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Representations of the Conformal Lie Algebra in the Space of Tensor Densities on the Sphere

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Abstract

Let $\mathcal{F}_\lambda(\mathbb{S}^n)$ be the space of tensor densities on $\mathbb{S}^n$ of degree λ. We consider this space as an induced module of the nonunitary spherical series of the group $\mathrm{SO}_0(n+1,1)$ and classify $(\mathrm{so}(n+1,1), \mathrm{SO}(n+1))$-simple and unitary submodules of $\mathcal{F}_\lambda(\mathbb{S}^n)$ as a function of λ.

1 Introduction and main result

Let $\mathcal{F}_\lambda(\mathbb{S}^n)$ be the space of tensor densities of degree $\lambda \in \mathbb{C}$ on the sphere $\mathbb{S}^n$, that is, of smooth sections of the line bundle

$$\Delta_\lambda(\mathbb{S}^n) = |\Lambda^n T^*\mathbb{S}^n|^{\otimes\lambda}$$

on $\mathbb{S}^n$. This space plays an important rôle in geometric quantization and, more recently, it has also been used in equivariant quantization (see [1]). This space is endowed with a structure of $\mathrm{Diff}(\mathbb{S}^n)$- and $\mathrm{Vect}(\mathbb{S}^n)$-module in the following way. As a vector space, it is isomorphic to the space $\mathcal{C}^\infty_{\mathbb{C}}(\mathbb{S}^n)$ of smooth complex-valued functions; the action of a vector field

$$Y = \sum_{i=1}^n Y_i \frac{\partial}{\partial x_i}$$

is given by the Lie derivative of degree λ

$$L^\lambda_Y(\varphi(x_1, \ldots, x_n)) = \sum_{i=1}^n \left(Y_i \frac{\partial\varphi}{\partial x_i} + \lambda \frac{\partial Y_i}{\partial x_i} \varphi \right)(x_1, \ldots, x_n) \tag{1.1}$$

in any coordinate system.

The Lie algebra $\mathrm{so}(n+1,1) \subset \mathrm{Vect}(\mathbb{S}^n)$ of infinitesimal conformal transformations, that we call the conformal Lie algebra, is generated by the vector fields

$$X_i = \frac{\partial}{\partial s_i}, \qquad X_{ij} = s_i \frac{\partial}{\partial s_j} - s_j \frac{\partial}{\partial s_i},$$
$$X_0 = \sum_i s_i \frac{\partial}{\partial s_i}, \qquad \bar{X}_i = \sum_j \left(s_j^2 \frac{\partial}{\partial s_i} - 2 s_i s_j \frac{\partial}{\partial s_j} \right), \tag{1.2}$$

where $(s_1, \ldots, s_n)$ are stereographic coordinates on the sphere $\mathbb{S}^n$.

The space $\mathcal{F}_\lambda(\mathbb{S}^n)$ is naturally an $\mathrm{so}(n+1,1)$-module; furthermore, the restriction of the action of the group $\mathrm{Diff}(\mathbb{S}^n)$, defines the action of the subgroup $\mathrm{SO}(n+1)$ given by the formula

$$(k_0.f)(k) = f(k_0^{-1}k), \qquad \text{where} \quad k_0 \in K, \quad k \in \mathbb{S}^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n).$$

Therefore, $\mathcal{F}_\lambda(\mathbb{S}^n)$ is also a $\mathrm{SO}(n+1)$-module.

Given a Lie group G and a compact subgroup $K \subset G$, let $\mathfrak{g}$ and $\mathfrak{k}$ be the corresponding Lie algebras. One calls $(\mathfrak{g}, K)$-module a complex vector space E endowed with actions of $\mathfrak{g}$ and K such that

1. $(\mathrm{Ad}k \cdot X) \cdot e = k \cdot X \cdot k^{-1} \cdot e \qquad \forall\, k \in K, \quad X \in \mathfrak{g}, \quad e \in E$
2. For all $e \in E$, the space $K \cdot e$ is finite-dimensional (i.e., e is a K-finite vector), the representation of K in F is continuous and one has for $X \in \mathfrak{k}$:
$$X \cdot e = \frac{d}{dt}(\exp tX) \cdot e|_{t=0}.$$

Put $G = \mathrm{SO}_0(n+1,1)$, the connected component of the identity in $\mathrm{SO}(n+1,1)$, $\mathfrak{g} = \mathrm{so}(n+1,1)$ and $K = \mathrm{SO}(n+1)$; let $\mathcal{H}(K)$ be the space of K-finite vectors in $\mathcal{F}_\lambda(\mathbb{S}^n)$. The main result of this note is a classification of simple and unitary $(\mathfrak{g}, K)$-submodules of $\mathcal{H}(K)$ as a function of λ.

Theorem 1. *1. If $\lambda \neq l/n$ for $l \in \mathbb{Z}$, or if, for $n > 1$, $\lambda \in \left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right\}$, then $\mathcal{F}_\lambda(\mathbb{S}^n)$ contains a unique simple $(\mathfrak{g}, K)$-module $\mathcal{H}(K)$, identified to the space of harmonic polynomials on $\mathbb{S}^n$. This module is unitary if and only if $\lambda = \frac{1}{2} + i\alpha$, $\alpha \in \mathbb{R}^*$, or $\lambda \in]0,1[\setminus \left\{\frac{1}{2}\right\}$.*

2. If $\lambda = -l/n$, $l \in \mathbb{N}$, $\mathcal{H}(K)$ contains a unique simple $(\mathfrak{g}, K)$-submodule, which is finite-dimensional and given by the elements of degree $\leq l$. It is unitary if and only if $\lambda = 0$.

3. If $n = 1$ and $\lambda = l$, $l \in \mathbb{N}^*$, $\mathcal{H}(K)$ contains two simple $(\mathfrak{g}, K)$-submodules, unitary and infinite-dimensional, and the direct sum of these modules consists of the elements of $\mathcal{H}(K)$ of degree $\geq l$.

4. If $n > 1$ and $\lambda = 1 + l/n$, $l \in \mathbb{N}$, $\mathcal{H}(K)$ contains a simple infinite-dimensional $(\mathfrak{g}, K)$-submodule consisting of the elements with degree $\geq l+1$. It is unitary if and only if $\lambda = 1$.

Remark 1. We described all the closed G-submodules of $\mathcal{F}_\lambda(\mathbb{S}^n)$ (cf. [3], Theorem 8.9), and, since G is connected, we obtained, in the case (2), every simple finite-dimensional $\mathfrak{g}$-submodules of $\mathcal{F}_\lambda(\mathbb{S}^n)$.

2 Nonunitary spherical series

The main ingredient of the proof of Theorem 1 is the identification of the modules $\mathcal{F}_\lambda(\mathbb{S}^n)$ with induced representations. Denote $G = KAN$ the Iwasawa decomposition of G and ρ the half-sum of the positive restricted roots of the pair $(\mathrm{so}(n+1,1), \mathfrak{a})$, $A = \exp \mathfrak{a}$.

Consider the representation $\mathrm{Ind}_{MAN}^{G}(0 \otimes \nu)$, induced from the minimal parabolic subgroup MAN of G, with the trivial representation of the subgroup $M = \mathrm{SO}(n)$ (the centralizer of A in K) and a one-dimensional representation μ of A such that, for $h \in A$, one has $\mu(h) = \exp(\nu(\log h))$, with a fixed $\nu \in \mathfrak{a}^*$. Abusing the notations, we identify an element ν in $\mathfrak{a}^*$ with $\nu(H)$, where H is the matricial element

$$H = \mathrm{E}_{n+1,n+2} + \mathrm{E}_{n+2,n+1} \quad \text{(with elementary matrices } \mathrm{E}_{ij}).$$

Therefore, $\rho = \frac{n}{2}$.

The Iwasawa decomposition shows that this induced representation acts on the space of functions in $\mathcal{L}^2(K/M) = \mathcal{L}^2(\mathbb{S}^n)$, and the operators of this representation are given, for $g \in G$, by

$$\mathrm{Ind}_{MAN}^{G}(0 \otimes \nu)(g) f(k) = \exp(-\nu(\log h)) f(k_g), \qquad \text{with} \quad g^{-1}k = k_g h n \in KAN.$$

Considering every value of ν in $\mathbb{C}$, we obtain the representations of the so-called *nonunitary spherical series*, that defines a structure of G-module on the space $\mathcal{L}^2(\mathbb{S}^n)$. We denote by $\mathcal{C}_\nu^\infty(\mathbb{S}^n)$ the submodule constituted of $\mathcal{C}^\infty$ elements.

Our proof is based on the following fact.

Theorem 2. *The $\mathfrak{g}$-modules $\mathcal{F}_\lambda(\mathbb{S}^n)$ and $\mathcal{C}_\nu^\infty(\mathbb{S}^n)$ are isomorphic if and only if $\nu = n\lambda$, and this isomorphism is compatible with the action of K.*

Let us give the main idea of the proof of Theorem 2. Denote by $d\,\mathrm{Ind}_{MAN}^{G}(0 \otimes \nu)$ the infinitesimal representation associated with $\mathrm{Ind}_{MAN}^{G}(0 \otimes \nu)$, L_X the Lie derivative along $X \in \mathfrak{g}$, and $(\theta_1, \dots, \theta_n)$ the spherical coordinates on $\mathbb{S}^n$. Straightforward but complicated computations lead to the following two facts, that use cohomological (elementary) notions.

Lemma 1. *For all $X \in \mathfrak{g}$ one has*

$$d\,\mathrm{Ind}_{MAN}^{G}(0 \otimes \nu)(X) = L_X + \nu\, c(X),$$

where c is the 1-cocycle on $\mathrm{so}(n+1,1)$ *with coefficients in $\mathcal{C}^\infty(\mathbb{R}^n)$ given, in spherical coordinates, by $c(X) = \frac{\partial X^n}{\partial \theta_n}$.*

It is known that the cohomology space $H^1(\mathrm{so}(n+1,1); \mathcal{C}^\infty(\mathbb{R}^n))$ is one-dimensional. We then have the following

Lemma 2. *The cocycle c is cohomological to the cocycle $\tilde{c}$ given in spherical coordinates by*

$$\tilde{c}(X) = \frac{1}{n}\,\mathrm{Div}\, X.$$

We now use the fact that two representations that are given by $L_X + c(X)$ and $L_X + \tilde{c}(X)$ are equivalent if the cocycles c and $\tilde{c}$ belong to the same cohomology class. Theorem 2 is proved.

As a consequence, $(\mathfrak{g}, K)$-modules of K-finite vectors in $\mathcal{F}_\lambda(\mathbb{S}^n)$ and $\mathcal{C}_\nu^\infty(\mathbb{S}^n)$ are isomorphic.

3 Classification of $(\mathfrak{g}, K)$-modules in $\mathcal{F}_\lambda(\mathbb{S}^n)$

Let us now use the results (and the notations) of [2] (see Appendix B.10). Let us put $k = \left[\frac{n+1}{2}\right]$ (where $[p]$ is the integral part of p), and denote by $D^{m_1,\ldots,m_k}$ the simple K-module with highest weight $m_k\varepsilon_1 + m_{k-1}\varepsilon_2 + \cdots + m_1\varepsilon_k$, where

$$\varepsilon_i(\lambda_1 H_1 + \cdots + \lambda_k H_k) = \lambda_i,$$

and the matricial $H_r = i(\mathrm{E}_{2r-1,2r} - \mathrm{E}_{2r,2r-1})$ generate a Cartan subalgebra of the Lie algebra $\mathfrak{k}$.

Consider the representation $\mathrm{Ind}_{MAN}^{G}(0 \otimes \nu + \rho)$ (which is unitary if and only if ν is pure imaginary), and describe the $(\mathfrak{g}, K)$-module $E_{0,\nu}$ of its K-finite vectors : the restriction of the latter to K is given by the direct sum of simple K-modules

$$E_{0,\nu}|_K = \bigoplus D^{0,\ldots,0,m}, \quad \left| \begin{array}{ll} m \in \mathbb{N} & \text{for} \quad n > 1; \\ m \in \mathbb{Z} & \text{for} \quad n = 1. \end{array} \right.$$

We use the isomorphism $E_{0,-\nu} \cong E_{0,\nu}^*$ (K-finite dual).

The module $E_{0,\nu}$ is unitary if and only if ν is pure imaginary, or $\nu \in \left]-\frac{n}{2}, \frac{n}{2}\right[\setminus \{0\}$.

In order to study simple $(\mathfrak{g}, K)$-submodules of $E_{0,\nu}$, we have to consider the following two cases.

- If $n = 1$, then the module $E_{0,\nu}$ is simple if and only if $\nu \notin \frac{1}{2} + \mathbb{Z}$.

  Otherwise, we have:

  - If $\nu < 0$, $E_{0,\nu}$ contains a unique simple$(\mathfrak{g}, K)$-submodule. It is finite-dimensional and given, as a K-module, by $\bigoplus_{|m| \le |\nu| - \frac{1}{2}} D^m$. This module is unitary for $\nu = -\frac{1}{2}$.
  - If $\nu > 0$, $E_{0,\nu}$ contains two simple infinite-dimensional $(\mathfrak{g}, K)$-submodules, given as K-modules by $\bigoplus_{\nu + \frac{1}{2} \le \pm m} D^m$. These modules are unitary.

- If $n > 1$, then the module $E_{0,\nu}$ contains a simple submodule if and only if $\nu = \pm\left(\frac{n}{2}, \frac{n}{2} + 1, \ldots\right)$. In this case, there exists a simple finite-dimensional $(\mathfrak{g}, K)$-module given, as a K-module, by $\bigoplus_{m \le |\nu| - \frac{n}{2}} D^{0,\ldots,0,m}$. This is a $(\mathfrak{g}, K)$-submodule of $E_{0,\nu}$ if $\nu < 0$, and a quotient-module if $\nu > 0$. It is unitary for $\nu = -\frac{n}{2}$.

Consider the space of smooth functions on $\mathbb{R}^{n+1} \setminus \{0\}$, homogeneous of degree $-\lambda(n+1)$. This space is a $\mathrm{Diff}(\mathbb{S}^n)$-module (and also a $\mathrm{Vect}(\mathbb{S}^n)$-module with the Lie derivative) isomorphic to the module $\mathcal{F}_\lambda(\mathbb{S}^n)$ (see [5]). Denote by $\mathcal{H}^{n+1,m}$ the K-module constituted of its elements of the form

$$\frac{P_m(x_0, \ldots, x_n)}{\left(x_0^2 + \cdots + x_n^2\right)^{\frac{m}{2} + \lambda \frac{n+1}{2}}},$$

where P_m is a harmonic polynomial homogeneous of degree m. We, finally, check the following facts:

- If $n = 1$, then $\mathcal{H}^{2,m} \cong D^{-m} \oplus D^m$. Indeed $\mathcal{H}^{2,m}$ is the direct sum of SO(2)-modules H_m and H_{-m}, respectively generated by

  $$(x_0 + ix_1)^m \left(x_0^2 + x_1^2\right)^{-\frac{m}{2}-\lambda}$$

  and its conjugate in $\mathbb{C}$, and we have $H_{\pm m} \cong D^{\pm m}$.

- If $n > 1$, then $\mathcal{H}^{n+1,m}$ is simple and we have $\mathcal{H}^{n+1,m} \cong D^{0,\ldots,0,m}$.

Consequently, the $(\mathfrak{g}, K)$-module of K-finite vectors of $\mathcal{F}_\lambda(\mathbb{S}^n)$ is given by

$$\mathcal{H}(K) \cong \bigoplus_{m\in\mathbb{N}} \mathcal{H}^{n+1,m}.$$

Let us apply the above results to the representation $\mathrm{Ind}_{MAN}^G(0 \otimes \nu)$. Substituting $\nu + \rho = n\lambda$ to Theorem 2, we obtain the assertions of Theorem 1.

Remark 2. The case $n = 1$ can be directly deduced from the classification of representations of $\mathrm{SL}(2,\mathbb{R})$: acting the same way as in [4], we observe that the space of K-finite vectors of $\mathcal{F}_\lambda(\mathbb{S}^1)$ is the direct sum $\bigoplus H_{2l}$, $l \in \mathbb{Z}$, where H_m is the space of the representation of $\mathrm{SO}(2) \subset \mathrm{SL}(2,\mathbb{R})$ with the character

$$\chi_m : \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \mapsto \mathrm{e}^{im\theta}.$$

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