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The Investigation into New Equations in (2 + 1) Dimensions

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Abstract

First of all, we show the existence of the Lax pair for the Calogero Korteweg-de Vries(CKdV) equation. Next we modify T operator that is one of the Lax pair for the CKdV equation for the search of the $(2+1)$-dimensional case and propose a new equation in $(2+1)$ dimensions. We call it the $(2+1)$-dimensional CKdV equation. And then we discuss the modification of L operator that is another of the Lax pair of the CKdV equation. Moreover we attempt the modification of L and T operators.

1 Introduction

It is well-known that the Lax representation [1] describes $(1+1)$-dimensional integrable equations as follows. Consider two operators L and T which are called the Lax pair and given by

$$L = L_{KdV} - \lambda, \tag{1}$$

$$T = \partial_x L_{KdV} + T'_{KdV} + \partial_t, \tag{2}$$

with $\partial_x \equiv \frac{\partial}{\partial x}$ and λ being a spectral parameter independent upon t. Then the commutator

$$[L, T] = 0, \tag{3}$$

contains a nonlinear evolution equation for suitable chosen L and T. Equation (3) is called the Lax equation, For example if we take

$$L_{KdV} = \partial_x^2 - u, \tag{4}$$

$$T'_{KdV} = \frac{1}{2}u\partial_x - \frac{1}{4}u_x, \tag{5}$$

then L_{KdV} and T'_{KdV} satisfy the Lax equation (3) provided that u satisfies the KdV equation

$$u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx} = 0. \tag{6}$$

The study of higher dimensional integrable system is one of the central themes in integrable systems. In this paper, we shall discuss extending a nonlinear integrable equation in $(1+1)$ dimensions, which is called the Calogero–Korteweg-de Vries (CKdV) equation in Ref. [2], to in $(2+1)$ dimensions. A typical way of constructing higher dimensional integrable systems is to modify the Lax pair for basic equation. The CKdV equation is a $(1+1)$-dimensional nonlinear equation having the form

$$w_t+\frac{1}{4}w_{xxx}+\frac{3}{8}\frac{w_x}{w^2}-\frac{3}{4}\frac{w_x w_{xx}}{w}+\frac{3}{8}\frac{w_x^3}{w^2}=0. \tag{7}$$

This equation is a special case of the equation (4.4.5) in [3] with an obvious change of the dependent variable.

This paper is organized as follows. In Section 2, we construct the Lax pair for equation (7) and propose a new equation in $(2+1)$ dimensions by the extension for T operator for the CKdV equation. We call it the $(2+1)$-dimensional CKdV equation. In Section 3, other dimensional extensions are performed by changing L operator and both L and T operators. A $(2+1)$-dimensional equation obtained here is, however, reduced to the KP equation. Section 4 is devoted to discussion.

2 The Lax pairs for the CKdV equation and the $(2+1)$-dimensional CKdV equation

We have the conjecture that the Lax pair for the CKdV equation has the form,

$$L=\partial_x^2+g[w]\partial_x+h[w]-\lambda \quad \Big(\equiv L_{CKdV}-\lambda\Big), \tag{8}$$

$$T=\partial_x L_{CKdV}+T'_{CKdV}+\partial_t, \tag{9}$$

where $g[w]$, $h[w]$ are functions of w, w_x, w_{xx} etc, and T'_{CKdV} is unknown operator. We can fix the forms of $g[w]$, $h[w]$ and T'_{CKdV} by the condition that the Lax equation (3) gives the CKdV equation. The result is

$$g[w]=\sigma\frac{1}{w}, \tag{10}$$

$$h[w]=-\frac{1}{4}\frac{1}{w^2}-\frac{1}{2}\sigma\frac{w_x}{w^2}, \tag{11}$$

$$T'_{CKdV}=\frac{1}{2}\sigma\frac{1}{w}\partial_x^2-\frac{1}{2}\frac{1}{w^2}\partial_x-\frac{3}{16}\sigma\frac{1}{w^3}+\frac{1}{4}\frac{w_x}{w^3}-\frac{1}{16}\sigma\frac{w_x^2}{w^3}+\frac{1}{8}\sigma\frac{w_{xx}}{w^2}, \tag{12}$$

with $\sigma\equiv\pm\imath$. Hence the Lax pair for the CKdV equation is expressed as

$$L=\partial_x^2+\sigma\frac{1}{w}\partial_x-\frac{1}{4}\frac{1}{w^2}-\frac{1}{2}\sigma\frac{w_x}{w^2}-\lambda \quad \Big(\equiv L_{CKdV}-\lambda\Big), \tag{13}$$

$$T=\partial_x L_{CKdV}+\frac{1}{2}\sigma\frac{1}{w}\partial_x^2-\frac{1}{2}\frac{1}{w^2}\partial_x-\frac{3}{16}\sigma\frac{1}{w^3}+\frac{1}{4}\frac{w_x}{w^3}-\frac{1}{16}\sigma\frac{w_x^2}{w^3}+\frac{1}{8}\sigma\frac{w_{xx}}{w^2}+\partial_t. \tag{14}$$

Next we construct a new equation in $(2+1)$ dimensions. For that, we modify the above T operator to include another spatial dimension(z) [4, 5, 6, 7, 8] as follows,

$$T=\partial_z L_{CKdV}+T''_{CKdV}+\partial_t, \tag{15}$$

where L_{CKdV} is same in operator (13). The Lax equation gives not only the form of T''_{CKdV} but also a new equation. They are

$$T''_{CKdV} = \frac{1}{2}\sigma\partial_x^{-1}\left(\frac{1}{w}\right)_z\partial_x^2 - \frac{1}{2}\frac{1}{w}\partial_x^{-1}\left(\frac{1}{w}\right)_z\partial_x + \frac{1}{8}\sigma\frac{w_{xz}}{w^2} - \frac{1}{8}\sigma\frac{w_xw_z}{w^3}$$
$$- \frac{1}{8}\sigma\frac{1}{w^2}\partial_x^{-1}\left(\frac{1}{w}\right)_z + \frac{1}{4}\frac{w_x}{w^2}\partial_x^{-1}\left(\frac{1}{w}\right)_z - \frac{1}{16}\sigma\frac{1}{w}\partial_x^{-1}\left(\frac{1}{w^2}\right)_z \tag{16}$$
$$+ \frac{1}{16}\sigma\frac{1}{w}\partial_x^{-1}\left(\frac{w_x^2}{w^2}\right)_z$$

and

$$w_t - \frac{1}{2}\left(\frac{1}{w}\right)_z - \frac{1}{8}\left[w\partial^{-1}\left(\frac{w_x^2-1}{w^2}\right)_z - 2w\left(\frac{w_x}{w}\right)_z\right]_x = 0, \tag{17}$$

respectively. Equation (17) is reduced to the CkdV equation setting $z = x$. So we call equation (17) the $(2+1)$-dimensional CKdV equation.

3 Other extensions

In this section, we also try to extend the CKdV equation by two other modification manners.

First let us change L operator such as one for the Kadomtsev–Petviashvili (KP) equation [9, 10, 11]. Namely we take L operator with another spatial dimension (y)

$$L = L_{CKdV} + \partial_y. \tag{18}$$

T operator corresponding to operator (18) should be of the form

$$T = \partial_x^3 + \frac{3}{2}\sigma\frac{1}{w}\partial_x^2 + \left\{-\frac{3}{4}\frac{1}{w^2} - \frac{3}{2}\sigma\frac{w_x}{w^2} - \frac{3}{4}\sigma\partial_x^{-1}\left(\frac{1}{w}\right)_y\right\}\partial_x$$
$$- \frac{1}{8}\sigma\frac{1}{w^3} + \frac{3}{4}\frac{w_x}{w^3} + \frac{3}{4}\sigma\frac{w_x^2}{w^3} - \frac{3}{8}\sigma\frac{w_{xx}}{w^2} + \frac{3}{8}\sigma\frac{w_y}{w^2} \tag{19}$$
$$+ \frac{3}{8}\partial_y^{-1}\left\{\frac{1}{w}\partial_x^{-1}\left(\frac{1}{w}\right)_{yy}\right\} + \partial_t.$$

The Lax equation with (18) and (19) leads to

$$w_t + \frac{1}{4}w_{xxx} + \frac{3}{2}\frac{w_x^3}{w^2} - \frac{3}{2}\frac{w_xw_{xx}}{w} - \frac{3}{4}\sigma\frac{w_y}{w} - \frac{3}{4}w^2\partial_x^{-1}\left(\frac{1}{w}\right)_{yy}$$
$$-\frac{3}{4}\sigma w_x\partial_x^{-1}\left(\frac{1}{w}\right)_y - \frac{3}{4}\sigma w^2\left(\partial_y^{-1}\left\{\frac{1}{w}\partial_x^{-1}\left(\frac{1}{w}\right)_{yy}\right\}\right)_x = 0. \tag{20}$$

However, the above equation is reduced to the KP equation

$$\left(u_t + \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x\right)_x + \frac{3}{4}u_{yy} = 0, \tag{21}$$

by the following transformation

$$w = -\frac{1}{2}\sigma \frac{1}{\partial_y^{-1} u_x}. \tag{22}$$

Thus we cannot construct a new $(2+1)$-dimensional equation in this method.

Next let us attempt the modification for L and T operators [7]. That is, we take the Lax operators

$$L = L_{CKdV} + \partial_y, \tag{23}$$

$$T = \partial_z L_{CKdV} + T'''_{CKdV} + \partial_t, \tag{24}$$

as the Lax operators which include other spatial dimension (y, z). However, we cannot fix the form of T'''_{CKdV} by the Lax equation and, therefore, cannot construct a new higher dimensional equation.

4 Conclusions

In this paper, we have obtained the Lax pair for the CKdV equation and searched the Lax pair for the higher dimensional CKdV equation using three methods. Our results are as follows.

(i) The first method is to modify T operator for the Lax pair for the CKdV equation, then we have obtained the $(2+1)$-dimensional CKdV equation and the Lax pair.

(ii) The second method is to modify L operator. We have constructed the Lax pair and a higher dimensional equation. The equation is, however, reduced to the KP equation. The Lax pair ia also reduced to one for the KP equation by the following transformation

$$f^{-1}\left(L_{CKdV} + \partial_y\right) f \mapsto L_{kdV} + \partial_y, \tag{25}$$

where $f = \exp\left(\frac{-\sigma}{2}\partial^{-1}\frac{1}{w}\right)$ and transformation (22).

(iii) The last method is to unify the first and second methods. Using this method, we can expect a new higher dimensional equation. It, however, gives no consistent Lax operators unlike in the first and second methods.

Finally, let us consider the higher dimensional extension for another Lax pair

$$L = L_0 - \lambda, \tag{26}$$

$$T = \partial_x L_0 + T_0 + \partial_t, \tag{27}$$

by taking

$$L_0 = \partial_x^2 + \sigma\frac{1}{w}\partial_x - \frac{1}{4}\frac{1}{w^2} - \frac{1}{2}\sigma\frac{w_x}{w^2}, \tag{28}$$

$$T_0 = \frac{1}{2}\sigma\frac{1}{w}\partial_x^2 - \frac{1}{2}\frac{1}{w^2}\partial_x - \frac{1}{8}\sigma\frac{1}{w^3} + \frac{1}{4}\frac{w_x}{w^3} - \frac{1}{16}\sigma\frac{w_x^2}{w^3} + \frac{1}{8}\sigma\frac{w_{xx}}{w^2}. \tag{29}$$

The Lax equation (3) gives a following $(1+1)$-dimensional equation

$$w_t + \frac{1}{4}w_{xxx} - \frac{3}{4}\frac{w_x w_{xx}}{w} + \frac{3}{8}\frac{w_x^3}{w^2} = 0. \tag{30}$$

By a integration with respect to x after the change of

$$w = \phi_x, \tag{31}$$

equation (30) is reduced to the Schwarz–Korteweg-de Vries (SKdV) equation

$$\frac{\phi_t}{\phi_x} + S[\phi; x] = 0, \tag{32}$$

where

$$S[\phi; x] \equiv \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{33}$$

is the Schwarz derivative of ϕ. We have constructed a new $(2+1)$–dimensional equation by using the Calogero manner. As our result, the equation in $(2+1)$ dimensions is

$$w_t + \frac{1}{4}w_{xxz} - \frac{1}{2}\frac{w_x w_{xz}}{w} - \frac{1}{4}\frac{w_{xx}w_z}{w} + \frac{1}{2}\frac{w_x^2 w_z}{w^2} - \frac{1}{8}w_x\partial_x^{-1}\left(\frac{w_x^2}{w^2}\right)_z = 0, \tag{34}$$

from the Lax equation (3) by taking

$$L_0 = \partial_x^2 + \sigma\frac{1}{w}\partial_x - \frac{1}{4}\frac{1}{w^2} - \frac{1}{2}\sigma\frac{w_x}{w^2}, \tag{35}$$

$$\begin{aligned} T_0 = {} & \frac{1}{2}\sigma\partial_x^{-1}\left(\frac{1}{w}\right)_z\partial_x^2 - \frac{1}{2}\frac{1}{w}\partial_x^{-1}\left(\frac{1}{w}\right)_z\partial_x + \frac{1}{8}\sigma\frac{w_{xz}}{w^2} - \frac{1}{8}\sigma\frac{w_x w_z}{w^3} \\ & - \frac{1}{8}\sigma\frac{1}{w^2}\partial_x^{-1}\left(\frac{1}{w}\right)_z + \frac{1}{4}\frac{w_x}{w^2}\partial_x^{-1}\left(\frac{1}{w}\right)_z + \frac{1}{16}\sigma\frac{1}{w}\partial_x^{-1}\left(\frac{w_x^2}{w^2}\right)_z. \end{aligned} \tag{36}$$

It is easy to see that, for a similar computation in $(1+1)$ dimensions, equation (34) is readily expressed as

$$\frac{\phi_t}{\phi_x} + S_{2+1}[\phi; x] = 0, \tag{37}$$

where ϕ denotes relation (31) and

$$S_{2+1}[\phi; x] \equiv \left(\frac{\phi_{xx}}{\phi_x}\right)_z - \frac{1}{2}\partial_x^{-1}\left(\frac{\phi_{xx}}{\phi_x}\right)^2_z. \tag{38}$$

The above $S_{2+1}[\phi; x]$ satisfies the following relation

$$\left(S_{2+1}[\phi; x]\right)_x = \left(S[\phi; x]\right)_z. \tag{39}$$

Setting $z = x$ reduces equation (34) or (37) to the SKdV equation (30) or (32), and $S_{2+1}[\phi; x]$ to $S[\phi; x]$, respectively. Therefor we shall call equation (37) the $(2+1)$–dimensional SKdV equation, and $S_{2+1}[\phi; x]$ the $(2+1)$–dimensional Schwarz derivative of ϕ.

For details of this work, we would ask you to see our work [12] and [13].

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