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Bilinear Approach to Supersymmetric KdV Equation

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Abstract

Extending the gauge-invariance principle for τ functions of the standard bilinear formalism to the supersymmetric case, we define $\mathcal{N}=1$ supersymmetric Hirota operators. Using them, we bilinearize SUSY KdV equation. The solution for multiple collisions of super-solitons is given.

1 Introduction

Supersymmetric integrable systems constitute a very interesting subject specially due to their potential applications in matrix models for 2D supergravity. As a consequence a number of well known integrable equations have been generalized into supersymmetric (SUSY) context. We mention the SUSY versions of sine-Gordon [1], nonlinear Schrödinger, KP-hierarchy [2], KdV [2, 3], Boussinesq [4] etc. There are also several algebraic approaches using the representation theory of affine Lie super-algebras in the papers of Kac and van der Leur [5], Kac and Medina [6] the super-conformal field theoretic approach of LeClair [7]. Anyway in these articles the bilinear hierarchies are not related to the SUSY hierarchies of nonlinear equations.

In this paper we consider a direct approach to SUSY KdV equation of Mathieu [3] namely extending the gauge-invariance principle of τ functions for classical Hirota operators. Our result generalize the Grammaticos-Ramani-Hietarinta [9] theorem, to SUSY case and we find $\mathcal{N}=1$ SUSY Hirota bilinear operators. With these operators one can obtain SUSY-bilinear forms for SUSY KdV-type equations [10].

2 Standard bilinear formalism

The Hirota bilinear operators were introduced as an antisymmetric extension of the usual derivative, because of their usefulness for the computation of multisoliton solution of nonlinear evolution equations. The bilinear operator $\mathbf{D}_x = \partial_{x_1} - \partial_{x_2}$, acts on a pair of functions (the so-called "dot product") antisymmetrically:

$$\mathbf{D}_x f \bullet g = (\partial_{x_1} - \partial_{x_2}) f(x_1) f(x_2)|_{x_1=x_2=x} = f'g - fg'. \tag{1}$$

And if we apply to KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2}$$

we obtain after the substitution $u = 2\partial_x^2 \log F$

$$\left(\mathbf{D}_x\mathbf{D}_t + \mathbf{D}_x^4\right) F \bullet F = 0. \tag{3}$$

The power of the bilinear formalism lies in the fact that for multisoliton solution F's are polynomials of exponentials.

A very important observation (which motivated the present paper) is the relation of the physical field $u = 2\partial_x^2 \log F$ of KdV equation with the Hirota function F: the gauge-transformation $F \to e^{px+\omega t}F$ leaves u invariant. This is a general property of all bilinear equations. Moreover, one can define the Hirota operators using the requirement of gauge-invariance. Let's introduce a general bilinear expression,

$$A_N(f, g) = \sum_{i=0}^{N} c_i \left(\partial_x^{N-i} f\right) \left(\partial_x^i g\right) \tag{4}$$

and ask to be invariant under the gauge-trasformation:

$$A_N\left(e^\theta f, e^\theta g\right) = e^{2\theta} A_N(f, g), \qquad \theta = kx+\omega t+\cdots \text{(linears)}, \tag{5}$$

where "linears" means that we have only linear terms in x and t. Anyway they can be absorbed in the definition of k and ω so we can write only $\theta = kx + \omega t$. Then we have the following [9]

Theorem. *$A_N(f, g)$ is gauge-invariant if and only if $A_N(f, g) = \mathbf{D}_x^N f \bullet g$ i.e.*

$$c_i = c_0(-1)^i \begin{pmatrix} N \\ i \end{pmatrix}$$

and c_0 is a constant and the brakets represent binomial coefficient.

3 Supersymmetry

The supersymmetric extension of a nonlinear evolution equation (KdV for instance) refers to a system of coupled equations for a bosonic $u(t, x)$ and a fermionic field $\xi(t, x)$ which reduces to the initial equation in the limit where the fermionic field is zero (bosonic limit). In the classical context, a fermionic field is described by an anticommuting function with values in an *infinitely* generated Grassmann algebra. However, supersymmetry is not just a coupling of a bosonic field to a fermionic field. It implies a transformation (supersymmetry invariance) relating these two fields which leaves the system invariant. In order to have a mathematical formulation of these concepts we have to extend the classical space (x, t) to a larger space (superspace) [3] (t, x, θ) where θ is a Grassmann variable and also to extend the pair of fiels (u, ξ) to a larger fermionic or bosonic superfield $\Phi(t, x, \theta)$. The derivative on the superspace will be defined in the form [3] $D = \partial_\theta + \theta\partial_x$. In order to have nontrivial extension for KdV we choose Φ to be fermionic, having the expansion

$$\Phi(t, x, \theta) = \xi(t, x) + \theta u(t, x). \tag{6}$$

Using the superspace formalism one can construct different supersymmetric extension of nonlinear equations. Thus the integrable (in the sense of Lax pair) variant of $\mathcal{N} = 1$ SUSY KdV is [2, 3]

$$\Phi_t + D^6\Phi + 3D^2(\Phi D\Phi) = 0, \tag{7}$$

which on the components has the form (notice that $D^2 = \partial_x$)

$$u_t = u_{xxx} - 6uu_x + 3\xi\xi_{xx},\\ \xi_t = -\xi_{xxx} - 3\xi_x u - 3\xi u_x. \tag{8}$$

If we choose $\Phi(x,t)$ to be a bosonic field i.e. to have the expansion

$$\Phi(t,x,\theta) = u(t,x) + \theta\xi(t,x). \tag{9}$$

then we have the following supersymmetric extension of the KdV equation [3]:

$$\Phi_t + D^6\Phi + 6\Phi D^2\Phi = 0, \tag{10}$$

or equivalently

$$u_t = -u_{xxx} - 6uu_x,\\ \xi_t = -\xi_{xxx} - 6(\xi u)_x. \tag{11}$$

This system is trivial in the sense that is linear in the fermionic field and, as a result, one of the two equation is simply the KdV equation itself.

4 Super-Hirota operators

We are going to consider the following general $\mathcal{N} = 1$ SUSY bilinear expression

$$S_N(f,g) = \sum_{i=0}^{N} c_i \left(D^{N-i} f\right)\left(D^i g\right), \tag{12}$$

where D is the covariant derivative and f, g are Grassmann valued functions (odd or even). We were able to prove the following

Theorem. *The general $\mathcal{N} = 1$ SUSY bilinear expression (12) is super-gauge invariant i.e. for $\Theta = kx + \omega t + \theta\hat{\zeta}$ (ζ is a Grassmann parameter)*

$$S_N\left(e^{\Theta} f, e^{\Theta} g\right) = e^{2\Theta} S_N(f,g),$$

if and only if

$$c_i = c_0(-1)^{i|f| + \frac{i(i+1)}{2}} \begin{bmatrix} N \\ i \end{bmatrix},$$

where the super-binomial coefficients are defined by:

$$\begin{bmatrix} N \\ i \end{bmatrix} = \begin{cases} \begin{pmatrix} [N/2] \\ [i/2] \end{pmatrix} & \text{if } (N,i) \neq (0,1) \bmod 2; \\ 0 & \text{otherwise}, \end{cases}$$

$|f|$ is the Grassmann parity of the function f defined by:

$$|f| = \begin{cases} 1 & \text{if } f \text{ is odd (fermionic)}; \\ 0 & \text{if } f \text{ is even (bosonic)} \end{cases}$$

and $[k]$ is the integer part of the real number k ($[k] \le k < [k]+1$).

We mention that the super-bilinear operator proposed by McArthur and Yung [8] is a particular case of the above super-Hirota operator. We shall note the bilinear operator as

$$S_N(f,g) := \mathbf{S}_x^N f \bullet g.$$

5 Bilinear SUSY KdV equation

In order to use the super-bilinear operators defined above we shall consider the following nonlinear substitution for the supersymmetric KdV superfield:

$$\Phi(t,x,\theta) = 2D^3 \log \tau(t,x,\theta), \tag{13}$$

where $\tau(t,x,\theta)$ is a bosonic superfield. With this substitution, (7) is transformed into the following super-bilinear form:

$$\left(\mathbf{S}_x \mathbf{D}_t + \mathbf{S}_x^7\right) \tau \bullet \tau = 0. \tag{14}$$

The 1 super-soliton solution has the following structure

$$\tau^{(1)} = 1 + e^{kx - k^3 t + \theta\hat{\zeta} + \eta^{(0)}}. \tag{15}$$

In order to find 2 super-soliton solution we are going to consider the form

$$\tau^{(2)} = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}} \tag{16}$$

and we have to find the factor $\exp A_{12}$, where $\eta_i = k_i x - k_i^3 t + \theta\hat{\zeta}_i + \eta_i^{(0)}$. The equation for $\exp A_{12}$ is the following:

$$\left[\left(\hat{\zeta}_1 - \hat{\zeta}_2\right) + \theta(k_1 - k_2)\right](k_1 - k_2) = \exp A_{12}\left[\left(\hat{\zeta}_1 + \hat{\zeta}_2\right) + \theta(k_1 + k_2)\right](k_1 + k_2). \tag{17}$$

We assume that $\exp A_{12}$ depends only on k_i, $\hat{\zeta}_i$, with $i = 1, 2$ and in the bosonic limit ($\hat{\zeta}_i = 0$) to have the standard form, $(k_1 - k_2)^2/(k_1 + k_2)^2$. Accordingly, in order to solve (17) we consider the ansatz:

$$\exp A_{12} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 + \hat{a}(k_1,k_2)\hat{\zeta}_1 + \hat{b}(k_1,k_2)\hat{\zeta}_2 + \gamma(k_1,k_2)\hat{\zeta}_1\hat{\zeta}_2, \tag{18}$$

where $\hat{a}$, $\hat{b}$ are odd Grassmann functions depending on k_1 and k_2 and γ is an even Grassmann function. Introducing (18) in (17) we shall find that

$$\hat{a}(k_1,k_2)\hat{\zeta}_1 + \hat{b}(k_1,k_2)\hat{\zeta}_2 + \gamma(k_1,k_2)\hat{\zeta}_1\hat{\zeta}_2 = 0$$

and

$$k_1\hat{\zeta}_2 = k_2\hat{\zeta}_1.$$

So, the interaction effect remains the same as in the bosonic case. One can easily verify that the N super-soliton solution is given by

$$\tau^{(N)} = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j} A_{ij}\mu_i\mu_j\right), \tag{19}$$

where

$$\eta_i = k_i x - k_i^3 t + \theta\hat{\zeta}_i + \eta_i^{(0)}, \qquad \exp A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2, \qquad k_i\hat{\zeta}_j = k_j\hat{\zeta}_i.$$

One can see that for $\hat{\zeta}_i = 0$, $\tau^{(N)}$ becomes a real function being exactly the N-soliton solution of the ordinary KdV equation.

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