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Peakon-Antipeakon Interaction

R BEALS [†1], *D H SATTINGER* [†2] *and J SZMIGIELSKI* [†3]

[†1] *Yale University, New Haven, CT 06520*

[†2] *University of Minnesota, Minneapolis 55455*

[†3] *University of Saskatchewan, Saskatoon, Saskatchewan, Canada*

Abstract

Explicit formulas are given for the multi-peakon-antipeakon solutions of the Camassa–Holm equation, and a detailed analysis is made of both short-term and long-term aspects of the interaction between a single peakon and single anti-peakon.

The strongly nonlinear equation

$$\left(1-\frac{1}{4}D^2\right)u_t=\frac{3}{2}\left(u^2\right)_x-\frac{1}{8}\left(u_x^2\right)_x-\frac{1}{4}(uu_{xx})_x, \qquad (1)$$

introduced by Camassa and Holm [5] as a possible model for dispersive waves in shallow water, admits solutions that are nonlinear superpositions of travelling waves (peakons) and troughs (antipeakons), having the form

$$u(x,t)=\sum_{j=1}^{n}p_j(t)\exp(-2|x-x_j(t)|), \qquad (2)$$

where $p_j(t)$ and $-\dot{x}_j$ are positive for peakons and negative for antipeakons.

Equation (1) can formally be integrated by the inverse scattering method, using the spectral problem $\left(D^2-z\,m-1\right)\psi=0$, where $m=2\left(1-\frac{1}{4}D^2\right)u$ [5]. It was shown in [1] that the spectral problem may be transformed to the string problem

$$v''(y)-z\,g(y)v(y)=0, \qquad -1<y<1; \qquad v(\pm 1,z)=0, \qquad (3)$$

where $m(x)=g(y)\,\mathrm{sech}^4(x)$, and $y=\tanh x$. The multi-peakons arise when $g(y)=\sum_{j=1}^{n}g_j\delta(y-y_j)$, $-1<y_1<\cdots<y_n<1$. The peakons correspond to $g_j>0$ and the anti-peakons to $g_j<0$.

In a previous note [2] we used a classical result of Stieltjes on continued fractions to obtain explicit formulas for the p_j, x_j in the pure multi-peakon or anti-peakon case; the functions are smooth functions of the scattering data for the related spectral problem. The particles do not collide, but remain separated for all time.

Moser [8] has used Stieltjes' theorem in an analysis of the interaction of particles in the Toda flows. The connection of the classical formulas of Frobenius and Stieltjes with the theta function of the associated singular curve has been discussed by McKean [7].

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In [9] the authors discussed a connection between the peakon sector of the Camassa-Holm equation and the finite Toda lattice thus making the appearance of continued fractions in the peakon sector of the Camassa–Holm equation less surprising. Further details regarding this connection can be found in [4].

The formulas [2] are valid also when both peakons and antipeakons are present, but in this case the particles collide and the associated amplitudes p_j become infinite at the instant of collision. Moreover, a general argument using differential inequalities shows that the slope becomes infinite at the point of collision [6]. On the other hand, the Hamiltonian $\int_{-\infty}^{\infty} u^2 + \frac{1}{4}u_x^2\,dx$ is conserved throughout the interaction [6]; and this shows not only that the solution itself, but also the L^2 norm of the derivative, remains uniformly bounded throughout the interaction.

In this note we use the explicit form of the peakon-antipeakon solution to derive detailed properties of the interaction. A treatment of the general multi-peakon/multi-antipeakon case will be given in a later paper.

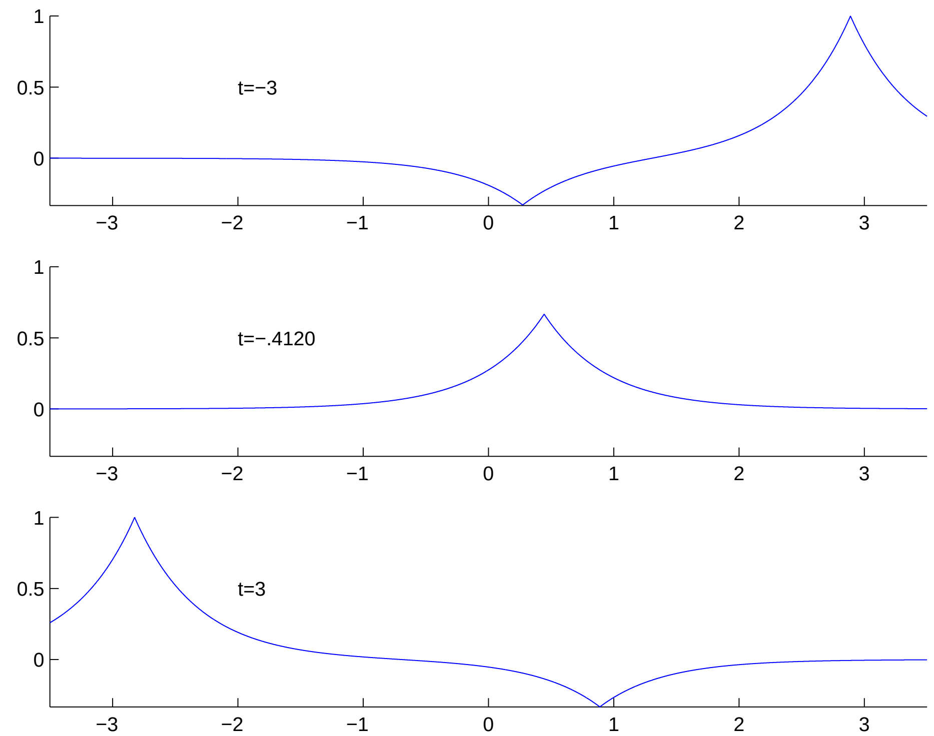

Figure 1. Peakon and anti-peakon interaction: $\lambda_- = -1$, $\lambda_+ = 3$; $a_\pm(0) = .4$.

For the single peakon-antipeakon case, $n = 2$ and $g(y) = g_1\,\delta(y - y_1) + g_2\,\delta(y - y_2)$, where $-1 < y_1 < y_2 < 1$, and $g_1 g_2 < 0$. We begin with the forward problem, using the interpretation of (3) given in [2]. As in [2] we let $l_j = y_{j+1} - y_j$, $j = 0, 1$, where $y_0 = -1$ and $y_3 = 1$.

Let φ and ψ be the solutions of (3) that satisfy the boundary conditions $\varphi(-1, z) = 0$, $\varphi'(-1, z) = 1$, $\psi(1, z) = 0$, and $\psi'(1, z) = -1$. Then $\varphi(1, z)$ is a polynomial of degree two whose roots $\lambda_+ > 0$, $\lambda_- < 0$ are the eigenvalues of (3). The scattering data consist of $\lambda_\pm$ and the associated coupling constants $c_\pm$, defined by $\varphi(z, \lambda_\pm) = c_\pm\psi(z, \lambda_\pm)$.

As in [2],

$$\frac{w(z)}{z} = \frac{1}{2z} + \frac{a_-}{z-\lambda_-} + \frac{a_+}{z-\lambda_+},$$

where w is the Weyl function $\varphi'(1,z)/\varphi(1,z)$, and $a_\pm = c_\pm \lambda_\mp / 2(\lambda_\mp - \lambda_\pm)$. In addition, $w(z)/z$ has a continued fraction expansion with the l_j and g_j as coefficients, and a Laurent expansion

$$\frac{w(z)}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k A_k}{z^{k+1}}, \qquad A_k = \begin{cases} \dfrac{1}{2} + a_- + a_+, & k = 0; \\ (-\lambda_-)^k a_- + (-\lambda_+)^k a_+, & k \geq 1. \end{cases}$$

A theorem of Stieltjes [10] shows that l_j and g_j can be recovered from the scattering data by

$$l_j = \frac{\left(\Delta^1_{n-j}\right)^2}{\Delta^0_{n-j}\Delta^0_{n-j+1}}, \qquad g_j = \frac{\left(\Delta^0_{n-j+1}\right)^2}{\Delta^1_{n-j+1}\Delta^1_{n-j}}. \tag{4}$$

Here $\Delta^0_0 = 1 = \Delta^1_0$ and Δ^0_k, Δ^1_k, $k \geq 1$, are the $k \times k$ minors of the Hankel matrix

$$H = \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & A_2 & A_3 & a_4 \\ A_2 & A_3 & A_4 & A_5 \\ A_3 & A_4 & A_5 & A_6 \end{pmatrix}.$$

whose upper left hand entries are, respectively, A_0 and A_1. It follows recursively from (4) and the positivity of l_2, l_1, l_0 that $\Delta^0_k > 0$. Moreover,

$$\Delta^0_1 = \frac{1}{2} + a_- + a_+ = \frac{1}{l_2} > \frac{1}{2},$$

$$\Delta^0_3 = \frac{1}{2} a_- a_+ \lambda_-^2 \lambda_+^2 (\lambda_+ - \lambda_-)^2 > 0;$$

and it follows that a_- and a_+ are positive.

Given the scattering data $\lambda_\pm$ and $c_\pm$, or equivalently $\lambda_\pm$ and $a_\pm$, we use (4) to recover l_j and g_j. If $a_\pm$ are positive, then calculation shows that the Δ^0_j are positive, $0 \leq j \leq 3$, and Δ^1_2 is negative. Thus the inverse problem has a solution provided $\Delta^1_1 = A_1 = -(\lambda_- a_- + \lambda_+ a_+) \neq 0$. Moreover, $g_1 \, g_2 < 0$, and, from (4), g_2 has the same sign as Δ^1_1. The zero of Δ^1_1 is simple, since $\dot{\Delta}^1_1 = -2\{a_-(0) + a_+(0)\} < 0$.

Under the evolution (1) the scattering data evolve according to $\dot{\lambda}_\pm = 0$, $\dot{a}_\pm = a_\pm / 2\lambda_\pm$. Combining this with the previous discussion we see that for any choice of $a_\pm(0) > 0$ and $\pm\lambda_\pm > 0$, the inverse problem has a (unique) solution except at the unique time t_0 at which Δ^1_1 changes sign.

The solution u of (1) has the same form as the two-peakon solution in [2], so that the asymptotic positions of the peak are

$$\frac{t}{\lambda_-} + \frac{1}{2}\log 2a_-(0), \qquad t \to -\infty;$$

$$\frac{t}{\lambda_-} + \frac{1}{2}\log 2a_-(0) + \log\left(1 - \frac{\lambda_-}{\lambda_+}\right), \qquad t \to +\infty.$$

while the asymptotic positions of the trough are

$$\frac{t}{\lambda_+} + \frac{1}{2}\log 2a_+(0) + \log\left(1 - \frac{\lambda_+}{\lambda_-}\right), \qquad t \to -\infty;$$

$$\frac{t}{\lambda_+} + \frac{1}{2}\log 2a_+(0), \qquad t \to +\infty.$$

Asymptotically, the peak has height $-1/\lambda_-$ and the trough has depth $-1/\lambda_+$.

We may rescale so that Δ_1^1 vanishes at time $t_0 = 0$. The previous formulas imply that the l_j are smooth, l_0 and l_2 are positive, and $y_2 - y_1 = l_1 = O\left(t^2\right)$. Therefore $x_2 - x_1 = O\left(t^2\right)$ and so $\exp(-2|x-x_2|) - \exp(-2|x-x_1|) = O\left(t^2\right)$, uniformly in x. It follows from the formulas in [2] that $P = p_1 + p_2 \equiv -(1/\lambda_- + 1/\lambda_+)$ (conservation of momentum). We write

$$u(x,t) = p_1 \exp(-2|x-x_1|) + p_2 \exp(-2|x-x_2|) = P\exp(-2|x-x_1|) + p_2\, O\left(t^2\right),$$

and deduce that as $t \to 0$,

$$\sup_x |u(x,t) - P\exp(-2|x-x_0|)| = O(|t|).$$

The behavior of the slope during the collision is given by

$$u_x = \begin{cases} -2P\exp(-2|x-x_1|) + O(t), & x < x_1; \\ 4p_2\exp(-2|x-x_1|) + O(1) & x_1 < x < x_2; \\ -2P\exp(-2|x-x_2|) + O(t) & x > x_2. \end{cases}$$

From (15), [2], we find that

$$p_2(t) = -\frac{1}{2}\frac{d}{dt}\log A_1 = -\frac{1}{2t} + O(1), \qquad t \to 0.$$

In the m-peakon/n-antipeakon case, it can be shown that triple collisions are excluded, so that the peakon/anti-peakon interactions occur in single peakon-antipeakon pairs, whose behavior is similar to what we have shown here. Details will be given elsewhere [3].

We close with the formulas for the n-peakon solution, valid independently of the signs of the λ_j. From [2], [10],

$$x_j = \frac{1}{2}\log\frac{2\widetilde{\Delta}^0_{n-j+1}}{\Delta^2_{n-j}}, \qquad p_j = \frac{\widetilde{\Delta}^0_{n-j+1}\Delta^2_{n-j}}{\Delta^1_{n-j+1}\Delta^1_{n-j}}, \tag{5}$$

where Δ_j^2 is the $j \times j$ minor of H beginning with A_2 in the $(1,3)$ slot, and $\widetilde{\Delta}_j^0$ is the $j \times j$ minor of the matrix H in which the $(1,1)$ term has been replaced by $\sum_{j=1}^{n} a_j$.

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