4. *Rational potential* for all of the generalized Calogero–Moser models.

$$V_{|\alpha|}(\alpha\cdot q)=\frac{1}{(\alpha\cdot q)^2},\quad \text{for all roots.} \tag{2.11}$$

These models are also integrable if a confining harmonic potential $\omega^2q^2/2$ is added to the Hamiltonian. The above degenerate potentials, (2.10) and (2.11) are obtained as one or both periods of the elliptic function diverge.

3 Lax pair operators

Here we construct a Lax pair for the generalized Calogero–Moser models in an operator form acting on an as yet unspecified vector space. The operators appearing in the Lax pair are naturally the reflection operators $\{\hat{s}_\alpha,\ \alpha\in\Delta\}$ of the root system. They act on a set of $\mathbf{R}^r$ vectors $\Gamma=\{\mu^{(k)}\in\mathbf{R}^r,\ k=1,\ldots\}$, which is permuted under the action of the reflection group. The totality of the vectors in Γ forms the representation space $\mathbf{V}$. Another set of operators $\{\hat{H}_j,\ j=1,\ldots,r\}$ is necessary. If $\hat{H}_j$ acts on a vector $\mu^{(k)}\in\Gamma$, the j-th component is returned:

$$\hat{H}_j\mu^{(k)}=\mu_j^{(k)}\mu^{(k)}.$$

These form the following operator algebra:

$$[\hat{H}_j,\hat{H}_k]=0,\qquad [\hat{H}_j,\hat{s}_\alpha]=\alpha_j\left(\alpha^\vee\cdot\hat{H}\right)\hat{s}_\alpha,\qquad \hat{s}_\alpha\hat{s}_\beta\hat{s}_\alpha=\hat{s}_{s_\alpha(\beta)}. \tag{3.1}$$

The first relation implies that the operators $\{\hat{H}_j\}$ form an Abelian subalgebra and the last relations are those for the finite reflection group associated with the root system Δ.

Next we describe the Lax pair and the corresponding Hamiltonian for the generalized Calogero–Moser model for the root system Δ. The Lax operators are

$$\begin{aligned}&L=p\cdot\hat{H}+X,\qquad X=i\sum_{\rho\in\Delta_+}g_{|\rho|}\left(\rho^\vee\cdot\hat{H}\right)x_{|\rho|}\left(\rho\cdot q,\left(\rho^\vee\cdot\hat{H}\right)\xi\right)\hat{s}_\rho,\\ &M=i\sum_{\rho\in\Delta_+}g_{|\rho|}\,y_{|\rho|}\left(\rho\cdot q,\left(\rho^\vee\cdot\hat{H}\right)\xi\right)\hat{s}_\rho.\end{aligned} \tag{3.2}$$

The function x for the untwisted models are (b is an arbitrary constant):

$$x(u) = x_L(u,w) = x_S(u,w) = \frac{\sigma(w-u)}{\sigma(w)\sigma(u)} \exp[b\,w\,u]. \tag{3.3}$$

For the twisted models except for G_2:

$$\begin{aligned} x_L(u,w) &= \frac{\sigma(w-u)}{\sigma(w)\sigma(u)} \exp[b\,w\,u], \\ x_S(u,w) &= \frac{\sigma(w/2-u|\{\omega_1, 2\omega_3\})}{\sigma(w/2|\{\omega_1,2\omega_3\})\sigma(u|\{\omega_1,2\omega_3\})} \exp\left[\left(b+\frac{e_1}{2}\right) w\,u\right], \\ &= \frac{x_L(u,w/2)x_L(u+\omega_1,w/2)}{x_L(\omega_1,w/2)}, \qquad e_1 \equiv \wp(\omega_1). \end{aligned} \tag{3.4}$$

For the twisted G_2 model:

$$\begin{aligned} x_L(u,w) &= \frac{\sigma(w-u)}{\sigma(w)\sigma(u)} \exp[b\,w\,u], \\ x_S(u,w) &= \frac{\sigma(w/3-u|\{2\omega_1/3, 2\omega_3\})}{\sigma(w/3|\{2\omega_1/3,2\omega_3\})\sigma(u|\{2\omega_1/3,2\omega_3\})} \exp\left[\left(b+\frac{2}{3}\wp(2\omega_1/3)\right) w\,u\right], \\ &= \frac{x_L(u,w/3)x_L(u+2\omega_1/3,w/3)x_L(u+4\omega_1/3,w/3)}{x_L(2\omega_1/3,w/3)x_L(4\omega_1/3,w/3)} \exp[b\,w\,u]. \end{aligned} \tag{3.5}$$

The function $y_{|\rho|}$ is defined by $y_{|\rho|}(u,w) \equiv \frac{\partial}{\partial u} x_{|\rho|}(u,w)$. Furthermore, $x_{|\rho|}(u,w)$ is odd: $x_{|\rho|}(-u,-w) = -x_{|\rho|}(u,w)$ so that L and M are independent of the choice of positive roots Δ_+. The function $x_{|\rho|}$ is a "square root" of the potential $V_{|\rho|}$

$$x_{|\rho|}(u,w)x_{|\rho|}(-u,w) = -V_{|\rho|}(u) + C_{|\rho|}(w). \tag{3.6}$$

The Hamiltonian for the theory is defined in terms of a representation of the operator L of (3.2) by $\mathcal{H} = Tr(L^2)/2C_\Gamma$ where the constant C_Γ, which depends on the representation, is defined by $Tr(\hat{H}_j\hat{H}_k) = C_\Gamma\,\delta_{jk}$. The resulting Hamiltonian is then (2.5) plus a constant.

The underlying idea of the Lax operator L, (3.2), is quite simple. As seen above, L is a "square root" of the Hamiltonian. Thus one part of L contains p which is not associated with roots and another part contains $x_{|\rho|}(\rho\cdot q)$, a "square root" of the potential $V_{|\rho|}(\rho\cdot q)$, which being associated with a root ρ is therefore accompanied by the reflection operator $\hat{s}_\rho$.

The equations of motion following from this Hamiltonian are

$$\dot{q}_j = \frac{\partial\mathcal{H}}{\partial p_j} = p_j, \tag{3.7}$$

$$\begin{aligned} \dot{p}_j &= -\frac{\partial\mathcal{H}}{\partial q_j} = -\frac{\partial}{\partial q_j}\left[\sum_{\rho\in\Delta} \frac{g_{|\rho|}^2}{|\rho|^2} V_{|\rho|}(\rho\cdot q)\right] \\ &= \sum_{\rho\in\Delta} \frac{g_{|\rho|}^2}{|\rho|^2}\rho_j \left[y_{|\rho|}(\rho\cdot q,w)x_{|\rho|}(-\rho\cdot q,w) - x_{|\rho|}(\rho\cdot q,w)y_{|\rho|}(-\rho\cdot q,w)\right]. \end{aligned} \tag{3.8}$$

Because of (3.6) the last expression in (3.8) is independent of w.

The Lax equation

$$\dot{L} = \frac{d}{dt}L = [L, M] \tag{3.9}$$

may be divided into three parts as

$$\frac{d}{dt}X = [p\cdot \hat{H}, M], \tag{3.10}$$

$$\frac{d}{dt}(p\cdot \hat{H}) = [X, M]_{\text{diagonal}}, \tag{3.11}$$

$$0 = [X, M]_{\text{off-diagonal}}. \tag{3.12}$$

We next prove that each of these equations is consistent with the equations of motion (3.7). The left-hand side of (3.10) is

$$\frac{d}{dt}X = i\sum_{\rho\in\Delta_+} g_{|\rho|}\left(\rho^\vee\cdot \hat{H}\right) y_{|\rho|}\left(\rho\cdot q, \left(\rho^\vee\cdot \hat{H}\right)\xi\right)(\rho\cdot\dot{q})\,\hat{s}_\rho \tag{3.13}$$

and the right-hand side is

$$\begin{aligned}[p\cdot \hat{H}, M] &= \left[p\cdot \hat{H}, i\sum_{\rho\in\Delta_+} g_{|\rho|}\, y_{|\rho|}\left(\rho\cdot q, \left(\rho^\vee\cdot \hat{H}\right)\xi\right)\hat{s}_\rho\right],\\ &= i\sum_{\rho\in\Delta_+} g_{|\rho|}\, y_{|\rho|}\left(\rho\cdot q, \left(\rho^\vee\cdot \hat{H}\right)\xi\right)\left(\rho^\vee\cdot \hat{H}\right)(\rho\cdot p)\,\hat{s}_\rho = \frac{d}{dt}X.\end{aligned} \tag{3.14}$$

The second line follows from the commutation relation (3.1) and the last equality follows from the equation of motion $\dot{q} = p$.

The left-hand side of (3.11), after using the equations of motion (3.8), is

$$\begin{aligned}\frac{d}{dt}\left(p\cdot \hat{H}\right) &= \sum_{\rho\in\Delta}\frac{g_{|\rho|}^2}{|\rho|^2}\left(\rho\cdot \hat{H}\right)\\ &\quad\times\left[y_{|\rho|}(\rho\cdot q, w)x_{|\rho|}(-\rho\cdot q, w) - x_{|\rho|}(\rho\cdot q, w)y_{|\rho|}(-\rho\cdot q, w)\right].\end{aligned} \tag{3.15}$$

The commutator $[X, M]$ reads

$$\begin{aligned}[X, M] &= -\left[\sum_{\rho\in\Delta_+} g_{|\rho|}\left(\rho^\vee\cdot \hat{H}\right)x_{|\rho|}\left(\rho\cdot q, \left(\rho^\vee\cdot \hat{H}\right)\xi\right)\hat{s}_\rho,\right.\\ &\qquad\left.\sum_{\sigma\in\Delta_+} g_{|\sigma|}\, y_{|\sigma|}\left(\sigma\cdot q, \left(\sigma^\vee\cdot \hat{H}\right)\xi\right)\hat{s}_\sigma\right],\\ &= -\sum_{\rho,\sigma\in\Delta_+} g_{|\rho|}g_{|\sigma|}\left[\left(\rho^\vee\cdot \hat{H}\right)x_{|\rho|}\left(\rho\cdot q, \left(\rho^\vee\cdot \hat{H}\right)\xi\right)\right.\\ &\qquad\times y_{|\sigma|}\left(\sigma\cdot q, \left(s_\rho(\sigma)^\vee\cdot \hat{H}\right)\xi\right)\hat{s}_\rho\hat{s}_\sigma - y_{|\sigma|}\left(\sigma\cdot q, \left(\sigma^\vee\cdot \hat{H}\right)\xi\right)\left(s_\sigma(\rho)^\vee\cdot \hat{H}\right),\\ &\qquad\left. x_{|\rho|}\left(\rho\cdot q, \left(s_\sigma(\rho)^\vee\cdot \hat{H}\right)\xi\right)\hat{s}_\sigma\hat{s}_\rho\right].\end{aligned} \tag{3.16}$$

Since $\hat{s}_\rho\hat{s}_\sigma$ and $\hat{s}_\sigma\hat{s}_\rho$ are rotations (except for $\rho=\sigma$, $\hat{s}_\rho^2=1$) they do not leave any real vectors in the rotation plane invariant. Thus $[X,M]$ is decomposed into the diagonal ($\rho=\sigma$) and the off-diagonal ($\rho\neq\sigma$) parts. The diagonal part gives the equation of motion

$$\begin{aligned}[X,M]_{\text{diag.}} &= \sum_{\rho\in\Delta_+} g_{|\rho|}^2\left(\rho^\vee\cdot\hat{H}\right)\Big[y_{|\rho|}\left(\rho\cdot q,\left(\rho^\vee\cdot\hat{H}\right)\xi\right)x_{|\rho|}\left(-\rho\cdot q,\left(\rho^\vee\cdot\hat{H}\right)\xi\right)\\ &\qquad -x_{|\rho|}\left(\rho\cdot q,\left(\rho^\vee\cdot\hat{H}\right)\xi\right)y_{|\rho|}\left(-\rho\cdot q,\left(\rho^\vee\cdot\hat{H}\right)\xi\right)\Big],\\ &= \frac{d}{dt}\left(p\cdot\hat{H}\right).\end{aligned}\tag{3.17}$$

Finally, the off-diagonal part vanishes:

$$\begin{aligned}0 &= [X,M]_{\text{off-diag.}}\\ &= \sum_{\rho\neq\sigma\in\Delta_+} g_{|\rho|}g_{|\sigma|}\Big[\left(\rho^\vee\cdot\hat{H}\right)x_{|\rho|}\left(\rho\cdot q,\left(\rho^\vee\cdot\hat{H}\right)\xi\right)y_{|\sigma|}\left(\sigma\cdot q,\left(s_\rho(\sigma)^\vee\cdot\hat{H}\right)\xi\right)\\ &\quad -\left(s_\rho(\sigma)^\vee\cdot\hat{H}\right)y_{|\rho|}\left(\rho\cdot q,\left(\rho^\vee\cdot\hat{H}\right)\xi\right)x_{|\sigma|}\left(\sigma\cdot q,\left(s_\rho(\sigma)^\vee\cdot\hat{H}\right)\xi\right)\Big]\hat{s}_\rho\hat{s}_\sigma.\end{aligned}\tag{3.18}$$

The right hand side is decomposed into a sum corresponding to a fixed rotation $\hat{R}_\psi\equiv\hat{s}_\rho\hat{s}_\sigma$ in each two-dimensional plane. The coefficient of each $\hat{R}_\psi\equiv\hat{s}_\rho\hat{s}_\sigma$ which corresponds to possible two-dimensional sub-root systems, A_2, B_2, G_2, and $I_2(m)$, separately vanishes. For the details of the proof and other materials, we refer to [4].

Acknowledgements

RS thanks A J Bordner and E Corrigan for fruitful collaboration, upon which the bulk of this talk is based. This work is partially supported by the Grant-in-aid from the Ministry of Education, Science and Culture, priority area (#707) "Supersymmetry and unified theory of elementary particles".

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