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Two-Photon Algebra and Integrable Hamiltonian Systems

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Abstract

The two-photon algebra h_6 is used to define an infinite class of N-particle Hamiltonian systems having $(N-2)$ additional constants of the motion in involution. By construction, all these systems are h_6-coalgebra invariant. As a straightforward application, a new family of (quasi)integrable N-dimensional potentials is derived.

1 Introduction

In a recent paper [1], a systematic construction of integrable Hamiltonians with coalgebra symmetry has been proposed. Such procedure can be applied to any Poisson coalgebra (A, Δ) with generators X_i, $i = 1, \ldots, l$ and Casimir element $\mathcal{C}(X_1, \ldots, X_l)$ as follows. Let us consider the N-th coproduct $\Delta^{(N)}(X_i)$ of the generators and the m-th order $(2 \leq m \leq N)$ coproducts $\Delta^{(m)}(\mathcal{C})$ of the Casimir operator of the coalgebra (recall that the m-th coproduct is an algebra homomorphism that maps $\Delta^{(m)} : A \to A \otimes A \otimes \cdots^{m)} \otimes A$). By making use of the structural properties of the coproduct it can be proven that

$$\left\{\Delta^{(m)}(\mathcal{C}), \Delta^{(N)}(X_i)\right\} = 0, \qquad i = 1, \ldots, l. \tag{1.1}$$

Therefore, the $(2 \leq m \leq N)$ coproducts of the Casimir operator commute with the N-th order coproduct of any generator of the coalgebra. This implies that, if $\mathcal{H}$ is an arbitrary (smooth/formal power series) function of the generators of the algebra A, any N-particle Hamiltonian defined as

$$H^{(N)} := \Delta^{(N)}(\mathcal{H}(X_1, \ldots, X_l)) = \mathcal{H}\left(\Delta^{(N)}(X_1), \ldots, \Delta^{(N)}(X_l)\right), \tag{1.2}$$

Poisson-commutes with all the $(N-1)$ functions $C^{(m)} = \Delta^{(m)}(\mathcal{C})$:

$$\left\{C^{(m)}, H^{(N)}\right\} = 0, \qquad 2 \leq m \leq N. \tag{1.3}$$

Furthermore, all the $C^{(m)}$ constants of the motion are in involution

$$\left\{C^{(m)}, C^{(n)}\right\} = \left\{\Delta^{(m)}(\mathcal{C}), \Delta^{(n)}(\mathcal{C})\right\} = 0, \qquad \forall\, m, n = 2, \ldots, N. \tag{1.4}$$

So far, this formalism has been considered for $sl(2)$, (1+1) Poincaré and oscillator h_4 Poisson coalgebras [1]–[3] under certain phase space realizations. In all these cases, the systems obtained through the coalgebra formalism turned out to be completely integrable due to the non triviality of the constants of the motion $C^{(m)}$ defined by the Casimir. Moreover, some special choices of the dynamical Hamiltonian $\mathcal{H}$ showed the coalgebra symmetry of integrable systems like the isotropic N-dimensional oscillator and the Gaudin–Calogero Hamiltonian [4]–[6].

Both the $sl(2)$ and h_4 algebras are distinguished subalgebras of the so-called two-photon algebra h_6 [7], which is isomorphic to the (1 + 1) Schrödinger Lie algebra [8]. Explicitly, the two-photon Lie–Poisson coalgebra (h_6, Δ) is spanned by the six generators $\{N, A_+, A_-, B_+, B_-, M\}$ together with the Poisson brackets

$$\begin{array}{lll} \{N, A_+\} = A_+, & \{N, A_-\} = -A_-, & \{A_-, A_+\} = M, \\ \{N, B_+\} = 2B_+, & \{N, B_-\} = -2B_-, & \{B_-, B_+\} = 4N + 2M, \\ \{A_+, B_-\} = -2A_-, & \{A_+, B_+\} = 0, & \{M, \cdot\} = 0, \\ \{A_-, B_+\} = 2A_+, & \{A_-, B_-\} = 0 & \end{array} \tag{1.5}$$

and the (non-deformed) two-body coproduct

$$\Delta^{(2)}(X) = X \otimes 1 + 1 \otimes X, \qquad X \in \{N, A_+, A_-, B_+, B_-, M\}. \tag{1.6}$$

The coproduct $\Delta \equiv \Delta^{(2)}$ is a Poisson algebra homomorphism between h_6 and $h_6 \otimes h_6$. It is important to recall that h_6 has two Casimir functions: the mass M and a fourth-order Casimir given by

$$\mathcal{C}_{h_6} = \left(MB_+ - A_+^2\right)\left(MB_- - A_-^2\right) - \left(MN - A_-A_+ + M^2/2\right)^2, \tag{1.7}$$

which will play a relevant role in what follows. The one-particle phase space realization D for h_6 that we shall use is given by

$$\begin{array}{ll} f_N^{(1)} = D(N) = q_1p_1 - \frac{1}{2}\mu_1, & f_{A_+}^{(1)} = D(A_+) = p_1, \\ f_{A_-}^{(1)} = D(A_-) = \mu_1 q_1, & f_M^{(1)} = D(M) = \mu_1, \\ f_{B_+}^{(1)} = D(B_+) = \frac{1}{\mu_1}p_1^2, & f_{B_-}^{(1)} = D(B_-) = \mu_1 q_1^2. \end{array} \tag{1.8}$$

This phase space representation is labelled by the values of the Casimirs:

$$f_M^{(1)} = D(M) = \mu_1, \qquad C_{h_6}^{(1)} = D(\mathcal{C}_{h_6}) = 0. \tag{1.9}$$

The aim of this contribution is to present a summary of the integrability properties of the h_6 systems with coalgebra symmetry obtained from [1] through the realization (1.8) and the Casimir (1.7).

2 Hamiltonians with h_6-coalgebra symmetry

Let us start with the construction of two-particle systems. In this case, the coproduct map $\Delta^{(2)}$ (1.6) gives us, under a $D \otimes D$ realization, six two-particle phase space functions:

$$\begin{aligned} f_N^{(2)} &= (D \otimes D)(\Delta(N)) = \left(q_1p_1 - \tfrac{1}{2}\mu_1\right) + \left(q_2p_2 - \tfrac{1}{2}\mu_2\right), \\ f_{A_+}^{(2)} &= (D \otimes D)(\Delta(A_+)) = p_1 + p_2, \\ f_{A_-}^{(2)} &= (D \otimes D)(\Delta(A_-)) = \mu_1 q_1 + \mu_2 q_2, \\ f_{B_+}^{(2)} &= (D \otimes D)(\Delta(B_+)) = \tfrac{1}{\mu_1}p_1^2 + \tfrac{1}{\mu_2}p_2^2, \\ f_{B_-}^{(2)} &= (D \otimes D)(\Delta(B_-)) = \mu_1 q_1^2 + \mu_2 q_2^2, \\ f_M^{(2)} &= (D \otimes D)(\Delta(M)) = \mu_1 + \mu_2. \end{aligned} \tag{2.1}$$

It is straightforward to check that these functions define a two-particle phase space realization of h_6, provided the canonical Poisson bracket $\{q_i, p_j\} = \delta_{ij}$ is considered.

By following the construction [1], any smooth function of the $f_X^{(2)}$ functions will Poisson-commute with the $\Delta^{(2)}$ map of the Casimirs of the h_6 algebra. However, in this particular case this statement does not provide any dynamical information, since the two-particle integrals of motion provided by the two Casimirs M and $\mathcal{C}$ are trivial:

$$f_M^{(2)} = (D\otimes D)(\Delta(M)) = \mu_1+\mu_2, \tag{2.2}$$

$$\begin{aligned} C_{h_6}^{(2)} = (D \otimes D)(\Delta(\mathcal{C}_{h_6})) = &\left(f_M^{(2)} f_{B_+}^{(2)} - \left(f_{A_+}^{(2)}\right)^2\right)\left(f_M^{(2)} f_{B_-}^{(2)} - \left(f_{A_-}^{(2)}\right)^2\right) \\ &- \left(f_M^{(2)} f_N^{(2)} - f_{A_-}^{(2)} f_{A_+}^{(2)} + \tfrac{1}{2}\left(f_M^{(2)}\right)^2\right)^2 = 0. \end{aligned} \tag{2.3}$$

However, this degeneracy is removed in the three particle case. The third-order coproduct $\Delta^{(3)}$ of any generator X reads

$$\Delta^{(3)}(X) = X\otimes 1\otimes 1+1\otimes X\otimes 1+1\otimes 1\otimes X, \tag{2.4}$$

so that the 3-dimensional phase space realization of (h_6, Δ) is

$$\begin{aligned} &f_N^{(3)} = (q_1p_1 + q_2p_2 + q_3p_3) - \tfrac{1}{2}(\mu_1 + \mu_2 + \mu_3), && f_M^{(3)} = \mu_1 + \mu_2 + \mu_3, \\ &f_{A_+}^{(3)} = p_1 + p_2 + p_3, && f_{A_-}^{(3)} = \mu_1 q_1 + \mu_2 q_2 + \mu_3 q_3, \\ &f_{B_+}^{(3)} = \tfrac{1}{\mu_1}p_1^2 + \tfrac{1}{\mu_2}p_2^2 + \tfrac{1}{\mu_3}p_3^2, && f_{B_-}^{(3)} = \mu_1 q_1^2 + \mu_2 q_2^2 + \mu_3 q_3^2. \end{aligned} \tag{2.5}$$

As in the previous case, the integrals of motion coming from M are trivial:

$$f_M^{(1)} = \mu_1, \qquad f_M^{(2)} = \mu_1+\mu_2, \qquad f_M^{(3)} = \mu_1+\mu_2+\mu_3, \tag{2.6}$$

but we find now a first non-trivial integral of motion provided by the Casimir $\mathcal{C}_{h_6}$:

$$C_{h_6}^{(3)} = \frac{\mu_1 + \mu_2 + \mu_3}{\mu_1\mu_2\mu_3} \left(p_1(q_2 - q_3)\mu_2\mu_3 + p_2(q_3 - q_1)\mu_1\mu_3 + p_3(q_1 - q_2)\mu_1\mu_2\right)^2 . \tag{2.7}$$

We stress that $C_{h_6}^{(3)}$ is, by construction, in involution with *any* function $H^{(3)}$ of the three-particle representation of the generators (2.5). Therefore, if $H^{(3)}$ is considered as the Hamiltonian of a three-particle system, we would need another integral of motion in involution (and functionally independent) from $C_{h_6}^{(3)}$ in order to ensure complete integrability.

The generalization to an arbitrary number of particles is straightforward. The m-th coproduct $\Delta^{(m)}$ is

$$\begin{aligned} \Delta^{(m)}(X) = &X \otimes 1 \otimes 1 \otimes \dots^{m-1)} \otimes 1 \\ &+ 1 \otimes X \otimes 1 \otimes \dots^{m-2)} \otimes 1 + \cdots + 1 \otimes 1 \otimes \dots^{m-1)} \otimes 1 \otimes X. \end{aligned} \tag{2.8}$$

Hence the m-dimensional particle phase space realization of (h_6, Δ) turns out to be:

$$\begin{aligned} &f_N^{(m)} = \sum_{i=1}^m (q_i p_i - \tfrac{1}{2}\mu_i), && f_M^{(m)} = \sum_{i=1}^m \mu_i, && f_{A_+}^{(m)} = \sum_{i=1}^m p_i, \\ &f_{A_-}^{(m)} = \sum_{i=1}^m \mu_i q_i, && f_{B_+}^{(m)} = \sum_{i=1}^m \tfrac{1}{\mu_i} p_i^2, && f_{B_-}^{(m)} = \sum_{i=1}^m \mu_i q_i^2. \end{aligned} \tag{2.9}$$

An N-dimensional Hamiltonian $H^{(N)}$ with h_6-coalgebra symmetry will be defined through an arbitrary smooth function $\mathcal{H}$ of the two-photon generators (2.9) for $m = N$:

$$H^{(N)} = \mathcal{H}\left(f_N^{(N)}, f_M^{(N)}, f_{A_+}^{(N)}, f_{A_-}^{(N)}, f_{B_+}^{(N)}, f_{B_-}^{(N)}\right). \tag{2.10}$$

The central generator M gives rise to N trivial integrals of motion

$$f_M^{(m)} = \sum_{i=1}^{m} \mu_i, \qquad m = 1, \ldots, N, \tag{2.11}$$

while the other Casimir provides $(N-2)$ non-trivial integrals of motion $C_{h_6}^{(m)}$ ($m = 3, \ldots, N$), which are in involution and given by

$$\begin{aligned} C_{h_6}^{(m)} = &\left(f_M^{(m)} f_{B_+}^{(m)} - \left(f_{A_+}^{(m)}\right)^2\right)\left(f_M^{(m)} f_{B_-}^{(m)} - \left(f_{A_-}^{(m)}\right)^2\right) \\ &- \left(f_M^{(m)} f_N^{(m)} - f_{A_-}^{(m)} f_{A_+}^{(m)} + \tfrac{1}{2}\left(f_M^{(m)}\right)^2\right)^2. \end{aligned} \tag{2.12}$$

Cumbersome computations lead to the following explicit expressions for $C_{h_6}^{(m)}$ in terms of m-pairs of dynamical variables

$$\begin{aligned} C_{h_6}^{(m)} &= \left(\sum_{s=1}^{m} \mu_s\right)\left(\sum_{l=1}^{m} \frac{p_l^2}{\mu_l} \sum_{\substack{r<s \\ r,s\neq l}}^{m} \mu_r \mu_s (q_r - q_s)^2 + 2\sum_{i<j}^{m} p_i p_j \sum_{k\neq i,j}^{m} \mu_k (q_k - q_i)(q_j - q_k)\right) \\ &= \left(\sum_{s=1}^{m} \mu_s\right) \sum_{\substack{i,j,k=1 \\ i<j<k}}^{m} \frac{[p_i(q_j - q_k)\mu_j\mu_k + p_j(q_k - q_i)\mu_i\mu_k + p_k(q_i - q_j)\mu_i\mu_j]^2}{\mu_i\mu_j\mu_k}. \end{aligned} \tag{2.13}$$

3 Integrability properties of h_6 systems

We have just shown that, given any dynamical Hamiltonian $\mathcal{H}$ defined on h_6, the associated N-particle system given by $H^{(N)} = \Delta^{(N)}(\mathcal{H})$ fulfills

$$\left\{C_{h_6}^{(m)}, H^{(N)}\right\} = 0, \qquad \left\{C_{h_6}^{(m)}, C_{h_6}^{(n)}\right\} = 0, \qquad m, n = 3, \ldots, N. \tag{3.1}$$

Therefore, for an arbitrary h_6 system there is only one integral of the motion left in order to ensure complete integrability. However, we should distinguish between two classes of h_6 Hamiltonians:

a) If $\mathcal{H}$ is defined on a *subalgebra* of h_6, $\mathcal{H}$ will be also in involution with the Casimir of the subalgebra. Therefore, provided the coproducts of this new Casimir are neither trivial under the realization (2.9) nor functionally dependent of the $C_{h_6}^{(m)}$ integrals, we obtain an additional set of constants of the motion. Under these conditions, these h_6 systems defined on subalgebras will be not only completely integrable, but superintegrable. In particular, it can be proven that this procedure gives a new algebraic construction of the superintegrability of the isotropic N-dimensional harmonic oscillator [9].

b) On the contrary, if $\mathcal{H}$ is not defined on a subalgebra of h_6, the formalism will provide only the $(N-2)$ integrals $C_{h_6}^{(m)}$ (we could say that in this case $H^{(N)}$ is a "quasi-integrable" Hamiltonian). An interesting example is provided by the dynamical Hamiltonian given by

$$\mathcal{H} = \frac{1}{2} B_+ + \mathcal{F}(A_-) + \mathcal{G}(B_-), \tag{3.2}$$

where $\mathcal{F}$ and $\mathcal{G}$ are arbitrary smooth functions. This choice for $\mathcal{H}$ defines a new (and very large) family of natural Hamiltonian systems of the type

$$H^{(N)} = \frac{1}{2}\left(\sum_{i=1}^{N} \frac{1}{\mu_i} p_i^2\right) + \mathcal{F}\left(\sum_{i=1}^{N} \mu_i q_i\right) + \mathcal{G}\left(\sum_{i=1}^{N} \mu_i q_i^2\right), \tag{3.3}$$

that will always Poisson-commute with the functions $C_{h_6}^{(m)}$. Obviously, this construction does not exclude that, for a certain choice of the functions $\mathcal{F}$ and $\mathcal{G}$, more independent integrals could exist. A more extensive description of this kind of Hamiltonians will be presented elsewhere [9].

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