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A Finite Dimensional Completely Integrable System Associated with the WKI- and Heisenberg Hierarchies

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Abstract

We consider the following spectral problem

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_x = \zeta \begin{bmatrix} -w & u \\ v & w \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \equiv M \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad (1)$$

where u, v, w are smooth functions. It produces a hierarchy of evolution equations with an arbitrary function A_{m-1}. This hierarchy includes the WKI [8] and Heisenberg [7] hierarchies by properly selecting the special function A_{m-1}. We derive this new evolution equations, and give the finite dimensional completely integrable systems (FDCIS) associated with theses equations.

1 The hierarchy and the Lax pair

Let

$$M = \zeta \begin{bmatrix} -w & u \\ v & w \end{bmatrix}, \qquad N_j = \begin{bmatrix} A_j & B_j \\ C_j & -A_j \end{bmatrix}$$

be the smooth matrix functions. Set the pair of Lenard's operators as

$$K = \begin{bmatrix} 0 & v & -u \\ 2u & -2w & 0 \\ -2v & 0 & 2w \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \partial_x,$$

and define the Lenard sequence $\{L_j\}$ $(j = 0, 1, 2, \ldots, m-1)$ recursively:

$$L_j = (-A_j, B_j, C_j)^T$$

with

$$L_0 = \frac{\alpha}{\sqrt{w^2+uv}} \begin{bmatrix} w \\ u \\ v \end{bmatrix} \equiv \frac{\alpha}{r} \begin{bmatrix} w \\ u \\ v \end{bmatrix}, \qquad (\alpha \neq 0 \text{ constant}),$$

$$JL_j = KL_{j+1}, \qquad j = 0, 1, 2, \ldots m-2. \qquad (2)$$

Lemma 1. *The commutator* $[M, N_j] = MN_j - N_jM$ *satisfies:*

$$[M, N_j] = N_{jx} + M_*(KL_j) - \zeta M_*(JL_j), \tag{3}$$

where

$$M_*(\delta w, \delta u, \delta v) = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} M(w+\varepsilon\delta w, u+\varepsilon\delta u, v+\varepsilon\delta v) = \zeta \begin{bmatrix} -\delta w & \delta u \\ \delta v & \delta w \end{bmatrix}.$$

Notice that A_{m-1} is an arbitrary function. $X_m \equiv JL_{m-1}$ is the m^{th} order vector field of the evolution equation hierarchy

$$\begin{bmatrix} w \\ u \\ v \end{bmatrix}_{t_m} = X_m, \qquad m = 1, 2, \ldots. \tag{4}$$

From Lemma 1 and the fact that $KL_0 = 0$ we get the following:

Theorem 1. *Let* $\{L_j\}$, $j = 0, 1, 2, \ldots$ *be a Lenard sequence. Each of the vector fields* X_m $(m = 1, 2, \ldots)$ *has a commutator representation:*

$$M_*(X_m) = V_{mx} + [V_m, M],$$

where

$$V_m = \sum_{j=0}^{m-1} N_j \zeta^{m-j}, \qquad \zeta_{t_m} \equiv 0.$$

Corollary. *Each evolution equation* (4) *has a Lax pair:*

$$\begin{cases} \phi_x = M\phi, \\ \phi_{t_m} = V_m\phi, \end{cases} \qquad \zeta_{t_m} = 0. \tag{5}$$

Special choices of the function A_{m-1} and the constraint give us either the WKI or the Heisenberg hierarchy. Indeed for $A_{m-1} = 0$ and $w = 1$, $r = \sqrt{uv+1}$ we obtain the WKI hierarchy whereas for $A_{m-1,x} = \frac{1}{2w}(uC_{m-1,x} + vB_{m-1,x})$ and $r = \sqrt{uv + w^2} = 1$ we obtain the Heisenberg hierarchy. For example, the Lenard sequence is $L_0 = \frac{\alpha}{r}(w, u, v)^T$, $L_1 = \frac{\alpha}{4r^3}\left(vu_x - uv_x, 2(uw_x - wu_x), 2(wv_x - vw_x)\right)^T$, etc. so the second order WKI equation is

$$\begin{bmatrix} u \\ v \end{bmatrix}_{t_2} = \frac{\alpha}{2} \begin{bmatrix} -\frac{u}{r} \\ \frac{v}{r} \end{bmatrix}_{xx}, \qquad r = \sqrt{1+uv},$$

the third order WKI equation is

$$\begin{bmatrix} u \\ v \end{bmatrix}_{t_3} = \frac{\alpha}{4} \begin{bmatrix} \frac{u_x}{r^3} \\ \frac{v_x}{r^3} \end{bmatrix}_{xx}.$$

The second order Heisenberg equation is

$$\begin{bmatrix} u \\ v \end{bmatrix}_{t_2} = \frac{\alpha}{2} \begin{bmatrix} uw_{xx} - wu_{xx} \\ wv_{xx} - vw_{xx} \end{bmatrix}, \qquad r = 1.$$

2 Associated FDCIS

Using the constraints and nonlinearization of the Lax pairs we derive finite dimensional completely integrable systems (FDCIS) associated with the infinite dimensional hierarchies. From the eigenvalue problem (1), we have

$$\operatorname{grad}\zeta_j = \begin{bmatrix} \delta\zeta_j/\delta w \\ \delta\zeta_j/\delta u \\ \delta\zeta_j/\delta v \end{bmatrix} = \gamma_j \begin{bmatrix} -\zeta_j y_{1j} y_{2j} \\ \zeta_j y_{2j}^2 \\ -\zeta_j y_{1j}^2 \end{bmatrix},$$

where

$$\gamma_j = \left(\int_\Omega \left(-v y_{1j}^2 - w y_{1j} y_{2j} + u y_{2j}^2\right) dx \right)^{-1}.$$

Let $G_0 = \sum_{j=1}^{N} \operatorname{grad}\zeta_j$. We get the constraint

$$\frac{\alpha}{r} w = -\langle Aq, p\rangle, \qquad \frac{\alpha}{r} u = \langle Aq, q\rangle, \qquad \frac{\alpha}{r} v = -\langle Ap, p\rangle,$$

where $p = (y_{11}, \dots, y_{1N})^T, q = (y_{21}, \dots, y_{2N})^T$ and $A = \operatorname{diag}(\zeta_1, \dots, \zeta_N)$. In the case of Heisenberg, $r = \sqrt{uv + w^2} = 1$, the constraint becomes

$$w = -\langle Aq, p\rangle, \qquad u = \langle Aq, q\rangle, \qquad v = -\langle Ap, p\rangle. \tag{6}$$

In the case of WKI, $w = 1$, $r = \sqrt{w^2 + uv} = \sqrt{1 + uv}$, we have the following constraint:

$$u = \langle Aq, p\rangle^{-1}\langle Aq, q\rangle, \qquad v = \langle Aq, p\rangle^{-1}\langle Ap, p\rangle. \tag{7}$$

Now we nonlinearize the Lax pairs by plugging the constraint (7) into the Lax pairs. Then (1) becomes the restricted flow

$$p_x = -wAp + uAp, \qquad q_x = vAp + wAq. \tag{8}$$

Now plugging in the constraint (7) into (8), we get

$$p_x = -Ap + \langle Aq, p\rangle^{-1}\langle Aq, q\rangle Aq, \qquad q_x = Aq - \langle Aq, p\rangle^{-1}\langle Ap, p\rangle Ap. \tag{9}$$

The equation (9) is the nonlinearization of the eigenvalue problem (1), which can be written in canonical Hamiltonian form

$$q_x = \frac{\partial H_0}{\partial p}, \qquad p_x = -\frac{\partial H_0}{\partial q} \tag{10}$$

where $H_0 = -\langle Aq, p\rangle + \sqrt{\alpha^2 + \langle Aq, q\rangle\langle Ap, p\rangle}$ with $\alpha \neq 0$ the constant given by the Lenard sequence (2).

Theorem 2. *The Hamiltonian system (10) is completely integrable (in the Liouville sense) with the following N functions in involution: $H_0, H_2, \dots H_N$*

$$\begin{aligned} H_m &= -\frac{1}{2}\sum_{j=0}^{m-1} \begin{vmatrix} \langle A^{j+1}q, q\rangle & \langle A^{j+1}q, p\rangle \\ \langle A^{m-j}p, q\rangle & \langle A^{m-j}p, p\rangle \end{vmatrix} + \langle A^m q, p\rangle H_0 \\ &= G_m + \langle A^m q, p\rangle H_0, \qquad m = 2, 3, \dots, N. \end{aligned}$$

where $\{G_m\}$ is a confocal involutive system, Cao [2] , Moser [5]. The level sets are

$$M_f = \left\{(q,p) \in \mathbf{R}^{2N} | H_0(q,p) = 0, H_m(q,p) = f_m \ , \ m = 2, 3, \dots N\right\}.$$

3 The relation between the higher order WKI and Heisenberg equations and the system (H_0)

Define the involutive solution of H_0 and H_m:

$$\begin{bmatrix} q \\ p \end{bmatrix} \equiv \begin{bmatrix} q(x,t_m) \\ p(x,t_m) \end{bmatrix} = g_0^x \circ g_m^{t_m} \begin{bmatrix} q(0,0) \\ p(0,0) \end{bmatrix}, \qquad \begin{bmatrix} q(0,0) \\ p(0,0) \end{bmatrix} \in M_f,$$

where g_0^x and $g_m^{t_m}$ are respectively the phase flows with Hamiltonian function (H_0) and (H_m).

Lemma 2. *Let* $\begin{bmatrix} q \\ p \end{bmatrix}$ *be an involutive solution of* (H_0) *and* (H_m)

$$(u,v)^T = \langle Aq,p\rangle^{-1}(\langle Aq,q\rangle, -\langle Ap,p\rangle)^T = f(q,p),$$

then there exist constants $\alpha_0, \alpha_1, \ldots, \alpha_{m-2}$ *such that*

$$\sum_{l=0}^{j} \alpha_j G_{j-l} = (-\langle A^{j+1}q,p\rangle, -\langle A^{j+1}q,q\rangle, \langle A^{j+1}p,p\rangle)^T, \qquad j=0,1,\ldots,m-2. \tag{11}$$

Theorem 3. *Let* $\begin{bmatrix} q \\ p \end{bmatrix}$ *be an involutive solution of* (H_0) *and* (H_m) ($m = 2,3,\ldots,N$) *on* M_f, *and* $\begin{bmatrix} u \\ v \end{bmatrix} = \langle Aq,p\rangle^{-1}(\langle Aq,q\rangle, -\langle Ap,p\rangle)^T = f(q,p)$. *Then*

1. (H_0) *and* (H_m) *are reduced to the spatial and time part respectively of the Lax pair of the higher order* WKI *equation (with potential* $(u,v)^T$*)*

$$\begin{bmatrix} u \\ v \end{bmatrix}_{t_m} = \sum_{l=0}^{m-2} \alpha_l \overline{X}_{m-l}(u,v) \tag{12}$$

($\alpha_0, \alpha_1, \ldots, \alpha_{m-2}$ *are given by Lemma 2)*

$$\begin{bmatrix} q \\ p \end{bmatrix}_x = \begin{bmatrix} -A & uA \\ vA & A \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = M(f(q,p)) \begin{bmatrix} q \\ p \end{bmatrix}, \tag{13}$$

$$\begin{bmatrix} q \\ p \end{bmatrix}_{t_m} = \sum_{l=0}^{m-2} \alpha_l V_{m-l}(f(q,p)) \begin{bmatrix} q \\ p \end{bmatrix}. \tag{14}$$

2. $\begin{bmatrix} u \\ v \end{bmatrix} = f(q,p)$ *satisfy the higher order* WKI *equation* (12).

Remark. As $\alpha > 0$, let $\begin{bmatrix} u \\ v \end{bmatrix} = -f(q,p)$, $\overline{H}_m = -H_m$ ($m = 0,2,\ldots,N$), Theorem 3 is still true.

Theorem 4. *Let* $\begin{bmatrix} q \\ p \end{bmatrix}$ *be an involutive solution of* (G_1) *and* (G_m) *of the canonical system:*

$$G_m = -\frac{1}{2} \sum_{i+j=m-1} \begin{vmatrix} \langle A^{j+1}q,q\rangle & \langle A^{j+1}q,p\rangle \\ \langle A^{i+1}p,q\rangle & \langle A^{i+1}p,p\rangle \end{vmatrix}, \qquad m=1,2,\ldots,N$$

on the level set $\Omega_h = \left\{(q,p) \in \mathbf{R}^{2N}, G_m = h_m,\ m = 1,\ldots,N,\ h_1 = \frac{1}{2}\right\}$ *and*

$$\left[\begin{array}{c} w \\ u \\ v \end{array}\right] \left[\begin{array}{c} -\langle Aq,p\rangle \\ -\langle Aq,q\rangle \\ \langle Ap,p\rangle \end{array}\right] = g(q,p).$$

Then

1. (G_1) *and* (G_m) *are reduced to the spatial and time part respectively of the Lax pair of higher order Heisenberg equation (with potential* (w,u,v)*,* $w^2 + uv = 1$*,* $\alpha = 1$*),*

$$\left[\begin{array}{c} w \\ u \\ v \end{array}\right]_{t_m} = \sum_{j=0}^{m-1} \beta_i \tilde{X}_{m-j}(w,u,v) \tag{15}$$

($\beta_0 = 1$*,* $\beta_n (n = 1,2,\ldots,m-1)$ *are some constants):*

$$\left[\begin{array}{c} q \\ p \end{array}\right]_x = \left[\begin{array}{cc} \langle Aq,p\rangle A & -\langle Aq,q\rangle A \\ \langle Ap,p\rangle A & -\langle Aq,p\rangle A \end{array}\right] \left[\begin{array}{c} q \\ p \end{array}\right], \tag{16}$$

$$\left[\begin{array}{c} q \\ p \end{array}\right]_{t_m} = \sum_{j=0}^{m-1} \beta_j V_{m-j}(g(q,p)) \left[\begin{array}{c} q \\ p \end{array}\right]. \tag{17}$$

2. $(w,u,v)^T = g(q,p)$ *satisfies the higher order Heisenberg equation* (15).

For details and proofs we refer to [6, 9].

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