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A Dynamical System: Mars and its Satellite

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Abstract

A description of the most accurate analytical theory of the motion of Phobos, so far constructed, is presented. Several elements of the gravitational field of Mars, gravitational interactions between Phobos and Mars, Deimos and Jupiter, as well as tidal effects due to the interaction between the Sun and Mars, never considered before, have been taken into account. The theory is based on the model of two fixed gravitational centers.

1 Introduction

In the nearest future several missions aimed at a detailed study of Mars are planned. Difficulties encountered during some of the previous experiments indicate that a more precise theoretical analysis of the forces which may act on the satellites close to Mars is necessary. In this context a precise analysis of the motion of Phobos, the moon rotating around Mars in a close orbit, seems to be particularly interesting.

In the present work, the problem is described in terms of the Hamiltonian formalism. The Hamiltonian K may be expressed as:

$$K = K_0 + K_p, \tag{1}$$

where K_0 is the Hamiltonian of a simplified, analytically solvable, model and K_p is a perturbation. In our case K_0 describes two fixed gravitational centers [1, 4, 5]. The perturbational potential mainly describes the interactions due to Mars, the Sun, Deimos and Jupiter.

The origin of the reference frame OXYZ has been selected at the mass center of Mars. The plane OXY is the plane of the mean equator of date of Mars. The OX axis is directed towards the ascending node of the mean orbit of date of Mars. The coordinate system constructed in this way is a non-inertial.

2 Formalism

Due to practical reasons both the integrable problem and the perturbational function have been written using a, e, s, l, g, h elements (close to the osculating elements). As in the case of the Kepler orbit, a stands for the semi-major axis, e — for the eccentricity, s — for the sine of the angle of inclination of the orbit to the reference plane, the angles l, g and h (up to J_2 and J_3/J_2, where J_2-second zonal harmonics of Mars and J_3-third zonal harmonics of Mars) are the counterparts of the mean anomaly, the argument of the pericentre and the longitude of the node of the satellite orbit. However, one has to remember, that for the analytical integration of the equations of motions one has to use the "Delauney variables" $(l,\ g,\ h,\ L,\ G,\ H)$.

The relations between the osculating elements and the Delauney ones may be found in ref. [4]. The main part of the Hamiltonian function, expressed in terms of the "osculating elements", is:

$$
\begin{aligned}
K_0 = \frac{fm}{2a} & \left[1 - \varepsilon^2 \left(1 - s^2\right)\left(1 - e^2\right) + \varepsilon^4 s^2 \left(1 - s^2\right)\left(1 - e^2\right)\left(3 + e^2\right)\right. \\
& + \varepsilon^4 \sigma^2 \left(1 - s^2\right)\left(1 - e^2\right)\left(-1 + 7s^2\right) \\
& \left. - \varepsilon^6 s^4 \left(1 - s^2\right)\left(1 - e^2\right)\left(5 + 10e^2 + e^4\right)\right] + \cdots ,
\end{aligned}
$$

where

$$
\varepsilon^2 = \frac{c^2}{a^2 \left(1 - e^2\right)^2}, \qquad \varepsilon\sigma = \frac{c\sigma}{a \left(1 - e^2\right)},
$$

and

$$
c = r_0 \sqrt{J_2 - \left(\frac{J_3}{2J_2}\right)^2}, \qquad \sigma = \frac{J_3}{2J_2 \sqrt{J_2 - \left(\frac{J_3}{2J_2}\right)^2}}.
$$

f is the universal gravitational constant and m is the sum of the masses of Mars and Phobos. With the limitation of the theory to the third order with respect to ε^2, $\varepsilon\sigma$ is a necessary compromise between an attempt to create a very accurate analytical theory and the practical limitation imposed by the affordable computational time. Under the conditions concerning the accuracy of the expansion accepted in this work, the perturbational function contains the following terms:

- Interactions of Phobos with the Sun, Deimos and Jupiter.
- The gravitational potential of Mars, in which all zonal harmonics up to the 12th order and the tesseral-sektorial harmonics up to the order and level 6 have been included.
- Tidal interactions between Mars and the Sun.
- Precession of the coordinate system (as a consequence of selecting a non-inertial coordinate system).

In order to express the perturbational function using the a, e, s, l, g, h variables, one has to know the relations between the above coordinates and the Cartesian ones.

An accurate description of the construction of the perturbational function may be found in paper [13]. The disturbing function due to j_4 (zonal harmonic of Mars) and the Sun is expanded up to the seventh order using e and s variables and those terms for which the numerical amplitudes are smaller than 10^{-10} are rejected. In the other cases the series are restricted up to the fifth order using e and s and the amplitude 10^{-8}. The perturbational function obtained in this way is composed of 22 000 terms ([13, 14]).

3 Analytical Integration

For the analytical integration I have chosen the method of averaging of variables based on the Lie canonical transformation [9]. In this method we are concerned with the transformation of the Hamiltonian rather than with solving the equations of motion. We are looking for new variables in which the Hamilton function does not depend upon the angular variables. The new Hamiltonian may be easily integrated using elementary methods. The relation between the new variables and the old ones may be defined by the following system of differential equations.

$$\frac{dx_i}{d\tau} = \frac{\partial W(x,y)}{\partial y_i}, \qquad \frac{dy_i}{d\tau} = -\frac{\partial W(x,y)}{\partial x_i}, \qquad i = 1, 2, \ldots, n$$

with a boundary condition:

$$x_i = q_i, \qquad y_i = p_i, \qquad \text{where} \quad \tau = 0.$$

$W(\mathbf{x}, \mathbf{y})$ ($\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$) does not contain the variable τ explicitly. An arbitrary function $F(\mathbf{x}, \mathbf{y})$ in the new variables may be written as:

$$F(\mathbf{x}, \mathbf{y}) = \exp(\varepsilon D) F(\mathbf{p}, \mathbf{q}) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} D^n F(\mathbf{p}, \mathbf{q}). \tag{2}$$

Transforming the function K to the new variables (L', G', H', l', g', h'), the generating function W which describes the connection between the old and the new variables is determined. After the transformation the new Hamiltonian E depends on neither the angular variables nor the time. Transforming function K one obtains a new function E containing 154 terms and the transforming function W containing 235 856 terms.

Using the averaged equations of motion, one can obtain the expressions for the secular perturbations in elements l', g' and h' as:

$$\frac{dl'}{dt} = -\frac{\partial E}{\partial L'},$$
$$\frac{dg'}{dt} = -\frac{\partial E}{\partial G'},$$
$$\frac{dh'}{dt} = -\frac{\partial E}{\partial H'},$$

$$\begin{aligned}
&r_0 = 3394 \text{ km (radius of Mars)},\\
&a' = 2.75924941\, r_0 \text{ (semi-major axis of the orbit of Phobos)},\\
&e' = 0.015 \text{ (eccentricity of the orbit of Phobos)},\\
&i' = 0.01931583828 \text{ rad (inclination of the orbit of Phobos)},\\
&M_M = 8154.455646055\ r_0^3\,\text{day}^{-2} \text{ (mass of Mars)},\\
&M_S = 3098710\, M_M \text{ (mass of Sun)},\\
&M_D = 2.801885 \times 10^{-9}\, M_M \text{ (mass of Deimos)},\\
&M_J = 2958.6052\, M_M \text{ (mass of Jupiter)},\\
&k_2 = 0.05 \text{ (number characterizing the non-elasticity of Mars)}.
\end{aligned} \tag{3}$$

Assuming (3) one can determine numerical values of the secular perturbations caused by appropriate orders:

order	dl'/dt [rad/day]	dg'/dt [rad/day]	dh'/dt [rad/day]
$0 = K_0$	19.702047193605	0.0151910203178	−0.0075961753821
I	0	0	0
$\mathrm{II} = j_4 + j_6 + \mathrm{Sun}$	4.9680511×10^{-6}	$-3.24483288 \times 10^{-5}$	$1.62289583 \times 10^{-5}$
III = the other forces	1.65162×10^{-8}	-9.57089×10^{-8}	-3.65653×10^{-8}
total	19.702052178172	0.0151584762801	−0.0075799829892

The values of the coefficients describing the gravitational field of Mars have been taken from Ref. [8].

4 Conclusions

The solar perturbations and the perturbations resulting from j_4 change the mean anomaly, the argument of the pericentre and the longitude of the node,

$$\left(\sqrt{(dl'/dt)^2 + (dg'/dt)^2 + (dh'/dt)^2}\right),$$

respectively, of the order 3×10^{-3} rad/year and 1×10^{-2} rad/year. Zonal harmonics $j_5, \ldots, j_{12}$ modify the mean motion, respectively, by approximately 1×10^{-5}, 1×10^{-3}, 2×10^{-6}, 1×10^{-4}, 3×10^{-7}, 6×10^{-5}, 8×10^{-8}, 7×10^{-6} rad/year. The tidal effects resulting from the interaction between the Sun and Mars are of the order 7×10^{-7} rad/year. Deimos, in spite of its small size, introduces perturbations in the mean motion of the order 6×10^{-6} rad/year. Non-inertiality of the coordinate system as well as the interaction between Phobos and Jupiter lead to a change in the mean motion of the order 3×10^{-5}–8×10^{-8} rad/year.

J_2^2 induces the greater secular effect as j_7, j_9, j_{11}, j_{12}, tide, Deimos, and Jupiter. Secular effect due to the perturbation by J_2^3 are of the same order as the secular effect due to the perturbation by the tide and j_9.

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