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The Kaldor–Kalecki Model of Business Cycle as a Two-Dimensional Dynamical System

Marek SZYDŁOWSKI † *and Adam KRAWIEC* ‡

† *Jagiellonian University, Astronomical Observatory, Orla 171, 30-244 Krakow, Poland*
E-mail: uoszydlo@cyf-kr.edu.pl

‡ *Jagiellonian University, Department of Economics*
Pl. Wszystkich Swietych 6, 31-004 Krakow, Poland
E-mail: uukrawie@cyf-kr.edu.pl

Abstract

In the paper we analyze the Kaldor–Kalecki model of business cycle. The time delay is introduced to the capital accumulation equation according to Kalecki's idea of delay in investment processes. The dynamics of this model is represented in terms of time delay differential equation system. In the special case of small time-to-build parameter the general dynamics is reduced to two-dimensional autonomous dynamical system. This system is examined in details by methods of qualitative analysis of differential equations. It is shown that there is a Hopf bifurcation leading to a limit cycle. Additionally stability of this solution is discussed.

1 Introduction

In this paper we consider a business cycle model and the problem of the investment. As was pointed out by Ichimura [2] the French physicist Le Corbeiller suggested the possibility of applying the theory of nonlinear oscillations to study business cycles.

The Kalecki model of business cycle [4] assumes that the saved part of profit is invested and the capital growth is due to past investment decisions. There is a time lag after which capital equipment is available for production.

We formulate the Kaldor–Kalecki model of business cycle [6] where Kalecki's assumption on time lag investment is incorporated into the Kaldor model of business cycle [3].

The equations of the Kaldor–Kalecki model are second order delay differential equations. Some general techniques of functional analysis and bifurcation theory are relevant to the study of our dynamical problems. One of the most fundamental tools we have is the Poincaré–Andronov–Hopf bifurcation theorem which has been generalized for functional differential equations including delay equations [1]. The power of the Hopf theorem is that, when its conditions are satisfied, it guarantees both the existence and uniqueness of periodic trajectories. We use the Hopf theorem and demonstrate that its conditions are fulfilled in the Kaldor–Kalecki model for a generic class of the parameters. It allows us to predict the occurrence of limit cycle bifurcation for the time delay parameter.

In the Kaldor–Kalecki model there are two mechanisms leading to cyclic behaviour, namely nonlinearity of investment function and the time delay in investment. The nonlinear investment function fulfills some conditions proposed by Kaldor [3]. It is assumed that investment function $I(Y)$ and saving function $S(Y)$ are increasing functions with respect to gross product Y. The derivative of investment function with respect to gross product I_Y changes in such a way that $I_Y < S_Y$ for lower value of product Y, $I_Y > S_Y$ for normal value of product Y and again $I_Y < S_Y$ for higher level of Y. The investment function is of s-shape.

As was demonstrated in our previous paper [6, 8] the existence of a limit cycle in the model is independent of the assumption that the investment function is s-shaped. In the present study we assumes the nonlinear function $I(Y)$. And in this framework we consider the case of a small time delay parameter. This approximation can be expressed by comparing a delay parameter with a natural characteristic parameter (a period of oscillations) as $T/P \ll 1$.

2 The Hopf bifurcation in the Kaldor–Kalecki model in small time delay approximation

The Kaldor–Kalecki model [6] is based on the Kaldor model [3] and Kalecki's idea of a time lag in the capital accumulation equation [4].

The Kaldor–Kalecki model is represented as the time-delay differential equation system

$$\frac{dY}{dt} = \alpha[I(Y(t),K(t)) - S(Y(t),K(t))], \tag{1a}$$

$$\frac{dK}{dt} = I(Y(t-T),K(t)) - \delta K(t), \tag{1b}$$

where I is the investment and S is the saving function, Y is gross product, K is capital stock, α is the adjustment coefficient in the goods market, and δ is the depreciation rate of capital stock. It is assumed that the investment function I is nonlinear (s-shaped) on Y [3], and the time delay $T = \text{const}$.

Equations (1) are functional differential equations of retarded type. It means that the current behaviour of the system depends on its past history.

Let I_Y, I_K, S_Y and S_K denote derivatives with respect to gross product Y and capital K. The saving function S depends only on Y and is linear such that $S_Y = \gamma \in (0,1)$. Additionally, we assume that the investment function $I(Y,K) = I(Y) + I(K)$ and $I_Y > 0$. Moreover $I(K)$ is linear such that $I_K = \beta < 0$ and then

$$I(Y,K) = I(Y) + \beta K.$$

With these assumptions dynamical system (1) has the form

$$\dot{Y} \equiv \frac{dY}{dt} = \alpha I(Y(t)) + \alpha\beta K(t) - \alpha\gamma Y(t), \tag{2a}$$

$$\dot{K} \equiv \frac{dK}{dt} = I(Y(t-T)) + (\beta - \delta)K(t). \tag{2b}$$

While the existence of closed orbit via mechanism of the Hopf bifurcation can be relatively easily established in most cases, the distinction between the subcritical and supercritical Hopf bifurcation is much more difficult. The standard procedure in determining which case prevails is based on the method of normal form and the theorem on the central manifold. The problem is that in the generic case the dynamical systems do not appear in the normal form and a transformation into normal form is necessary for the stability analysis.

System (2) has the same critical points as the original one for time delay $T = 0$. Let (Y^*, K^*) be critical points for which $\dot{Y} = \dot{K} = 0$. To avoid some obstacles in the application of the assumptions of some theorems about existence of limit cycles it is useful to perform a coordinate transformation such that the system is centered at the stationary equilibrium (Y^*, K^*). Let $y = Y - Y^*$, $k = K - K^*$, $i = I - I^*$ and $s = S - S^*$. System (2) then turns into

$$\dot{y} = \alpha[i(y(t)) + \beta k(t) - \gamma y(t)], \tag{3a}$$

$$\dot{k} = i(y(t-T)) + (\beta - \delta)k. \tag{3b}$$

As the time delay parameter T is small (if comparing with the characteristic period of oscillation), $i(y(t-T))$ can be derived in this approximation from a linear Taylor expansion, i.e.

$$\begin{aligned} i(y(t-T)) &= i(y - T\dot{y}) = i(y) - Ti_y\dot{y} \\ &= i(y) - \alpha T i_y[i(y(t)) + \beta k(t) - \gamma y(t)]. \end{aligned}$$

Therefore in this approximation system (3) is reduced to the form of 2-dimensional autonomous dynamical system

$$\dot{y}(t) = \alpha[i(y) - \gamma y + \beta k], \tag{4a}$$

$$\dot{k}(t) = i(y) - \alpha T i_y(0)[i(y(t) + \beta k(t) - \gamma y(t)] + (\beta - \delta)k. \tag{4b}$$

The Jacobian of (4) at critical points is

$$J = \begin{bmatrix} \alpha(i_y(0) - \gamma) & \alpha\beta \\ i_y(0) - \alpha T i_y(0)(i_y(0) - \gamma) & (\beta - \delta) - \alpha\beta T i_y(0) \end{bmatrix} \tag{5}$$

with the determinant

$$\det J = \alpha(\beta - \delta)(i_y(0) - \gamma) - \alpha\beta i_y(0) - \alpha(i_y(0) - \gamma)T(\beta - i_y(0))$$

and the trace

$$\operatorname{tr} J = \alpha(i_y(0) - \gamma) + (\beta - \delta) - \alpha\beta T i_y(0).$$

Therefore the characteristic equation is $\lambda^2 - \operatorname{tr} \mathrm{J}\lambda + \det \mathrm{J} = 0$ and the equilibrium is locally stable if and only if the real parts of eigenvalues λ are negative. Then the equilibria are asymptotically stable if $\alpha(i_y(0) - \gamma) + (\beta - \delta) - \alpha\beta T i_y(0) < 0$. According to the Hopf bifurcation theorem a single bifurcation occurs if the complex conjugate roots cross the imaginary axis. Apparently, the roots are complex conjugate with zero real parts if $\operatorname{tr} \mathrm{J} = 0$. As there are no other roots in this 2-dimensional system the consideration of the existence of closed orbits is complete if the eigenvalues cross an imaginary axis with nonzero speed at the bifurcation point.

If $i_y - s_y > 0$ at a stationary equilibrium (Kaldor explicitly assumed that this condition is satisfied) there exists a value $T = T_{\rm bif}$ for which $\alpha(i_y - s_y) + i_k - \delta - \alpha\beta T_{\rm bif} i_y = 0$ and the complex conjugate roots cross the imaginary axis transversally. As for $T > T_{\rm bif}$ the real parts are becoming positive, $T_{\rm bif}$ is indeed a bifurcation value of the Kaldor–Kalecki model, where

$$T_{\rm bif} = \frac{\alpha(i_y(0) - \gamma) + \beta - \delta}{\alpha\beta i_y(0)} > 0,$$

where it is assumed that $\alpha(i_y(0) - \gamma) + \beta - \delta > 0$ because $T_{\rm bif} > 0$. Let us note that existence of positive value of time parameter $T_{\rm bif}$ means that the corresponding Chang–Smith condition of asymptotic stability is not fulfilled, while $i_y(0) > 0$. This confirms that that the stability analysis for small T is true (it is not true in general that stability for small T holds).

The centered dynamical system (4) evaluated at the bifurcation point reads

$$\begin{pmatrix} \dot{y} \\ \dot{k} \end{pmatrix} = J \begin{pmatrix} y \\ k \end{pmatrix} + g(y,k), \tag{6}$$

where

$$J = \begin{bmatrix} \alpha(i_y(0) - \gamma) & \alpha\beta \\ i_y(0) - (i_y(0) - \gamma)\dfrac{\alpha(i_y(0) - \gamma) + \beta - \delta}{\beta} & \beta - \delta - \dfrac{\alpha(i_y(0) - \gamma) + \beta - \delta}{\beta}\beta \end{bmatrix}$$

As we assumed that $i(y)$ is the only term involving nonlinearity, the nonlinear part $g(y,k)$ reduces to

$$g^1(y,k) = \alpha i(y) - \alpha(i_y(0) - \gamma), \tag{7a}$$

$$\begin{aligned} g^2(y,k) &= i(y) - \alpha T_{\rm bif} i_y(0)[i(y) + \beta k - \gamma y] \\ &\quad - [i_y(0) - \alpha T_{\rm bif} i_y(0)(i_y(0) - \gamma)]y + \alpha\beta T_{\rm bif} i_y(0) k. \end{aligned} \tag{7b}$$

In order to carry out the necessary coordinate transformation for the reduced dynamical system to the normal form [5, 7], let us consider the (2×2)-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{8}$$

and let

$$a_{11} = 0, \qquad a_{12} = 1, \qquad a_{21} = \frac{\sqrt{-\frac{1}{4}(f_{11} - f_{22})^2 - f_{12}f_{21}}}{f_{12}}, \qquad a_{22} = -\frac{f_{11} - f_{22}}{2f_{12}},$$

where f_{ij} are the entries in Jacobian (5) evaluated at the bifurcation point.

Therefore the inverse matrix of (8) is

$$A^{-1} = -\frac{1}{a_{21}} \begin{pmatrix} -a_{22} & -1 \\ -a_{21} & 0 \end{pmatrix} \tag{9}$$

and the matrix A can be used to write the linear part of the system in off-diagonal form, i.e. in the new coordinates we obtain

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A^{-1}JA \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -a_{21}f_{12} \\ a_{21}f_{12} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{10}$$

where $a_{21}f_{12} = \sqrt{-\alpha\beta\gamma - \alpha\beta(i_y(0) - \gamma)}$ and

$$J(T_{\mathrm{bif}}) = \begin{pmatrix} \alpha(i_y(0) - \gamma) & \alpha\beta \\ i_y(0) - \left[\frac{\alpha}{\beta} i_y(0) - \gamma - \frac{\delta}{\beta} + 1\right] & \frac{1}{\beta}(i_y(0) - \gamma - \alpha(i_y(0) - \gamma) \end{pmatrix}.$$

Finally we obtain the transformation to the normal coordinates in the form

$$y = v, \qquad k = a_{21}u + a_{22}v$$

and in these coordinates the nonlinear terms of the dynamical system takes the form

$$\begin{pmatrix} g^1(u,v) \\ g^2(u,v) \end{pmatrix} = \frac{1}{a_{21}} \begin{pmatrix} a_{22} & 1 \\ a_{21} & 0 \end{pmatrix} \begin{pmatrix} g^1(y(u,v), k(u,v)) \\ g^2(y(u,v), k(u,v)) \end{pmatrix}.$$

Let us note that $a_{21}f_{12} = \sqrt{\det[A^{-1}JA]} = \omega$ assumes a very simple form if a constant of depreciation vanishes, namely

$$a_{21}f_{12} = \sqrt{-\alpha\beta\gamma} \Rightarrow a_{21} = \frac{\sqrt{-\alpha\beta\gamma}}{\alpha\beta}.$$

The stability properties of the limit cycles depend on the nonlinear terms $g(y,k)$ because in the Hopf bifurcation the real parts of eigenvalues of the Jacobi matrix J, in the linear approximation, vanish. It can be shown that the stability of the cycle depends on derivatives up to third-order of the nonlinear function g. Consider the expression $V'''(T_{\mathrm{bif}})$

$$b = \frac{1}{16}\left(g^1_{xxx} + g^1_{xyy} + g^2_{yyy}\right) + \frac{1}{16\omega}\left[g^1_{xy}\left(g^1_{xx} + g^1_{yy}\right) - g^2_{xy}\left(g^2_{xx} + g^2_{yy}\right) - g^1_{xx}g^2_{xx} + g^1_{yy}g^2_{yy}\right]$$

and

$$g^1(u,v) = \alpha i(v) - L^1(v),$$
$$g^2(u,v) = i(v) - \alpha T_{\mathrm{bif}} i_v[i(v) - \gamma v + \beta(a_{21}u + a_{22}v) - L^2(u,v),$$

where $L^i(u,v)$, $i = 1,2$, denotes the linear parts of the system.

Now we obtain the final formula for the stability parameter in the form

$$V'''(0, T_{\mathrm{bif}}) = i_{vvv}(0) - 3\alpha T_{\mathrm{bif}} i_{vvv}(0)[2(i_v(0) - \gamma + i_v(0)],$$

where we assume that at the inflection point $i_{vv}(0) = 0$.

Therefore emerging cycle is attracting if $V'''(0, T_{\mathrm{bif}}) < 0$, i.e. $i_{vvv}(0) < 0$ like in the classical Kaldor model, because the contribution comes from the existence of time-to-build are second order if α is small (it is significant if α is large) and then the corresponding terms produce positive contribution.

Let us note that interpretation of sufficient conditions for the presence of a stable limit cycle is very simple. Namely if (i) the investment function is increasing with respect to its argument at the equilibrium point $i_y(0) > 0$; and (ii) the investment function changes second derivatives from positive sign to negative; then the third derivative of the investment function in the equilibrium point is negative.

It is interesting to observe that if the following Kaldor assumptions are satisfied

(i) $I_Y > S_Y$ for normal levels of income;

(ii) $I_Y < S_Y$ for high or low levels of income;

(iii) the stationary state equilibrium has a normal level of income, then the corresponding values of T_{bif} always exist because there is a pair of conjugate imaginary values $\lambda = \sigma \pm i\omega$, i.e. if

$$3\alpha^2(i_y(0)-\gamma)^2-\alpha\beta\left[\frac{\delta}{\beta}i_y(0)+\gamma\right]>0$$

(for $\delta = 0$ such values always exist), and moreover if

$$\frac{\partial\sigma}{\partial T}|_{T=T_{\text{bif}}} = \frac{\partial \operatorname{tr} A}{\partial T}|_{T=T_{\text{bif}}} = -\alpha\beta i_y(0) > 0$$

i.e. the transversality condition is satisfied at a bifurcation point then $T = T_{\text{bif}}$ is indeed a bifurcation value of the model. As $T > T_{\text{bif}}$ the real parts are becoming positive.

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