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Quasi-Bi-Hamiltonian Systems Obtained from Constrained Flows

Yunbo ZENG

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
E-mail: yzengtsinghua.edu.cn

Abstract

An infinite number of families of quasi-bi-Hamiltonian (QBH) systems can be constructed from the constrained flows of soliton equations. The Nijenhuis coordinates for the QBH systems are proved to be exactly the same as the separation variables introduced by the Lax matrices for the constrained flows.

1 Introduction

For a finite-dimensional integrable Hamiltonian system (FDIHS) with degree $2n$, a vector field, X, is said to be a QBH vector field with respect to two compatible and nondegenerated Poisson tensors, θ_0 and θ_1, if there exist two integrals of motion F_1, E_1 and an integrating factor ρ, such that [1, 2]

$$X = \theta_0 \bigtriangledown F_1 = \frac{1}{\rho}\theta_1 \bigtriangledown E_1. \tag{1.1}$$

The Nijenhuis tensor $\Phi = \theta_1\theta_0^{-1}$ has n distinct eigenvalues $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ [3]. One can construct a canonical transformation $(\boldsymbol{q}, \boldsymbol{p}) \mapsto (\boldsymbol{\mu}, \boldsymbol{\nu})$ ($(\boldsymbol{\mu}, \boldsymbol{\nu})$ referred to as the Nijenhuis coordinates) and the FDIHS in the Nijenhuis coordinates is separable. Several QBH systems are presented and some relationship between BH and QBH structure is discussed in [1, 2, 4, 5]. The aims of this paper is to show how to construct an infinite number of families of QBH systems from the constrained flows of soliton equations [6–10] and to prove that the Nijenhuis coordinates for the underlying families of QBH systems are exactly the same as the separated variables introduced by the Lax matrices [11–13].

2 New QBH system

For Jaulent–Miodek (JM) spectral problem [14]

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U(u,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad U(u,\lambda) = \begin{pmatrix} 0 & 1\\ \lambda^2 - u_1\lambda - u_0 & 0\end{pmatrix}, \quad u = \begin{pmatrix}u_1\\ u_0\end{pmatrix}, \tag{2.1}$$

the Jaulent–Miodek hierarchy reads

$$u_{t_n} = \begin{pmatrix}u_1\\ u_0\end{pmatrix}_{t_n} = J\begin{pmatrix}b_{n+2}\\ b_{n+1}\end{pmatrix} = J\frac{\delta H_n}{\delta u}, \qquad n = 1, 2, \ldots, \tag{2.2}$$

where $H_n = \frac{1}{n}(2b_{n+3} - u_1 b_{n+2})$

$$b_0 = b_1 = 0, \qquad b_2 = 1, \qquad \begin{pmatrix} b_{k+2} \\ b_{k+1} \end{pmatrix} = L \begin{pmatrix} b_{k+1} \\ b_k \end{pmatrix}, \qquad k = 1, 2, \ldots,$$

$$L = \begin{pmatrix} u_1 - \frac{1}{2}D^{-1}u_{1,x} & \frac{1}{4}D^2 + u_0 - \frac{1}{2}D^{-1}u_{0,x} \\ 1 & 0 \end{pmatrix},$$

$$D = \frac{\partial}{\partial x}, \qquad DD^{-1} = D^{-1}D = 1.$$

Under zero boundary condition we have $\frac{\delta\lambda}{\delta u} = \binom{\lambda\psi_1^2}{\psi_1^2}$. The constrained flows of (2.2) are defined by [6-10]:

$$\Psi_{1,x} = \Psi_2, \qquad \Psi_{2,x} = \Lambda^2\Psi_1 - u_1\Lambda\Psi_1 - u_0\Psi_1, \tag{2.3a}$$

$$\frac{\delta H_l}{\delta u} - \frac{1}{2}\sum_{j=1}^{N} \frac{\delta\lambda_j}{\delta u} = \begin{pmatrix} b_{l+2} \\ b_{l+1} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \langle\Lambda\Psi_1, \Psi_1\rangle \\ \langle\Psi_1, \Psi_1\rangle \end{pmatrix} = 0. \tag{2.3b}$$

Hereafter we denote the inner product in $\mathbf{R}^N$ by $\langle\cdot,\cdot\rangle$ and $\Psi_i = (\psi_{i1}, \ldots, \psi_{iN})^T$, $i = 1, 2$, $\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_N)$ for N distinct λ_j.

The third constrained flow ($l = 4$) can be transformed into a FDIHS

$$P_x = \theta_0 \bigtriangledown F_1, \tag{2.4a}$$

$$P = \left(\Psi_1^T, q_1, q_2, \Psi_2^T, p_1, p_2\right)^T, \qquad \theta_0 = \begin{pmatrix} 0 & I_{(N+2)\times(N+2)} \\ -I_{(N+2)\times(N+2)} & 0 \end{pmatrix},$$

$$\begin{aligned} F_1 = {} & \frac{1}{2}\langle\Psi_2, \Psi_2\rangle - \frac{1}{2}\langle\Lambda^2\Psi_1, \Psi_1\rangle + \frac{1}{2}q_1\langle\Lambda\Psi_1, \Psi_1\rangle + \frac{1}{2}q_2\langle\Psi_1, \Psi_1\rangle \\ & - 8p_1p_2 + 10q_1p_2^2 - \frac{5}{16}q_1^3q_2 - \frac{3}{8}q_1q_2^2 - \frac{7}{128}q_1^5. \end{aligned} \tag{2.4b}$$

The Lax matrix $Q \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}$ for FDIHS (2.4) is given by the method [7, 8]

$$\begin{aligned} A(\lambda) &= 2p_2\lambda + 2p_1 - 2q_1p_2 - \frac{1}{2}\sum_{j=1}^{N} \frac{\psi_{1j}\psi_{2j}}{\lambda - \lambda_j}, \\ B(\lambda) &= \lambda^2 + \frac{1}{2}q_1\lambda + \frac{3}{8}q_1^2 + \frac{1}{2}q_2 + \frac{1}{2}\sum_{j=1}^{N} \frac{\psi_{1j}^2}{\lambda - \lambda_j}, \\ C(\lambda) &= \lambda^4 - \frac{1}{2}q_1\lambda^3 - \left(\frac{1}{2}q_2 + \frac{1}{8}q_1^2\right)\lambda^2 + \left(\frac{1}{4}q_1^3 + \frac{1}{2}q_1q_2 - \frac{1}{2}\langle\Psi_1, \Psi_1\rangle\right)\lambda \\ & \quad + \frac{1}{4}q_2^2 - \frac{5}{64}q_1^4 - 4p_2^2 - \frac{1}{2}\langle\Lambda\Psi_1, \Psi_1\rangle + \frac{1}{2}q_1\langle\Psi_1, \Psi_1\rangle - \frac{1}{2}\sum_{j=1}^{N} \frac{\psi_{2j}^2}{\lambda - \lambda_j}. \end{aligned} \tag{2.5}$$

We can show the following propositions.

Proposition 1. *The FDIHS (2.4) possesses the QBH representation (1.1) with*

$$\rho = B(\lambda)|_{\lambda=0} = \frac{3}{8}q_1^2 + \frac{1}{2}q_2 - \frac{1}{2}\langle \Lambda^{-1}\Psi_1, \Psi_1\rangle,$$

$$E_1 = \left[A^2(\lambda) + B(\lambda)C(\lambda)\right]|_{\lambda=0}, \qquad \theta_1 = \begin{pmatrix} 0_{(N+2)\times(N+2)} & A_1 \\ -A_1^T & B_1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} \lambda & -\frac{1}{4}\Psi_1 & 0_{N\times 1} \\ 0_{1\times N} & q_1 & 1 \\ 2\Psi_1^T & -\frac{1}{2}q_2 - \frac{15}{8}q_1^2 & -\frac{3}{2}q_1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0_{N\times N} & \frac{1}{4}\Psi_2 & 0_{N\times 1} \\ -\frac{1}{4}\Psi_2^T & 0 & p_2 \\ 0_{1\times N} & -p_2 & 0 \end{pmatrix}. \tag{2.6}$$

The eigenvalues $\mu_1, \ldots, \mu_{N+2}$ of $\theta_1\theta_0^{-1}$ are defined by the roots of the equation $f(\lambda) = |\lambda I - A_1| = 0$ which gives rise to [4,5]

$$\mu_j = f_j(\Psi_1, q_1, q_2), \quad \psi_{1j} = g_j(\boldsymbol{\mu}), \quad q_1 = g_{N+1}(\boldsymbol{\mu}), \quad q_2 = g_{N+2}(\boldsymbol{\mu}), \tag{2.7a}$$

$$\nu_j = \frac{\partial S}{\partial \mu_j} = \sum_{j=1}^{N} \psi_{2j}\frac{\partial g_j}{\partial \mu_j} + p_1\frac{\partial g_{N+1}}{\partial \mu_j} + p_2\frac{\partial g_{N+2}}{\partial \mu_j}, \qquad j = 1, \ldots, N+2. \tag{2.7b}$$

It is known [11–13] that the separated variables $(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\nu}})$ for (2.4) can be constructed by means of the Lax matrix in the following way:

$$B(\lambda) = \frac{R(\lambda)}{K(\lambda)}, \qquad R(\lambda) = \prod_{k=1}^{N+2}(\lambda - \bar{\mu}_k), \qquad K(\lambda) = \prod_{k=1}^{N}(\lambda - \lambda_k), \tag{2.8a}$$

$$\bar{\nu}_k = -A(\bar{\mu}_k), \qquad k = 1, \ldots, N+2. \tag{2.8b}$$

Proposition 2. *The Nijenhuis coordinates* $(\boldsymbol{\mu}, \boldsymbol{\nu})$ *defined by Eqs. (2.7) are exactly the same as the separated variables* $(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\nu}})$ *defined by Eqs. (2.8).*

3 Infinite numbers of families of QBH systems

Consider the following polynomial second order spectral problem [15]

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U(u, \lambda)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad U(u, \lambda) = \begin{pmatrix} 0 & 1 \\ -\sum_{i=0}^{m} u_i\lambda^i & 0 \end{pmatrix}. \tag{3.1}$$

In the same way, one obtains the first constrained flows associated with spectral problem (3.1) [10]

$$P_x = \theta_0 \bigtriangledown F_1, \tag{3.2}$$

where

$$P = \left(\Psi_1^T, \Psi_2^T\right)^T, \qquad \theta_0 = \begin{pmatrix} 0 & I_{N\times N} \\ -I_{N\times N} & 0 \end{pmatrix},$$

$$F_1 = \frac{1}{2}\langle \Psi_2, \Psi_2\rangle + \sum_{j=0}^{m}\left(-\frac{1}{2}\right)^{j+1} \sum_{l_1+\cdots+l_{j+1}=m-j} \langle \Lambda^{l_1}\Psi_1, \Psi_1\rangle \cdots \langle \Lambda^{l_{j+1}}\Psi_1, \Psi_1\rangle.$$

Proposition 3. *The first family of FDIHS (3.2) ($m = 1, 2, \ldots$) possesses the QBH representation (1.1) with* $\theta_1 = \begin{pmatrix} 0_{N\times N} & A_1 \\ -A_1^T & B_1 \end{pmatrix}$ *and*

$$A_1 = \lambda - \frac{1}{2}\Psi_1\Psi_1^T, \qquad B_1 = \frac{1}{2}\Psi_2\Psi_1^T - \frac{1}{2}\Psi_1\Psi_2^T,$$

$$\rho = B(\lambda)|_{\lambda=0} = 1 - \frac{1}{2}\langle\Lambda^{-1}\Psi_1, \Psi_1\rangle,$$

$$E_1 = \frac{1}{2}\langle\Lambda^{-1}\Psi_2, \Psi_2\rangle + \frac{1}{4}\left[\langle\Lambda^{-1}\Psi_1, \Psi_2\rangle^2 - \langle\Lambda^{-1}\Psi_1, \Psi_1\rangle\langle\Lambda^{-1}\Psi_2, \Psi_2\rangle\right]$$
$$+ \sum_{j=0}^{m}\left(-\frac{1}{2}\right)^{j+1} \sum_{l_1+\cdots+l_{j+1}=m-j} \langle\Lambda^{l_1}\Psi_1, \Psi_1\rangle \cdots \langle\Lambda^{l_{j+1}-1}\Psi_1, \Psi_1\rangle.$$

The second constrained flows associated with Eq. (3.1) is [10]

$$P_x = \theta_0 \bigtriangledown F_1, \tag{3.3}$$

where

$$P = \left(\Psi_1^T, q, \Psi_2^T, p\right)^T, \qquad \theta_0 = \begin{pmatrix} 0 & I_{(N+1)\times(N+1)} \\ -I_{(N+1)\times(N+1)} & 0 \end{pmatrix},$$

$$F_1 = \frac{1}{2}\langle\Psi_2, \Psi_2\rangle + \left(-\frac{1}{2}q\right)^{m+2} - 4p^2$$
$$+ \sum_{i=0}^{m} q^i \sum_{j=1}^{\left[\frac{m+2-i}{2}\right]} \beta_{i,j} \sum_{l_1+\cdots+l_j=m+2-i-2j} \langle\Lambda^{l_1}\Psi_1, \Psi_1\rangle \cdots \langle\Lambda^{l_j}\Psi_1, \Psi_1\rangle.$$

Proposition 4. *The second family of FDIHS (3.3) ($m = 1, 2, \ldots$) possesses the QBH representation (1.1) with*

$$\theta_1 = \begin{pmatrix} 0_{(N+1)\times(N+1)} & A_1 \\ -A_1^T & B_1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \lambda & -\frac{1}{4}\Psi_1 \\ 2\Psi_1^T & -\frac{1}{2}q \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0_{N\times N} & \frac{1}{4}\Psi_2 \\ -\frac{1}{4}\Psi_2^T & 0 \end{pmatrix}, \qquad \rho = B(\lambda)|_{\lambda=0} = \frac{1}{2}q - \frac{1}{2}\langle\Lambda^{-1}\Psi_1, \Psi_1\rangle,$$

$$E_1 = 2p\langle\Lambda^{-1}\Psi_1, \Psi_2\rangle + \frac{1}{4}q\langle\Lambda^{-1}\Psi_2, \Psi_2\rangle + 4p^2 - \left(-\frac{1}{2}q\right)^{m+2}$$
$$- \sum_{i=0}^{m} q^i \sum_{j=1}^{\left[\frac{m+2-i}{2}\right]} \beta_{i,j} \sum_{l_1+\cdots+l_j=m+2-i-2j} \langle\Lambda^{l_1}\Psi_1, \Psi_1\rangle \cdots \langle\Lambda^{l_j}\Psi_1, \Psi_1\rangle$$
$$+ \frac{1}{4}\left[\langle\Lambda^{-1}\Psi_1, \Psi_2\rangle^2 - \langle\Lambda^{-1}\Psi_1, \Psi_1\rangle\langle\Lambda^{-1}\Psi_2, \Psi_2\rangle\right] - \frac{1}{2}\sum_{i=0}^{m+1} q^i \sum_{j=0}^{\left[\frac{m+1-i}{2}\right]} \beta_{i,j}$$
$$\times \sum_{l_1+\cdots+l_{j+1}=m+1-i-2j} \langle\Lambda^{l_1}\Psi_1, \Psi_1\rangle \cdots \langle\Lambda^{l_j}\Psi_1, \Psi_1\rangle\langle\Lambda^{l_{j+1}-1}\Psi_1, \Psi_1\rangle.$$

The separation of variables for FDIHSs (3.2) and (3.3) was studied in [10]. We can show that

Proposition 5. *The Nijenhuis coordinates* $(\boldsymbol{\mu}, \boldsymbol{\nu})$ *for FDIHS (3.2) and (3.3) are exactly the same as the generalized elliptic and the generalized parabolic coordinates* $(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\nu}})$ *defined by Lax matrix for (3.2) and (3.3), respectively.*

In general, we can obtain infinite number of families of QBH systems from the higher-order constrained flows associated with Eq. (3.1).

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