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A Pivotal Model for the (1 + 1)-Dimensional Heisenberg and Sigma Models

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Abstract

An integrable interpolative (Pivotal) model for the (1 + 1)-dimensional Hyperbolic Heisenberg and Hyperbolic sigma models is proposed and some solutions classifiable by an integer winding number examined.

1 Introduction

Both the Heisenberg and sigma models in (1 + 1) dimensions are well known integrable systems and have been discussed to date in some detail with respect to various target manifolds and aspects of their integrability (see for example [1–8]). Whilst both systems admit the same static equations, if time dependence is introduced the Galilean invariant Heisenberg model admits an equation of motion of parabolic type whereas the equation of motion for the Lorentz invariant sigma model resides in the class of hyperbolic equations. It therefore seems unlikely that any direct interpolation between the time dependent models might exist, however, this is indeed possible; in this note a third integrable model is proposed which *contains* both the Heisenberg and Sigma models and facilitates, via a single scalar parameter, just such an interpolation between the models and at least some of their solutions.

The paper is set out as follows: to begin with the "Pivotal" equation is stated where, in keeping with the discussions of the Heisenberg and sigma models of [9, 10], we choose the hyperboloid of one sheet as the target manifold. The integrability of the model is then established in the sense that a suitable Lax pair is shown to exist, and some conserved quantities of the motion are briefly discussed. Some solutions classifiable by an integer winding number are then given and their interpolative limits examined.

2 The Hyperbolic Pivotal model (HPM)

The field $\vec{\psi}(t,x) = \left(\psi^1,\psi^2,\psi^3\right)$ takes its values on the hyperboloid of one sheet H^2 in $\mathbb{R}^{2+1}$ and satisfies the constraint $\eta_{ab}\psi^a\psi^b = 1$, where $\eta_{ab} = \mathrm{diag}\,(1,1,-1)$. Here $t \in \mathbb{R}$, and $x \in X$ is such that either $X = \mathbb{R}$ and the boundary condition $\vec{\psi}(t,\infty) = \vec{\psi}(t,-\infty)$ is imposed or, X has finite period and $\vec{\psi}$ is periodic in x. With the metric on H^2 taken to be that induced by η_{ab}, the manifold is a symmetric space $SO(2,1)/SO(1,1)$ with fundamental group $\mathbb{Z}$ so for each fixed t, $\vec{\psi}$ is a continuous mapping from a circle into H^2 with winding number N and may be visualized as a closed string wrapped around H^2 and evolving in time.

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Arising as a variation on an integrable extension of the non-linear $O(3)$ sigma model on Hermitian symmetric target spaces proposed in [11], the Hyperbolic Pivotal model (HPM) has the following equation of motion:

$$(1-\omega)\psi^a_t = ?\epsilon^a_{bc}?\left(\psi^b\psi^c_{xx} - \omega\psi^b\psi^c_{tt}\right), \tag{1}$$

where hyperbolic refers to the target manifold and the scalar parameter ω is such that $0 \leq \omega \leq 1$. In the limit $\omega = 0$ the equation

$$\psi^a_t = ?\epsilon^a_{bc}?\psi^b\psi^c_{xx} \tag{2}$$

is produced which is exactly the Hyperbolic Heisenberg model (HHM) equation. And in the limit $\omega = 1$ the result is

$$?\epsilon^a_{bc}?\left(\psi^b\psi^c_{xx} - \psi^b\psi^c_{tt}\right) = 0; \tag{3}$$

when the hyperboloid is parametrized in terms of either stereographic coordinates or polar angles (θ, ϕ), (3) is precisely the Hyperbolic sigma model (HSM) equation of [9, 10].

The integrability of the HPM may be established in the following way: using the identity $[S, [S, \partial_\mu S]] = -\partial_\mu S$ (cf. [11]), where

$$S = \begin{pmatrix} \psi_1 & \psi_2 + \psi_3 \\ \psi_2 - \psi_3 & -\psi_1 \end{pmatrix} \in SL(2, \mathbb{R}).$$

And with

$$\alpha = \frac{2\omega}{\lambda^2 - \omega}, \qquad \beta = \frac{2\lambda}{\lambda^2 - \omega},$$

$$\gamma_0 = \beta^2(1-\omega) = \frac{4\lambda^2(1-\omega)}{(\lambda^2 - \omega)^2}$$

and

$$\gamma_1 = \beta(\omega-1)(1+\alpha) = \frac{2\lambda(\omega - 1)\left(\lambda^2 + \omega\right)}{(\lambda^2 - \omega)^2},$$

(λ being the spectral parameter); the pair

$$U = \alpha\left[S, \partial_x S\right] - \beta\omega\left[S, \partial_t S\right] + \gamma_1 S, \tag{4a}$$

$$V = \alpha\left[S, \partial_t S\right] - \beta\left[S, \partial_x S\right] + \gamma_0 S \tag{4b}$$

satisfy the equation

$$\partial_t U - \partial_x V + \left[V, U\right] = 0$$

if and only if ψ^a satisfy the HPM equation (1). Hence the HPM equation satisfies the zero curvature condition with the U, V pair (4) and is in this sense, integrable. Furthermore, the integrability of both the HHM and HSM models may be verified similarly by taking the limits $\omega = 0, 1$, respectively.

Conserved quantities of the motion may be found as follows: by taking the Laurent expansion in λ with $F(x;\lambda) = \sum_{n=0}^{\infty} \frac{F_n}{\lambda^n}$, $(F_0 = 1)$, in the linear problem $(\partial_\mu + A_\mu)F(x;\lambda) = 0$,

($A_\mu \sim U, V$), and defining the conserved current as $J^{(n)}_\mu = \epsilon_{\mu\nu}\partial^\nu F_n(x)$, which satisfy the current conservation $\partial_\mu J^\mu = 0$; to first order one has

$$(\partial_0 - \partial_1)F_1 - 2\Big([S, \partial_1 S] - \omega[S, \partial_0 S] - (1-\omega)S\Big) = 0.$$

The current conservation here is just the equation of motion (1) and is in fact, a local conservation law since (1) is a vector divergence equation. To second order one has

$$\begin{aligned} 0 &= (\partial_0 - \partial_1)F_2 + 2\omega\big([S, \partial_0 S] - [S, \partial_1 S]\big) + 4(1-\omega)S \\ &\quad - 2F_1\big([S, \partial_1 S] - \omega[S, \partial_0 S] - (1-\omega)S\big) \end{aligned} \tag{5}$$

which is non-local as are the higher order currents which may be similarly obtained ad infinitum.

3 Topological solitons

Since our main concern for the HHM and HSM in [9, 10] was the existence of topological solitons, let us expand this investigation to the Pivotal model case; this may also give some insight into how the interpolation holds up under scrutiny with respect to solutions of the model.

3.1 Travelling waves

The hyperboloid H^2 may be parametrized in terms of the polar angles (θ, ϕ) where $\theta \in (-\infty, \infty)$, $\phi \in [0, 2\pi]$ such that

$$\vec{\psi} = (\cosh\theta\cos\phi, \cosh\theta\sin\phi, \sinh\theta).$$

Further, using the characteristic variable $\xi = x - vt$ so that $\theta(t, x)$ and $\phi(t, x)$ are replaced by $f(\xi)$ and $g(\xi)$ respectively; in (1) and with the substitution $p = \sinh f$ one finds the following equations for p and g:

$$2\left(1 - \omega v^2\right)^2 p'^2 = 4qp^2 - v(1-\omega)kp + 4q - 2\left[(1-\omega)^2 v^2 - k^2\right] \tag{6}$$

and

$$\left(1 - \omega v^2\right) g' = \frac{vp(\omega - 1) + k}{1 + p^2}, \tag{7}$$

where "prime" denotes differentiation with respect to ξ and q and k are constants. On integration, (6) yields the solution

$$p(\xi) = \sinh f = \sqrt{\frac{A}{B}}\sin\left[\frac{\sqrt{B}(\xi - \xi_0)}{(1 - \omega v^2)}\right] + p_0, \tag{8}$$

where the constants $A = 2q - v^2(1-\omega)^2 + k^2 - \frac{2k^2v^2(1-\omega)}{q}$, $B = -2q$ are both positive and $p_0 = \frac{kv(1-\omega)}{2q}$. This solution is of topological type with $X = S^1$, which may be checked by substituting (8) into (7) and integrating with respect to x. Analyzing the result (cf. similar result in [10]) one finds that the solution does indeed wind around the hyperboloid an integer number of times. Furthermore, in the limits of the parameter ω the solution reduces to one for both the HHM and HSM models where in the sigma model case the resultant solution is related to a Lorentz boost of the static case[1]. Hence the above is a topological soliton of travelling wave type for $X = S^1$ and is a solution of all three models which can be seen by varying the parameter ω.

Investigation into the existence of travelling waves where X is the real line is omitted due to space considerations and will be covered in a subsequent paper, however, in the next section it will be shown that time dependent solutions not of travelling wave type are possible for $X = \mathbb{R}$.

3.2 Solutions from a stereographic parametrization of H^2

If the hyperboloid is parametrized in terms of a stereographic projection such that $\vec{\psi} = \zeta^{-1}\left(1 - u^2 + v^2, 2u, 2v\right)$, where $\zeta = 1 + u^2 - v^2$; in the relevant limits of ω the HPM equations reduce to both the HHM and HSM equations. Letting $u^2 - v^2 = f(x)^2$ and choosing

$$u(t,x) = f(x)\cosh(mt), \tag{9a}$$

$$v(t,x) = f(x)\sinh(mt) \tag{9b}$$

with m constant; substitution of (9) into the parametrized system produces a second order equation for f which, on integration (cf. [12]) results in the first order equation

$$f'^2 = \frac{1}{4}\left(\frac{m^2\omega}{2} - k\right)\left(1+f^2\right)^2 + \frac{m(1-\omega)}{2}\left(f^2+1\right)\left(f^2-1\right) - \frac{m^2\omega}{4}\left(f^2-1\right)^2. \tag{10}$$

This is satisfied by the elliptic solution

$$f(x) = A\,\mathrm{sc}\left[B(x-x_0)|M\right], \tag{11}$$

where

$$B^2 = \frac{1}{8}\left[2l + 3m^2\omega \pm 2m\sqrt{2\left[2\omega l + m^2\omega^2 + 2(1-\omega)^2\right]}\right], \tag{12a}$$

$$A^2 = \frac{3m^2\omega + 2l \mp 2m\sqrt{2\left[2\omega l + m^2\omega^2 + 2(1-\omega)^2\right]}}{2l - m^2\omega + 4m(1-w)}, \tag{12b}$$

$$M = \frac{\pm 4m\sqrt{2\left[2\omega l + m^2\omega^2 + 2(1-\omega)^2\right]}}{2l + 3m^2\omega \pm 2m\sqrt{2\left[2\omega l + m^2\omega^2 + 2(1-\omega)^2\right]}}, \tag{12c}$$

$l = -k > 0$, M is the elliptic function parameter and the signs are ordered throughout. Taking $m > 0$ and the positive square root in B^2, one then has both M and $B^2 \geq 0$ and the only constraints arise from $A^2 \geq 0$ and the fact that, with respect to the elliptic function, things are simplified if $0 \leq M \leq 1$. Both stipulations result in the single constraint

$$2l - m^2\omega \geq 4m(1-\omega). \tag{13}$$

Taking the limit of the parameter $M = 0$, one then has $A = \pm 1$ and $B = \pm\frac{\sqrt{l}}{2}$ so that if $l = 4$, the static solution $f(x) = \tan x$ is recovered (and is a solution for all three models).

[1]Note that all static solutions of the HPM are also solutions for the other two models and vice versa since the static equations are the same for all three models.

If x is now shifted by the half period K : $x \longrightarrow K - x$ then

$$A\,\mathrm{sc}\,[K - x|M] = \mu\,\mathrm{cs}\,[Bx|M] \tag{14}$$

(where $\mu = \frac{A}{\sqrt{1-M}}$) and the solution

$$f(x) = \mu\,\mathrm{cs}\,(Bx|M) \tag{15}$$

is well defined with the parameters (12) and the constraint (13). Further, with (15) and u, v as in (9), $\vec{\psi}$ is a topological (t dependent) soliton with period $2K$. Taking the limit $M = 1$ requires $2l - m^2\omega = \pm 4m(1-\omega)$ and choosing the positive case results in $\mu = \sqrt{\frac{m\omega}{(1-\omega)}}$, $B = \sqrt{m^2\omega + m(1-\omega)}$ and the solution

$$f(x) = \pm\left(\frac{m\omega}{(1-\omega)} + 1\right)^{\frac{1}{2}} \mathrm{cosech}\left[\sqrt{m^2\omega + m(1-\omega)}(x - x_0)\right]. \tag{16}$$

This passes through the points $\vec{\psi} = (\pm 1, 0, 0)$ if $X = \mathbb{R}$ and hence winds once around the hyperboloid, so that (16) is a topological t dependent soliton where X is the real line.

It remains now to examine how (15) behaves in the limiting models which we consider separately as follows:

(i) If $\omega = 0$ (i.e. for HHM), one has $B = \pm\frac{\sqrt{l+2m}}{2}$, $A = \sqrt{\frac{2l-4m}{2l+4m}}$, $M = \frac{8m}{2l+4m}$ and $\mu = \sqrt{M}$. Hence, if $l \geq 2m$, (15) has a satisfactory reduction to a topological soliton in terms of elliptic functions for the Heisenberg model. Taking the limit $M = 1$ requires $l = 2m$ and the corresponding solution (putting $m = 1$ for example), is given by $f(x) =$ cosech x which again winds once around the hyperboloid for $X = \mathbb{R}$.

(ii) In the $w = 1$ (i.e. HSM) limit the situation is not quite so straightforward; here one has

$$B^2 = \frac{1}{8}\left(2l + 3m^2 + 2m\sqrt{2(2l + m^2)}\right),$$

$$A^2 = \frac{3m^2 + 2l - 2m\sqrt{2(2l + m^2)}}{2l - m^2},$$

$$M = \frac{4m\sqrt{2(2l + m^2)}}{2l + 3m^2 + 2m\sqrt{2(2l + m^2)}}$$

resulting in $\mu = \sqrt{\frac{4m[2(2l+m^2)]^{\frac{1}{2}}}{2l-m^2}}$. If $2l > m^2$ the solution (15) is perfectly valid for the sigma model and is topological in its elliptic form (and of course, in the static ($M = m = 0$) case). However, taking the limit $M = 1$ requires $2l = m^2$ so that $f(x)$ diverges. Nevertheless, it is interesting to note that as M gets *close* to unity the solution does actually come close to a “cosech”. For example, taking $l = 2.1$ and $m = 2$ one has $M \approx 0.999962$ and $\mu \approx 161.99$ (with $B \approx 4$) so that such a solution is approximated as the limit is approached.

4 Concluding remarks

There are few known parabolic or hyperbolic systems which are both integrable *and* admit topological solitons; the Heisenberg and sigma models are of this type. The Pivotal system proposed here is also of this type and moreover, contains a parabolic system (the HHM) which is Galilean invariant, *and* a Lorentz invariant hyperbolic system (the HSM). Furthermore, as documented, at least some of the solutions and properties of this Pivotal model "carry through" to the two constituent models via the scalar parameter ω. With these combined properties it is therefore believed that the Pivotal model is unique among known systems. Since the model is new, there are obviously many possibilities for further investigation, for example, its complete integrability via the inverse scattering transform, the symmetries of the model and its applications, however, these must be left for the future.

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