Representations of the Conformal Lie Algebra in the Space of Tensor Densities on the Sphere

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Received June 11, 2002; Accepted August 2, 2002

Abstract

Let $\mathcal{F}_\lambda(S^n)$ be the space of tensor densities on $S^n$ of degree $\lambda$. We consider this space as an induced module of the nonunitary spherical series of the group $SO_0(n+1,1)$ and classify $(so(n+1,1), SO(n+1))$-simple and unitary submodules of $\mathcal{F}_\lambda(S^n)$ as a function of $\lambda$.

1 Introduction and main result

Let $\mathcal{F}_\lambda(S^n)$ be the space of tensor densities of degree $\lambda \in \mathbb{C}$ on the sphere $S^n$, that is, of smooth sections of the line bundle

$$\Delta_\lambda(S^n) = |\Lambda^n T^* S^n|^{\otimes \lambda}$$

on $S^n$. This space plays an important rôle in geometric quantization and, more recently, it has also been used in equivariant quantization (see [1]). This space is endowed with a structure of $\text{Diff}(S^n)$- and $\text{Vect}(S^n)$-module in the following way. As a vector space, it is isomorphic to the space $C^\infty_c(S^n)$ of smooth complex-valued functions; the action of a vector field

$$Y = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i}$$

is given by the Lie derivative of degree $\lambda$

$$L_Y^\lambda(\varphi(x_1,\ldots,x_n)) = \sum_{i=1}^{n} \left( Y_i \frac{\partial \varphi}{\partial x_i} + \lambda \frac{\partial Y_i}{\partial x_i} \varphi \right)(x_1,\ldots,x_n)$$

(1.1)

in any coordinate system.

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The Lie algebra so\((n + 1, 1)\) \(\subset\) Vect\((S^n)\) of infinitesimal conformal transformations, that we call the conformal Lie algebra, is generated by the vector fields

\[
X_i = \frac{\partial}{\partial s_i}, \quad X_{ij} = s_i \frac{\partial}{\partial s_j} - s_j \frac{\partial}{\partial s_i},
\]

\[
X_0 = \sum_i s_i \frac{\partial}{\partial s_i}, \quad \bar{X}_i = \sum_j \left( s_j^2 \frac{\partial}{\partial s_i} - 2s_i s_j \frac{\partial}{\partial s_j} \right),
\]

where \((s_1, \ldots, s_n)\) are stereographic coordinates on the sphere \(S^n\).

The space \(\mathcal{F}_\lambda(S^n)\) is naturally an \(\text{so}(n + 1, 1)\)-module; furthermore, the restriction of the action of the group \(\text{Diff}(S^n)\), defines the action of the subgroup \(\text{SO}(n + 1)\) given by the formula

\[
(k_0 f)(k) = f(k_0^{-1} k), \quad \text{where} \quad k_0 \in K, \ k \in S^n \simeq \text{SO}(n + 1)/\text{SO}(n).
\]

Therefore, \(\mathcal{F}_\lambda(S^n)\) is also a \(\text{SO}(n + 1)\)-module.

Given a Lie group \(G\) and a compact subgroup \(K \subset G\), let \(g\) and \(\mathfrak{k}\) be the corresponding Lie algebras. One calls \((g, K)\)-module a complex vector space \(E\) endowed with actions of \(g\) and \(K\) such that

1. \((\text{Ad}k \cdot X) \cdot e = k \cdot X \cdot k^{-1} \cdot e \quad \forall k \in K, \ X \in g, \ e \in E\)
2. For all \(e \in E\), the space \(K \cdot e\) is finite-dimensional (i.e., \(e\) is a \(K\)-finite vector), the representation of \(K\) in \(F\) is continuous and one has for \(X \in \mathfrak{k}\):

\[
X \cdot e = \frac{d}{dt} (\exp t X) \cdot e|_{t=0}.
\]

Put \(G = \text{SO}_0(n + 1, 1)\), the connected component of the identity in \(\text{SO}(n + 1, 1)\), \(g = \text{so}(n + 1, 1)\) and \(K = \text{SO}(n + 1)\); let \(\mathcal{H}(K)\) be the space of \(K\)-finite vectors in \(\mathcal{F}_\lambda(S^n)\). The main result of this note is a classification of simple and unitary \((g, K)\)-submodules of \(\mathcal{H}(K)\) as a function of \(\lambda\).

**Theorem 1.**

1. If \(\lambda \neq l/n\) for \(l \in \mathbb{Z}\), or if, for \(n > 1\), \(\lambda \in \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\}\), then \(\mathcal{F}_\lambda(S^n)\) contains a unique simple \((g, K)\)-module \(\mathcal{H}(K)\), identified to the space of harmonic polynomials on \(S^n\). This module is unitary if and only if \(\lambda = \frac{1}{2} + i \alpha, \ \alpha \in \mathbb{R}^*, \) or \(\lambda \in [0, 1]\setminus \{\frac{1}{2}\}\).
2. If \(\lambda = -l/n, \ l \in \mathbb{N}\), \(\mathcal{H}(K)\) contains a unique simple \((g, K)\)-submodule, which is finite-dimensional and given by the elements of degree \(\leq 1\). It is unitary if and only if \(\lambda = 0\).
3. If \(n = 1\) and \(\lambda = l, \ l \in \mathbb{N}^*\), \(\mathcal{H}(K)\) contains two simple \((g, K)\)-submodules, unitary and infinite-dimensional, and the direct sum of these modules consists of the elements of \(\mathcal{H}(K)\) of degree \(\geq l\).
4. If \(n > 1\) and \(\lambda = 1 + l/n, \ l \in \mathbb{N}\), \(\mathcal{H}(K)\) contains a simple infinite-dimensional \((g, K)\)-submodule consisting of the elements with degree \(\geq l + 1\). It is unitary if and only if \(\lambda = 1\).

**Remark 1.** We described all the closed \(G\)-submodules of \(\mathcal{F}_\lambda(S^n)\) (cf. [3], Theorem 8.9), and, since \(G\) is connected, we obtained, in the case (2), every simple finite-dimensional \(g\)-submodules of \(\mathcal{F}_\lambda(S^n)\).
2 Nonunitary spherical series

The main ingredient of the proof of Theorem 1 is the identification of the modules $\mathcal{F}_\lambda(S^n)$ with induced representations. Denote $G = KAN$ the Iwasawa decomposition of $G$ and $\rho$ the half-sum of the positive restricted roots of the pair $(\text{so}(n+1,1), A)$. Let $A = \exp \lambda$.

Consider the representation $\text{Ind}^G_{MAN}(0 \otimes \nu)$, induced from the minimal parabolic subgroup $MAN$ of $G$, with the trivial representation of the subgroup $M = \text{SO}(n)$ (the centralizer of $A$ in $K$) and a one-dimensional representation $\mu$ of $A$ such that, for $h \in A$, one has $\mu(h) = \exp(\nu(\log h))$, with a fixed $\nu \in \mathfrak{a}^*$. Abusing the notations, we identify an element $\nu$ in $\mathfrak{a}^*$ with $\nu(H)$, where $H$ is the matricial element $H = E_{n+1,n+2} + E_{n+2,n+1}$ (with elementary matrices $E_{ij}$).

Therefore, $\rho = \frac{n}{2}$.

The Iwasawa decomposition shows that this induced representation acts on the space of functions in $L^2(K/M) = L^2(S^n)$, and the operators of this representation are given, for $g \in G$, by

$$\text{Ind}^G_{MAN}(0 \otimes \nu)(g)f(k) = \exp(-\nu(\log h))f(kg), \quad \text{with} \quad g^{-1}k = kghn \in KAN.$$ 

Considering every value of $\nu$ in $\mathbb{C}$, we obtain the representations of the so-called nonunitary spherical series, that defines a structure of $G$-module on the space $L^2(S^n)$. We denote by $C^\infty_\nu(S^n)$ the submodule constituted of $C^\infty$ elements.

Our proof is based on the following fact.

**Theorem 2.** The $\mathfrak{g}$-modules $\mathcal{F}_\lambda(S^n)$ and $C^\infty_\nu(S^n)$ are isomorphic if and only if $\nu = n\lambda$, and this isomorphism is compatible with the action of $K$.

Let us give the main idea of the proof of Theorem 2. Denote by $d\text{Ind}^G_{MAN}(0 \otimes \nu)$ the infinitesimal representation associated with $\text{Ind}^G_{MAN}(0 \otimes \nu)$, $L_X$ the Lie derivative along $X \in \mathfrak{g}$, and $(\theta_1, \ldots, \theta_n)$ the spherical coordinates on $S^n$. Straightforward but complicated computations lead to the following two facts, that use cohomological (elementary) notions.

**Lemma 1.** For all $X \in \mathfrak{g}$ one has

$$d\text{Ind}^G_{MAN}(0 \otimes \nu)(X) = L_X + \nu c(X),$$

where $c$ is the 1-cocycle on $\text{so}(n+1,1)$ with coefficients in $C^\infty(\mathbb{R}^n)$ given, in spherical coordinates, by $c(X) = \frac{\partial X^n}{\partial \theta_n}$.

It is known that the cohomology space $H^1(\text{so}(n+1,1); C^\infty(\mathbb{R}^n))$ is one-dimensional. We then have the following

**Lemma 2.** The cocycle $c$ is cohomological to the cocycle $\tilde{c}$ given in spherical coordinates by

$$\tilde{c}(X) = \frac{1}{n} \text{Div } X.$$ 

We now use the fact that two representations that are given by $L_X + c(X)$ and $L_X + \tilde{c}(X)$ are equivalent if the cocycles $c$ and $\tilde{c}$ belong to the same cohomology class. Theorem 2 is proved.

As a consequence, $(\mathfrak{g}, K)$-modules of $K$-finite vectors in $\mathcal{F}_\lambda(S^n)$ and $C^\infty_\nu(S^n)$ are isomorphic.
3 Classification of \((\mathfrak{g}, K)\)-modules in \(\mathcal{F}_\lambda(S^n)\)

Let us now use the results (and the notations) of [2] (see Appendix B.10). Let us put 
\[ k = \left\lceil \frac{a+1}{2} \right\rceil \] (where \(p\) is the integral part of \(p\)), and denote by \(D^{m_1, \ldots, m_k}\) the simple \(K\)-module with highest weight \(m_k \varepsilon_1 + m_{k-1} \varepsilon_2 + \cdots + m_1 \varepsilon_k\), where

\[ \varepsilon_i(\lambda_1 H_1 + \cdots + \lambda_k H_k) = \lambda_i, \]

and the matricial \(H_r = i(E_{2r-1,2r} - E_{2r,2r-1})\) generate a Cartan subalgebra of the Lie algebra \(\mathfrak{k}\).

Consider the representation \(\text{Ind}_{\mathfrak{k}}^{\mathfrak{g}} \otimes \nu + \rho\) (which is unitary if and only if \(\nu\) is pure imaginary), and describe the \((\mathfrak{g}, K)\)-module \(E_{0,\nu}\) of its \(K\)-finite vectors: the restriction of the latter to \(K\) is given by the direct sum of simple \(K\)-modules

\[ E_{0,\nu}|_K = \bigoplus D^{0,\ldots,0,m}, \quad m \in \mathbb{N} \text{ for } n > 1; \]
\[ m \in \mathbb{Z} \text{ for } n = 1. \]

We use the isomorphism \(E_{0,-\nu} \cong E_{0,\nu}^*\) (\(K\)-finite dual).

The module \(E_{0,\nu}\) is unitary if and only if \(\nu\) is pure imaginary, or \(\nu \in \left] \frac{-n}{2}, \frac{n}{2} \right[ \setminus \{0\}\).

In order to study simple \((\mathfrak{g}, K)\)-submodules of \(E_{0,\nu}\), we have to consider the following two cases.

- If \(n = 1\), then the module \(E_{0,\nu}\) is simple if and only if \(\nu \not\in \frac{1}{2} + \mathbb{Z}\).

Otherwise, we have:

- If \(\nu < 0\), \(E_{0,\nu}\) contains a unique simple \((\mathfrak{g}, K)\)-submodule. It is finite-dimensional and given, as a \(K\)-module, by \(\bigoplus_{|m| \leq |\nu|-\frac{1}{2}} D^m\). This module is unitary for \(\nu = \frac{1}{2}\).

- If \(\nu > 0\), \(E_{0,\nu}\) contains two simple infinite-dimensional \((\mathfrak{g}, K)\)-submodules, given as \(K\)-modules by \(\bigoplus_{\nu+\frac{1}{2} \leq m} D^m\). These modules are unitary.

- If \(n > 1\), then the module \(E_{0,\nu}\) contains a simple submodule if and only if \(\nu = \pm \left( \frac{n}{2}, \frac{n}{2} + 1, \ldots \right)\). In this case, there exists a simple finite-dimensional \((\mathfrak{g}, K)\)-module given, as a \(K\)-module, by \(\bigoplus_{m \leq |\nu|-\frac{n}{2}} D^{0,\ldots,0,m}\). This is a \((\mathfrak{g}, K)\)-submodule of \(E_{0,\nu}\) if \(\nu < 0\), and a quotient-module if \(\nu > 0\). It is unitary for \(\nu = -\frac{n}{2}\).

Consider the space of smooth functions on \(\mathbb{R}^{n+1} \setminus \{0\}\), homogeneous of degree \(-\lambda(n+1)\). This space is a \(\text{Diff}(S^n)\)-module (and also a \(\text{Vect}(S^n)\)-module with the Lie derivative) isomorphic to the module \(\mathcal{F}_\lambda(S^n)\) (see [5]). Denote by \(\mathcal{H}^{n+1,m}\) the \(K\)-module constituted of its elements of the form

\[ \frac{P_m(x_0, \ldots, x_n)}{(x_0^2 + \cdots + x_n^2)^{\frac{n}{2} + \lambda + \frac{1}{2}}}, \]

where \(P_m\) is a harmonic polynomial homogeneous of degree \(m\). We, finally, check the following facts:
• If \( n = 1 \), then \( \mathcal{H}^{2,m} \cong D^{-m} \oplus D^{m} \). Indeed \( \mathcal{H}^{2,m} \) is the direct sum of SO(2)-modules \( H_{m} \) and \( H_{-m} \), respectively generated by

\[
(x_0 + ix_1)^m \left( x_0^2 + x_1^2 \right)^{-\frac{m}{2}-\lambda}
\]

and its conjugate in \( \mathbb{C} \), and we have \( H_{\pm m} \cong D^{\pm m} \).

• If \( n > 1 \), then \( \mathcal{H}^{n+1,m} \) is simple and we have \( \mathcal{H}^{n+1,m} \cong D^{0,\ldots,0,m} \).

Consequently, the \((\mathfrak{g}, K)\)-module of \( K \)-finite vectors of \( \mathcal{F}_\lambda(S^n) \) is given by

\[
\mathcal{H}(K) \cong \bigoplus_{m \in \mathbb{N}} \mathcal{H}^{n+1,m}.
\]

Let us apply the above results to the representation \( \text{Ind}_{GAN}^{G}(0 \otimes \nu) \). Substituting \( \nu + \rho = n\lambda \) to Theorem 2, we obtain the assertions of Theorem 1.

**Remark 2.** The case \( n = 1 \) can be directly deduced from the classification of representations of \( \text{SL}(2, \mathbb{R}) \): acting the same way as in [4], we observe that the space of \( K \)-finite vectors of \( \mathcal{F}_\lambda(S^1) \) is the direct sum \( \bigoplus H_{2l}, \ l \in \mathbb{Z} \), where \( H_{m} \) is the space of the representation of \( \text{SO}(2) \subset \text{SL}(2, \mathbb{R}) \) with the character

\[
\chi_m : \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \mapsto e^{im\theta}.
\]

**Acknowledgments**

We are grateful to Valentin Ovsienko, Thierry Levasseur and Alain Guichardet, so as to Ranee Brylinski, Patrick Delorme and Pierre Lecomte.

**References**


