Multiple Hamiltonian Structures and Lax Pairs for Bogoyavlensky–Volterra Systems

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Abstract

Results on the Volterra model which is associated to the simple Lie algebra of type $A_n$ are extended to the Bogoyavlensky–Volterra systems of type $B_n$, $C_n$ and $D_n$. In particular we find Lax pairs, Hamiltonian and Casimir functions and multi-Hamiltonian structures. Moreover, we investigate recursion operators, higher Poisson brackets and master symmetries.

1 Introduction

The purpose of this paper is to investigate the integrable systems constructed by Bogoyavlensky in 1988 [2, 3]. These systems are connected with simple Lie algebras and are generalizations of the well known Volterra system. In particular, the Volterra system (also known as the KM system) is related to the root system of a simple Lie algebra of type $A_n$.

This system and its Poisson structure is treated in detail in [10]. The equations of motion are

$$\frac{dv_i}{dt} = v_i (v_{i+1} - v_{i-1}), \quad i = 1, 2, \ldots, n,$$

(1.1)

where $v_0 = v_{n+1} = 0$. These equations were studied originally by Volterra in [29] to describe population evolution in a hierarchical system of competing individuals. The importance of this system derives from the fact that it can be considered as a discrete analogue of the Korteweg-de Vries equation. It is also associated with a lattice deformation of the Virasoro algebra [11]. This system was solved by Kac and Van Moerbeke [18] using a discrete version of inverse scattering. There is also an explicit solution by Moser in [21]. The integrability of the periodic KM system is considered in [13]. Finally, Damianou [5] constructed Multi Hamiltonian structures and master symmetries for the system.

Equations (1.1) can be written as a Lax pair $L = [B, L]$, where $L$ is the Jacobi matrix

$$L = \begin{pmatrix}
0 & \sqrt{v_1} & 0 & \cdots & 0 \\
\sqrt{v_1} & 0 & \sqrt{v_2} & \ddots & \vdots \\
0 & \sqrt{v_2} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & \sqrt{v_n} \\
0 & \cdots & 0 & \sqrt{v_n} & 0
\end{pmatrix},$$

(1.2)
and

\[
B = \begin{pmatrix}
0 & 0 & \frac{1}{2}\sqrt{v_1 v_2} & 0 \\
0 & 0 & 0 & \frac{1}{2}\sqrt{v_2 v_3} \\
-\frac{1}{2}\sqrt{v_1 v_2} & 0 & \ddots & \ddots \\
-\frac{1}{2}\sqrt{v_2 v_3} & \ddots & \ddots & 0 & \frac{1}{2}\sqrt{v_{n-1} v_n} \\
0 & \ddots & 0 & 0 & 0 \\
& \ddots & 0 & 0 & 0 \\
0 & & & & -\frac{1}{2}\sqrt{v_{n-1} v_n} & 0 & 0 \\
\end{pmatrix}.
\]

It follows that the functions \(H_k = \frac{1}{k} \text{Tr} L^{2k}\) are constants of motion.

We present the Poisson structure of the Volterra system following [5]. We denote by \(\pi_j\) the bracket of degree \(j\). The bracket \(\pi_2\) is defined by

\[
\{v_i, v_{i+1}\} = -\{v_{i+1}, v_i\} = v_i v_{i+1},
\]

and all other brackets are zero. In this bracket the Hamiltonian is \(H_1 = \text{Tr} L^2\) and \(\det L\) is the Casimir.

The Poisson bracket \(\pi_3\) is defined by

\[
\{v_i, v_{i+1}\} = v_i v_{i+1} (v_i + v_{i+1}), \quad \{v_i, v_{i+2}\} = v_i v_{i+1} v_{i+2},
\]

and all other brackets are zero.

We assume that \(n\) is odd. In order to define the bracket \(\pi_1\) we define the vector field

\[
Y_{-1} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial v_i},
\]

where \(f_i\) is determined by

\[
f_1 = -1, \quad f_{2i} = -\frac{v_{2i}}{v_{2i-1}} f_{2i-1}, \quad f_{2i-1} = -f_{2i-2} - 1,
\]

and we define \(\pi_1 = L_{Y_{-1}} (\pi_2)\). In this bracket the Hamiltonian is \(H_2\) and the Casimir is \(H_1\).

In [5] there is a construction of an infinite sequence of vector fields \(Y_n\), for \(n \geq -1\), and an infinite sequence of Poisson brackets \(\pi_n\), \(n \geq 1\), satisfying:

(i) \(\pi_j\) are all Poisson.
(ii) \(\pi_i, \pi_j\) are compatible for all \(i, j\).
(iii) The functions \(H_j\) are in involution with respect to all of the \(\pi_i\).
(iv) \(Y_i (H_j) = (i + j) H_{i+j}\).
(v) \(L_{Y_i} (\pi_j) = (i - j + 2) \pi_{i+j}\).
(vi) \(\pi_i \nabla H_j = \pi_{i-1} \nabla H_{j+1}\).
(vii) \([Y_i, X_j] = (j - 1) X_{j+1}\), where \(X_j\) is the Hamiltonian vector field generated by \(H_j\) with respect to \(\pi_1\).
In this paper we obtain similar results for the generalized Volterra systems of Bogoyavlensky.

We would like to comment on the relation between the Volterra systems in this paper and the well known Toda systems generalized on simple Lie groups also by Bogoyavlensky [1].

There is a transformation due to Hénon which maps the Bogoyavlensky–Volterra system of type $A_{2n}$ to the usual $A_n$ Toda lattice. The mapping is given by

$$a_i = -\frac{1}{2} \sqrt{v_{2i}v_{2i-1}}, \quad 1 \leq i \leq n-1, \quad b_i = \frac{1}{2} (v_{2i-1} + v_{2i-2}), \quad 1 \leq j \leq n. \quad (1.7)$$

The equations satisfied by the new variables $a_i, b_j$ are given by:

$$\dot{a}_i = a_i (b_{i+1} - b_i), \quad \dot{b}_i = 2 \left( a_i^2 - a_{i-1}^2 \right). \quad (1.8)$$

These are precisely the equations for the finite nonperiodic Toda lattice. Note that the number of variables for the Toda lattice is odd and this justifies our choice to consider the KM system with an odd number of variables.

In order to generalize the Hénon correspondence from generalized Volterra to generalized Toda it is necessary to work not with the original variables $b_j$ of Bogoyavlensky but rather with some new variables $v_j$ which also appear in [2]. However, Bogoyavlensky did not give a Lax pair for these systems in the variables $v_j$ and this is the main construction of this paper.

The relation between the Volterra system of type $B_{2n+1}$ (or $C_{2n+1}$) and Toda $B_n$ ($C_n$) is due to Damianou and Fernandes [7], in 2002. The results of the present paper are essential for the calculations in [7]. The connection between Volterra $D_{2n+1}$ and Toda $D_n$ is still an open problem. In any case, the multi-Hamiltonian structure for the Toda $D_n$ system is a recent development and can be found in [8].

We have to point out that since the $B_n$ Toda lattice involves only an even number of variables it is natural to consider only Volterra $B_{2n+1}$ systems with an even number of variables. It is not a coincidence that we have obtained results only in this particular case.

In Section 2 we present the necessary background on Poisson manifolds, bi-Hamiltonian systems and master symmetries.

In Section 3 we describe the construction of the systems. We obtain the Bogoyavlensky–Volterra (BV) system for each simple Lie algebra $G$.

In Section 4 we investigate the BV system of type $B_{n+1}$. We find a Lax-pair $(L, B)$ for every $n \geq 2$. When $n$ is even, we define two compatible brackets $\pi_1, \pi_3$ which define a recursion operator $R = \pi_3 \pi_1^{-1}$. This recursion operator produces compatible Poisson brackets $\pi_{2j+1} = R^j \pi_1$ and the constants of motion are in involution for every $j = 1, 2, 3, \ldots$.

In Section 5 we find master symmetries of the BV $B_{n+1}$ system as well as the relations which they satisfy. We do not present the analogous results for the $C_{n+1}$ system since it is equivalent to the $B_{n+1}$ system.

In Section 6 we investigate the BV system of type $D_{n+1}$. We find again a Lax pair $(L, B)$ for every $n \geq 4$ and, when $n$ is odd, we define two compatible Poisson brackets $\pi_1, \pi_3$. We also describe the Hamiltonian formulation and compute the Casimirs.
2 Background

2.1 Poisson manifolds

We begin with a brief review of Poisson Manifolds. See for example [19, 30, 28].

Let \( M \) be a \( C^\infty \) manifold and \( C^\infty (M) \) the space of \( C^\infty \) real valued functions on \( M \). A contravariant, antisymmetric tensor of order \( p \) will be called a \( p \)-tensor for short.

A manifold \( M \) with a bilinear map \( \{ \ , \} : C^\infty (M) \times C^\infty (M) \to C^\infty (M) \) which satisfies the following conditions:

\[
\{ f, g \} = - \{ g, f \}, \tag{skew-symmetry}
\]
\[
\{ f, \{ g, h \} \} + \{ g, \{ h, f \} \} + \{ h, \{ f, g \} \} = 0, \tag{Jacobi identity}
\]
\[
\{ f, g \cdot h \} = \{ f, g \} \cdot h + \{ f, h \} \cdot g \tag{Leibniz rule}
\]

is called a Poisson manifold. The bilinear form \( \{ \ , \} \) is called Poisson bracket or Poisson structure on \( M \).

The Poisson bracket gives rise to a 2-tensor \( \pi \) such that

\[
\{ f, g \} = \langle \pi, df \wedge dg \rangle
\]

where \( \langle \ , \ \rangle \) is the pairing between the 2-tensors and the differential 2-forms. In local coordinates \((x_1, x_2, \ldots, x_n)\), \( \pi \) is given by

\[
\pi = \sum_{1 \leq i,j \leq n} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{2.1}
\]

and

\[
\{ f, g \} = \langle \pi, df \wedge dg \rangle = \sum_{1 \leq i,j \leq n} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \tag{2.2}
\]

where \( \pi_{ij} = \{ x_i, x_j \} \). Hence if we know the Poisson matrix \( \pi_{ij} \) we know the bracket \( \{ f, g \} \) of two arbitrary functions as well.

The Poisson bracket allows one to associate a vector field to each element \( f \in C^\infty (M) \). Leibniz’s rule implies that \( \{ f, \cdot \} \) is a derivation of \( C^\infty (M) \). Therefore, for each \( f \in C^\infty (M) \) there exists a well defined vector field \( X_f \) defined by the formula

\[
X_f (g) = \{ f, g \}. \tag{2.3}
\]

It is called Hamiltonian vector field generated by \( f \). A nonconstant function \( f \) such that \( X_f = 0 \) is called a Casimir.

A symplectic manifold is a pair \((M, \omega)\), where \( \omega \) is closed, nondegenerate 2-form. In local coordinates \((x_1, \ldots, x_n)\) we have

\[
\omega = \sum_{i,j} \omega_{ij} (x) \, dx_i \wedge dx_j. \tag{2.4}
\]

The nondegeneracy condition means that there exists an inverse (skew-symmetric) matrix \( \omega^{-1} \), so the dimension of \( M \) is even. The condition for \( \omega \) to be closed (i.e. \( d\omega = 0 \)) is equivalent to the Jacobi identity for the tensor \( \omega^{-1} \) (see [25]). Therefore we can define a Poisson bracket by

\[
\{ f, g \} = \omega (X_f, X_g), \tag{2.5}
\]

which is called the symplectic bracket.
Let $\pi_1, \pi_2$ two Poisson brackets on manifold $M$. The two brackets are called compatible if $\pi_1 + \pi_2$ is also Poisson. For example if $\pi_1$ is Poisson and $\pi_2 = L_X\pi_1$ for some vector field $X$ then it is easy to prove that $\pi_1, \pi_2$ are compatible; see [6]. If $\pi_1$ is symplectic then we can define a Recursion operator: It is the $(1,1)$-tensor $R$ defined by

$$R = \pi_2\pi_1^{-1}.$$  
(2.6)

Recursion operators were introduced by Olver in [24].

A bi-Hamiltonian system is defined by specifying two Hamiltonian functions $H_1, H_2$ satisfying:

$$\pi_1 \nabla H_2 = \pi_2 \nabla H_1.$$  
(2.7)

The notion of bi-Hamiltonian system is due to F Magri [20]. We have the following result, see [9, 15]:

Suppose we have a bi-Hamiltonian system on a manifold $M$, whose first cohomology group is trivial. Then there exists a hierarchy of mutually commuting functions $H_1, H_2, \ldots$ all in involution with respect to both brackets. They generate mutually commuting bi-Hamiltonian flows $X_i, i = 1, 2, \ldots$ satisfying the Lenard recursion relations

$$X_{i+j} = \pi_i \nabla H_j,$$  
(2.8)

where $\pi_{i+1} = R^i\pi_1$ are the higher order Poisson tensors.

### 2.2 Master symmetries

Master symmetries were introduced by Fokas and Fuchssteiner in [14]. For further details on bi-Hamiltonian systems relevant to this paper see [12, 26, 27, 22]. The technique of generating master symmetries for bi-Hamiltonian systems can be found in [16].

We recall the definition and basic properties of master symmetries following Fuchssteiner [17].

Consider a differential equation on a manifold $M$

$$\dot{x} = X(x).$$  
(2.9)

A vector field $Y = Y(x)$ is a symmetry of (2.9) if

$$[Y, X] = 0.$$  
(2.10)

The condition for $Z$ to be a master symmetry is:

$$[[Z, X], X] = 0, \quad \text{and} \quad [Z, X] \neq 0.$$  
(2.11)

We consider a bi-Hamiltonian system defined by the compatible Poisson tensors $J_0, J_1$ and the Hamiltonians $h_0, h_1$. Assume that $J_0$ is symplectic. We define the recursion operator $R = J_1J_0^{-1}$, the higher flows

$$X_{i+1} = R^iX_1, \quad \text{where} \quad X_1 = J_1dh_0 = J_0dh_1,$$  
(2.12)
and the higher order Poisson tensors

\[ J_i = \mathcal{R}^i J_0, \quad i = 1, 2, \ldots \] (2.13)

Master symmetries preserve constants of motion, Hamiltonian vector fields and generate hierarchies of Poisson structures. For a nondegenerate bi-Hamiltonian system, master symmetries can be generated using a method due to W Oevel [23].

**Theorem 1.** Suppose that \( Z_0 \) is a conformal symmetry for both \( J_0, J_1 \) and \( h_0 \), i.e. for some scalars \( \alpha, \beta, \) and \( \gamma \) we have

\[ L_{Z_0} J_0 = \alpha J_0, \quad L_{Z_0} J_1 = \beta J_1, \quad L_{Z_0} h_0 = \gamma h_0. \] (2.14)

Then the vector fields

\[ Z_i = \mathcal{R}^i Z_0, \quad i = 1, 2, \ldots \] (2.15)

are master symmetries, the \( J_i \) are Poisson and they satisfy

\[ (i) \quad [Z_i, X_j] = (\beta + \gamma + (j - 1)(\beta - \alpha)) X_{i+j}, \]

\[ (ii) \quad [Z_i, Z_j] = (\beta - \alpha)(j - i) Z_{i+j}, \]

\[ (iii) \quad L_{Z_i} J_j = (\beta + (j - i - 1)(\beta - \alpha)) J_{i+j}. \]

### 3 Definition of the systems

We consider the system

\[ \frac{du_i}{dt} = u_i(u_{i+1} - u_{i-1}), \quad i = 1, \ldots, n, \] (3.1)

where \( u_0 = u_{n+1} = 0 \). This is the Volterra system, also known as the KM system and is related to the root system of a simple Lie algebra of type \( A_{n+1} \). Bogoyavlensky constructed integrable dynamical systems connected with simple Lie algebras that generalize the Volterra system. For more details see [2, 3].

We outline the construction of the systems:

Let \( \mathcal{G} \) be a simple Lie algebra (rank \( \mathcal{G} = n \)) and \( \Pi = \{ \omega_1, \omega_2, \ldots, \omega_n \} \) the Cartan–Weyl basis of simple roots in \( \mathcal{G} \) ([4]). There are unique positive integers \( k_i \) such that

\[ k_0 \omega_0 + k_1 \omega_1 + \cdots + k_n \omega_n = 0, \] (3.2)

where \( k_0 = 1 \) and \( \omega_0 \) is the minimal negative root.

We consider the following Lax pairs:

\[ \dot{L} = [B, L], \]

\[ L(t) = \sum_{i=1}^{n} b_i(t) e_{\omega_i} + e_{\omega_0} + \sum_{1 \leq i < j \leq n} [e_{\omega_i}, e_{\omega_j}], \]

\[ B(t) = \sum_{i=1}^{n} \frac{k_i}{b_i(t)} e_{-\omega_i} + e_{-\omega_0}. \] (3.3)
Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$. For every root $\omega_a \in \mathcal{H}^*$ there is a unique $H_{\omega_a} \in \mathcal{H}$ such that $\omega(h) = k(H_{\omega_a}, h) \forall h \in \mathcal{H}$, where $k$ is the Killing form and $\mathcal{H}^*$ is the dual space of $\mathcal{H}$. We also have an inner product on $\mathcal{H}^*$ such that $\langle \omega_a, \omega_b \rangle = k(H_{\omega_a}, H_{\omega_b})$. We set

$$c_{ij} = \begin{cases} 1 & \text{if } \langle \omega_i, \omega_j \rangle \neq 0 \text{ and } i < j, \\ 0 & \text{if } \langle \omega_i, \omega_j \rangle = 0 \text{ or } i = j, \\ -1 & \text{if } \langle \omega_i, \omega_j \rangle \neq 0 \text{ and } i > j. \end{cases} \quad (3.4)$$

The vector equation (3.3) is equivalent to the dynamical system

$$\dot{b}_i = -\sum_{j=1}^n k_j c_{ij} b_j. \quad (3.5)$$

We determine the skew-symmetric variables

$$x_{ij} = c_{ij} b_i b_j^{-1}, \quad x_{ji} = -x_{ij}, \quad x_{jj} = 0, \quad (3.6)$$

which correspond to the edges of the Dynkin diagram for the Lie algebra $\mathcal{G}$, connecting the vertices $\omega_i$ and $\omega_j$.

The dynamical system (3.5) in the variables $x_{ij}$ takes the form

$$\dot{x}_{ij} = x_{ij} \sum_{s=1}^n k_s (x_{is} + x_{js}). \quad (3.7)$$

We recall that the vertices $\omega_i$, $\omega_j$ of the Dynkin diagram are joined by edges only if $\langle \omega_i, \omega_j \rangle \neq 0$. Hence $x_{ij} = 0$ if there are no edges connecting the vertices $\omega_i$ and $\omega_j$ of the diagram. We call the equations (3.7) Bogoyavlensky–Volterra system associated with $\mathcal{G}$ (BV system for short).

We shall now describe the BV system for each simple Lie algebra $\mathcal{G}$. The number of independent variables $x_{ij}(t)$ is equal to $n - 1$ and is one less than the number of variables $b_j(t)$. We use the standard numeration of vertices of the Dynkin diagram and define the variables $u_k(t) = x_{ij}(t)$ corresponding to the edges of the Dynkin diagram with increasing order of the vertices ($i < j$).

The phase space consists of variables $u_i$, $1 \leq i \leq n - 1$, with $u_i > 0$.

**$A_{n+1}$**

$$\omega_0 = -\left(\omega_1 + \omega_2 + \cdots + \omega_{n+1}\right)$$

$$k_i = 1, \quad i = 1, \ldots, n + 1$$

$$c_{ij} = \begin{cases} 0, & |i - j| \neq 1, \\ 1, & j = i + 1, \\ -1, & j = i - 1 \end{cases}$$

$$u_i = x_{i,i+1} = \frac{1}{b_i b_{i+1}}, \quad i = 1, \ldots, n$$

\[\text{(BV $A_{n+1}$)}\]

$$\dot{u}_1 = u_1 u_2$$

$$\dot{u}_n = -u_{n-1} u_n$$

$$\dot{u}_i = u_i (u_{i+1} - u_{i-1}) \quad 2 \leq i \leq n - 1$$
\(B_{n+1}\)

\[
\begin{array}{ccccccc}
\omega_1 & u_1 & \omega_2 & u_2 & \omega_3 & \cdots & \omega_n & u_n \\
\omega_1 & u_1 & \omega_2 & u_2 & \omega_3 & \cdots & \omega_n & u_n \\
\end{array}
\]

\(\omega_0 = -(\omega_1 + 2\omega_2 + \cdots + 2\omega_{n+1})\)

\(k_1 = 1, k_i = 2, \quad i = 2, \ldots, n + 1\)

\[c_{ij} = \begin{cases} 
0, \ |i - j| \neq 1, \\
1, \ j = i + 1, \\
-1, \ j = i - 1
\end{cases} \]

\[u_i = x_{i,i+1} = \frac{1}{b_i b_{i+1}} , \quad i = 1, \ldots, n\]

\(\text{(BV } B_{n+1})\)

\(C_{n+1}\)

\[
\begin{array}{ccccccc}
\omega_1 & u_1 & \omega_2 & u_2 & \omega_3 & \cdots & \omega_n & u_n \\
\omega_1 & u_1 & \omega_2 & u_2 & \omega_3 & \cdots & \omega_n & u_n \\
\end{array}
\]

\(\omega_0 = -(2\omega_1 + \cdots + 2\omega_n + \omega_{n+1})\)

\(k_i = 2 , \quad i = 1, \ldots, n, \quad k_{n+1} = 1\)

\[c_{ij} = \begin{cases} 
0, \ |i - j| \neq 1, \\
1, \ j = i + 1, \\
-1, \ j = i - 1
\end{cases} \]

\[u_i = x_{i,i+1} = \frac{1}{b_i b_{i+1}} , \quad i = 1, \ldots, n\]

\(\text{(BV } C_{n+1})\)

\(D_{n+1}\)

\[
\begin{array}{ccccccc}
\omega_1 & u_1 & \omega_2 & u_2 & \omega_3 & \cdots & \omega_{n-2} & u_{n-2} \\
\omega_1 & u_1 & \omega_2 & u_2 & \omega_3 & \cdots & \omega_{n-2} & u_{n-2} \\
\end{array}
\]

\(\omega_0 = -(\omega_1 + 2\omega_2 + \cdots + 2\omega_{n-1} + \omega_n + \omega_{n+1})\)

\(k_1 = 1, k_n = 1, k_{n+1} = 1, k_i = 2, \quad 2 \leq i \leq n - 1\)

\[c_{ij} = -c_{ji} = \begin{cases} 
1, \ 2 \leq j = i + 1 \leq n, \\
0, \ (i,j) = (n,n+1), \\
0, \ 3 \leq i + 2 \leq j \leq n, \\
1, \ (i,j) = (n-1,n+1)
\end{cases} \]

\[u_i = x_{i,i+1} = \frac{1}{b_i b_{i+1}} , \quad i = 1, \ldots, n - 1, \quad u_n = x_{n-1,n+1} = \frac{1}{b_{n-1} b_{n+1}}\]

\[\dot{u}_1 = u_1 (2u_2 + u_1), \quad \dot{u}_2 = u_2 (2u_3 - u_1), \quad \text{(BV } D_{n+1})\]
\[ \dot{u}_i = u_i(u_{i+1} - u_{i-1}), \quad 3 \leq i \leq n - 3, \quad \dot{u}_{n-2} = u_{n-2}(u_n + u_{n-1} - 2u_{n-3}), \]
\[ \dot{u}_{n-1} = u_{n-1}(u_n - u_{n-1} - 2u_{n-2}), \quad \dot{u}_n = -u_n(u_n - u_{n-1} + 2u_{n-2}). \]

\[
E_6 \quad \omega_0 = -(\omega_1 + 2\omega_2 + 2\omega_3 + 3\omega_4 + 2\omega_5 + \omega_6)
\]

\[
E_7 \quad \omega_0 = -(2\omega_1 + 2\omega_2 + 3\omega_3 + 4\omega_4 + 3\omega_5 + 2\omega_6 + \omega_7)
\]

\[
E_8 \quad \omega_0 = -(2\omega_1 + 3\omega_2 + 4\omega_3 + 6\omega_4 + 5\omega_5 + 4\omega_6 + 3\omega_7 + 2\omega_8)
\]

\[
F_4 \quad \omega_0 = -(2\omega_1 + 3\omega_2 + 4\omega_3 + 2\omega_4)
\]

\[
\dot{u}_1 = u_1(3u_3 + u_1), \quad \dot{u}_2 = u_2(2u_4 - 2u_3 + u_2), \quad (BV E_6)
\]
\[
\dot{u}_3 = u_3(2u_4 + u_3 - 2u_2 - u_1), \quad \dot{u}_4 = u_4(u_5 - u_4 - 2u_3 - 2u_2), \quad (BV E_7)
\]
\[
\dot{u}_5 = -u_5(u_5 + 3u_4). \quad (BV E_6)
\]

\[
\dot{u}_1 = u_1(4u_3 + u_1), \quad \dot{u}_2 = u_2(3u_4 - 3u_3 + 2u_2), \quad (BV E_7)
\]
\[
\dot{u}_3 = u_3(3u_4 + u_3 - 2u_2 - 2u_1), \quad \dot{u}_4 = u_4(2u_5 - u_4 - 3u_3 - 2u_2), \quad (BV E_8)
\]
\[
\dot{u}_5 = u_5(u_6 - u_5 - 4u_4), \quad \dot{u}_6 = -u_6(3u_5 + u_6). \quad (BV E_8)
\]

\[
\dot{u}_1 = 2u_1(3u_3 + u_1), \quad \dot{u}_2 = u_2(5u_4 - 4u_3 + 3u_2), \quad (BV E_8)
\]
\[
\dot{u}_3 = u_3(5u_4 + 2u_3 - 3u_2 - 2u_1), \quad \dot{u}_4 = u_4(4u_5 - u_4 - 4u_3 - 3u_2), \quad (BV E_8)
\]
\[
\dot{u}_5 = u_5(3u_6 - u_5 - 6u_4), \quad \dot{u}_6 = u_6(2u_7 - u_6 - 5u_5), \quad \dot{u}_7 = -u_7(4u_6 + u_7). \]

\[
\dot{u}_1 = u_1(4u_2 + u_1), \quad \dot{u}_2 = u_2(2u_3 + u_2 - 2u_1), \quad (BV F_4)
\]
\[
\dot{u}_3 = -u_3(2u_3 + 3u_2). \quad (BV F_4)
\]
\[ G_2 \quad \omega_0 = -(3\omega_1 + 2\omega_2) \]

\[
\begin{array}{c}
\omega_1 \\
\omega_2
\end{array}
\]

\[ \dot{u}_1 = -u_1^2 \quad \text{(BV } G_2) \]

Note that the systems (3.7) and the BV system for every simple Lie algebra \( G \) are special cases of the corresponding periodic systems which were constructed by Bogoyavlensky ([2], Section 7), when

\[ \mu_{ij} = c_{ij}, \quad \mu_{k,0} = \mu_{0,k} = 0, \quad 1 \leq i, j \leq n, \quad 0 \leq k \leq n, \quad (3.8) \]

\[ b_0 = 1, \quad u_0 = 0. \]

In this paper we restrict our attention to the Bogoyavlensky–Volterra systems associated with the classical Lie algebras.

4 The Bogoyavlensky–Volterra \( B_n \) system and its Poisson bracket

In this section we investigate the BV \( B_{n+1} \) system. We find a Lax-pair \((L, B)\) for every \( n \geq 2 \). When \( n \) is even, we define two brackets \( \pi_1, \pi_3 \) which define a recursion operator \( R = \pi_3 \pi_1^{-1} \) so that the Poisson brackets \( \pi_{2j+1} = R^j \pi_1 \) are compatible and the constants of motion are in involution in each bracket \( \pi_{2j+1} \).

Recall the BV \( B_{n+1} \) system \((u_i > 0)\)

\[
\begin{align*}
\dot{u}_1 &= u_1(u_1 + 2u_2), \\
\dot{u}_2 &= u_2(2u_3 - u_1), \\
\dot{u}_i &= 2u_i(u_{i+1} - u_{i-1}), \quad i = 3, \ldots, n-1, \\
\dot{u}_n &= -2u_{n-1}u_n.
\end{align*}
\]

(4.1)

We rescale the coordinates

\[
\begin{align*}
v_1 &= u_1, \\
v_i &= 2u_i, \quad i = 2, \ldots, n,
\end{align*}
\]

(4.2)

to obtain the equivalent system

\[
\begin{align*}
\dot{v}_1 &= v_1(v_1 + v_2), \\
\dot{v}_i &= v_i(v_{i+1} - v_{i-1}), \quad i = 2, \ldots, n-1, \\
\dot{v}_n &= -v_{n-1}v_n.
\end{align*}
\]

(4.3)

Before giving the Lax pair for the system (4.3) we introduce some matrix notations:

\[
X_i = \begin{pmatrix} \sqrt{v_i} & 0 \\ 0 & i\sqrt{v_i} \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
Y_i = \frac{1}{2} \begin{pmatrix} \sqrt{v_{i+1}} & 0 \\ 0 & \sqrt{v_{i+1}} \end{pmatrix}, \quad Y_0 = \frac{i}{2} \begin{pmatrix} 0 & v_1 \\ -v_1 & 0 \end{pmatrix}.
\]

(4.4)
It turns out the equations (4.3) are equivalent to the Lax pair \( \dot{L} = [L, B] \), where \( L \) and \( B \) are \((n + 1) \times (n + 1)\) matrices

\[
L = \begin{bmatrix}
0 & 0 & \cdots & 0 & \sqrt{v_1} & i\sqrt{v_1} \\
0 & O & X_n & O & \cdots & O \\
\vdots & X_n & O & \ddots & \ddots & \vdots \\
0 & O & \ddots & \ddots & X_3 & O \\
\sqrt{v_1} & \vdots & \ddots & X_3 & O & X_2 \\
i\sqrt{v_1} & \vdots & \ddots & O & X_2 & O
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0 & \cdots & \cdots & 0 & -\frac{1}{2}\sqrt{v_1v_2} & -\frac{1}{2}\sqrt{v_1v_2} & 0 & 0 \\
\vdots & O & O & Y_{n-1} & O & \cdots & \cdots & O \\
\vdots & O & O & O & \ddots & \ddots & \ddots & \vdots \\
\frac{1}{2}\sqrt{v_1v_2} & O & \ddots & \ddots & \ddots & O & Y_3 & O \\
\frac{1}{2}\sqrt{v_1v_2} & \vdots & \ddots & -Y_4 & O & O & O & Y_2 \\
0 & \vdots & O & -Y_3 & O & O & O & \vdots \\
0 & O & \cdots & \cdots & O & -Y_2 & O & Y_0
\end{bmatrix}.
\] (4.5)

We note that the elements of the matrices \( L, B \) are the \( 2 \times 2 \) matrices \( X_i, Y_j \) and \( O \) except for the elements of the first row and the first column which are scalars.

The functions \( H_{2k} = \frac{1}{2\pi} \text{Tr} (L^{4k}) \), \( k = 1, 2, \ldots \) are constants of motion for the system. We use the old variables \( b_j \) appearing in the equations (3.5) in order to find a cubic bracket \( \pi_3 \) for the system. The equations (3.5) in the case of the Lie algebra \( B_{n+1} \) become

\[
\dot{b}_1 = -2b_2^{-1}, \quad \dot{b}_2 = -2b_3^{-1} + b_1^{-1}, \\
\dot{b}_j = -2 \left( b_{j-1}^{-1} - b_{j-1}^{-1} \right), \quad j = 3, \ldots, n, \quad \dot{b}_{n+1} = 2b_1^{-1}.
\] (4.6)

The dynamical system (4.6) can be written in Hamiltonian form \( \dot{b}_j = \{b_j, H\} \), with Hamiltonian \( H = \log b_1 + 2 \sum_{j=2}^{n+1} \log b_j \) and a constant Poisson bracket

\[
\{b_j, b_{j+1}\} = -\{b_{j+1}, b_j\} = 1, \quad \text{for } j = 1, 2, \ldots, n.
\] (4.7)

All other brackets are zero. In terms of the variables \( v_j \) \((v_1 = b_1^{-1}b_2^{-1}, v_k = 2b_k^{-1}b_{k+1}^{-1}, k = 2, \ldots, n)\) the above skew-symmetry bracket, which we denote by \( \pi_3 \), is given by

\[
\{v_1, v_2\} = v_1v_2(2v_1 + v_2), \\
\{v_i, v_{i+1}\} = v_iv_{i+1}(v_i + v_{i+1}), \quad i = 2, \ldots, n - 1, \\
\{v_i, v_{i+2}\} = v_iv_{i+1}v_{i+2}, \quad i = 1, \ldots, n - 2,
\] (4.8)

and all other brackets are zero.
Suppose that \( n \) is even \((n = 2l)\) and we look for a bracket \( \pi_1 \) which satisfies
\[
\pi_3 \nabla H_2 = \pi_1 \nabla H_4.
\]
We define the skew-symmetric matrix
\[
\omega = \begin{pmatrix}
0 & -\frac{1}{v_1} & \cdots & -\frac{1}{v_1} & -\frac{1}{v_1} \\
\frac{1}{v_1} & 0 & -\frac{1}{v_2} & \cdots & -\frac{1}{v_2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{1}{v_1} & \frac{1}{v_2} & \cdots & 0 & -\frac{1}{v_{n-2}} \\
\frac{1}{v_1} & \frac{1}{v_2} & \cdots & \frac{1}{v_{n-2}} & 0 \\
\frac{1}{v_1} & \frac{1}{v_2} & \cdots & \frac{1}{v_{n-2}} & \frac{1}{v_{n-1}} & 0
\end{pmatrix},
\] (4.9)
and we define \( \pi_1 = \omega^{-1} \) (i.e. \( \{v_i, v_j\}_{\pi_1} = (\omega^{-1})_{ij} \)).

**Theorem 2.** The brackets \( \pi_1, \pi_3 \) satisfy:

(i) \( \pi_1, \pi_3 \) are Poisson.

(ii) The function
\[
\frac{1}{4} H_2 = \frac{1}{8} \text{Tr}(L^4) = \sum_{i=2}^{n} \left( \frac{1}{2} v_i^2 + v_{i-1} v_i \right)
\]
is the Hamiltonian of the BV \( B_{n+1} \) system with respect to the bracket \( \pi_1 \).

(iii) \( \pi_1, \pi_3 \) are compatible.

**Proof.** (i) Changing variables in the Poisson tensor (4.7) preserves the Jacobi identity and therefore \( \pi_3 \) is a Poisson bracket.

In order to prove that \( \pi_1 \) is a Poisson bracket we consider the 2-form
\[
\omega = \frac{1}{2} \sum_{i,j=1}^{n} \omega_{ij} dv_i \wedge dv_j = \sum_{1 \leq i < j \leq n} -\frac{1}{v_i} dv_i \wedge dv_j.
\] (4.10)

Since the 2-form \( \omega \) is closed, (i.e. \( d\omega = 0 \)), \( \pi_1 = \omega^{-1} \) satisfies the Jacobi identity (see [25], page 11) and therefore \( \pi_1 \) is Poisson.

(ii) follows from simple calculations.

(iii) It is well-known, see [6], that if a Poisson tensor is a Lie derivative of another, then the two tensors are compatible. We will see later, in the next section, that \( \pi_3 \) is the Lie derivative of \( \pi_1 \) in the direction of a master symmetry and this fact makes \( \pi_1, \pi_3 \) compatible. ■

Finally, we define a sequence of Poisson brackets \( \pi_{2j-1}, j = 1, 2, \ldots \) which are compatible and the constants of motion are in involution with respect to each \( \pi_{2j-1} \). Since the 2-tensor \( \pi_1 \) is invertible we can define the recursion operator \( R = \pi_3 \pi_1^{-1} \). We define the higher order Poisson tensors
\[
\pi_{2j+1} = R^j \pi_1, \quad j = 1, 2, \ldots,
\] (4.11)

Using standard theory of recursion operators [6, 20, 23] we obtain the following theorem.
Theorem 3. The sequence of higher Poisson tensors and invariants satisfy:
(i) $\pi_{2j+1} \nabla H_{2i} = \pi_{2j-1} \nabla H_{2i+2}$, $\forall \ i,j$.
(ii) $H_{2i}$ are in involution with respect to all Poisson brackets.
(iii) $\pi_{2j+1}$ are all compatible Poisson brackets.

Remark. Since the functions $H_2, H_4, \ldots, H_{2l}$ are independent and in involution the BV $B_{2l+1}$ system is integrable.

5 Master symmetries of the Bogoyavlensky–Volterra $B_n$ system

In this section we find master symmetries for the system (4.3) and derive the relations which they satisfy.

We consider

$$\pi_3 = v_1v_2(2v_1 + v_2) \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} + \sum_{i=2}^{n-1} v_iv_{i+1} (v_i + v_{i+1}) \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_{i+1}}$$

$$+ \sum_{i=2}^{n-2} v_iv_{i+1}v_{i+2} \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_{i+2}}$$

$$\pi_i^{-1} = \sum_{1 \leq i < j \leq n} -\frac{1}{v_i} dv_i \wedge dv_j, \quad H_2 = \sum_{i=2}^{n} (2v_i^2 + 4v_{i-1}v_i).$$

The recursion operator is then

$$R = \pi_3\pi_1^{-1} = \sum_{i,j=1}^{n} \alpha_{ij} dv_j \otimes \frac{\partial}{\partial v_i}, \quad (5.1)$$

where

$$\alpha_{11} = v_2(2v_1 + v_2 + v_3), \quad \alpha_{12} = v_1v_3,$$

$$\alpha_{13} = -v_1(2v_1 + v_2), \quad \alpha_{21} = \frac{v_2v_3}{v_1}(v_2 + v_3 + v_4),$$

$$\alpha_{ii} = v_i^2 + v_{i+1}^2 + 2v_{i-1}v_i + v_iv_{i+1} + v_{i+1}v_{i+2}, \quad i = 2,3,\ldots,n,$$

$$\alpha_{i,i+1} = v_i(v_{i+2} + v_i + 2v_{i-1}), \quad i = 2,3,\ldots,n-1,$$

$$\alpha_{i,i+2} = -v_i(v_{i-1} - 2v_{i-1}), \quad i = 2,3,\ldots,n-2,$$

$$\alpha_{i,i-1} = \frac{v_i}{v_{i-1}}(v_{i+1}^2 + v_{i-1}^2 + v_iv_{i+1} + v_{i+1}v_{i+2}), \quad i = 3,4,\ldots,n,$$

$$\alpha_{i,i-2} = \frac{v_i}{v_{i-2}}(v_{i+1}^2 - v_{i-1}^2 - v_{i-1}v_i + v_iv_{i+1} + v_{i+1}v_{i+2}), \quad i = 3,4,\ldots,n,$$

$$\alpha_{ij} = \frac{v_i}{v_j}(v_{i+1}^2 - v_{i-1}^2 - v_{i-2}v_{i-1} - v_{i-1}v_i + v_iv_{i+1} + v_{i+1}v_{i+2}), \quad i-j > 2,$$

$$\alpha_{ij} = -2\dot{v}_i, \quad j-i > 2.$$

(We assume $v_{n+1} = v_{n+2} = v_{n+3} = \cdots = 0$.)
We now prove that $\pi_1$ and $\pi_3$ are compatible. It is enough to show that $\pi_3 = L_{Z_1} \pi_1$ for some vector field $Z_1$. We define

$$Z_1 = R(Z_0) = \left( \sum_{i,j} \alpha_{ij}dv_j \otimes \frac{\partial}{\partial v_i} \right)(Z_0) = \sum_{i,j} \alpha_{ij}dv_j(Z_0) \otimes \frac{\partial}{\partial v_i}$$

$$= \sum_{i,j} \alpha_{ij}v_j \frac{\partial}{\partial v_i} = \sum_{i=1}^n \left( \sum_{j=1}^n v_j a_{ij} \right) \frac{\partial}{\partial v_i}, \tag{5.2}$$

where $Z_0$ is the Euler vector field

$$Z_0 = \sum_{i=1}^n v_i \frac{\partial}{\partial v_i}. \tag{5.3}$$

Using the formula

$$\{f,g\}_L = X\{f,g\}_\pi - \{f,X(g)\}_\pi - \{X(f),g\}_\pi \tag{5.4}$$

it is easy to check that

$$L_{Z_1}(\pi_1) = -3\pi_3, \tag{5.5}$$

and therefore $\pi_3$ is the Lie-derivative of $\pi_1$ in the direction of the vector field $Z_1$. This makes $\pi_1$ compatible with $\pi_3$ and completes the proof of Theorem 2.

Using the recursion operator we generate the master symmetries

$$Z_i = R^i Z_0. \tag{5.6}$$

One calculates that

$$L_{Z_0}(\pi_1) = -\pi_1, \quad L_{Z_0}(\pi_3) = \pi_3, \quad L_{Z_0}(H_2) = 2H_2. \tag{5.7}$$

Therefore $Z_0$ is a conformal symmetry for $\pi_1$, $\pi_3$, and $H_2$. The constants appearing in Oevel’s Theorem are

$$\alpha = -1, \quad \beta = 1, \quad \gamma = 2. \tag{5.8}$$

We use the notation $h_0 = H_2$, $J_0 = \pi_1$, $X_1 = J_1 dh_0 = J_0 dh_1$ and in general $h_i = H_{2i+2}$, $J_j = \pi_{2j+1}$, $X_i = R^{i-1}X_1$. It follows from Theorem 1 that the higher order Poisson tensors satisfy the following deformation relations:

$$(i) \quad [Z_i, X_j] = (1 + 2j)X_{i+j},$$

$$(ii) \quad [Z_i, Z_j] = 2(j - i)Z_{i+j},$$

$$(iii) \quad L_{Z_i} J_j = (2j - 2i - 1)J_{i+j} \iff L_{Z_i}(\pi_{2j+1}) = (2j - 2i - 1)\pi_{2(i+j)+1}. \tag{5.9}$$

We also have

$$Z_i(H_{2j}) = 2(i + j)H_{2(i+j)}. \tag{5.10}$$

We will not present the results for the BV $C_{n+1}$ system. In fact the BV $C_{n+1}$ system is equivalent to the BV $B_{n+1}$ system through the transformation

$$u_1 \mapsto u_n, \quad u_2 \mapsto u_{n-1}, \quad \ldots, \quad u_{n-1} \mapsto u_2, \quad u_n \mapsto u_1. \tag{5.11}$$
6 The Bogoyavlensky–Volterra $D_n$ system and its Poisson bracket

We recall the BV $D_{n+1}$ system ($u_i > 0$)

\[
\begin{align*}
\dot{u}_1 &= u_1 (u_1 + 2u_2), \\
\dot{u}_2 &= u_2 (2u_3 - u_1), \\
\dot{u}_i &= 2u_i(u_{i+1} - u_{i-1}), \quad i = 3, \ldots, n - 3, \\
\dot{u}_{n-2} &= u_{n-2}(u_n + u_{n-1} - 2u_{n-3}), \\
\dot{u}_{n-1} &= u_{n-1}(u_n - u_{n-1} - 2u_{n-2}), \\
\dot{u}_n &= -u_n(u_n - u_{n-1} + 2u_{n-2}).
\end{align*}
\]  

(6.1)

We make a linear transformation

\[
\begin{align*}
v_1 &= u_1, \\
v_i &= 2u_i, \quad i = 2, \ldots, n - 2, \\
v_{n-1} &= u_{n-1}, \\
v_n &= u_n,
\end{align*}
\]  

(6.2)

to obtain the equivalent system

\[
\begin{align*}
\dot{v}_1 &= v_1 (v_1 + v_2), \\
\dot{v}_i &= v_i(v_{i+1} - v_{i-1}), \quad i = 2, \ldots, n - 3, \\
\dot{v}_{n-2} &= v_{n-2}(v_n + v_{n-1} - v_{n-3}), \\
\dot{v}_{n-1} &= v_{n-1}(v_n - v_{n-1} - v_{n-2}), \\
\dot{v}_n &= -v_n(v_n - v_{n-1} + v_{n-2}).
\end{align*}
\]  

(6.3)

We consider again the $2 \times 2$ matrices which were defined in (4.4) and we also set

\[
X = \left( \begin{array}{cc} \sqrt{v_1} & i\sqrt{v_1} \\ -\sqrt{v_{n-1}} & i\sqrt{v_{n-1}} \end{array} \right), \quad Y = \frac{1}{2} \left( \begin{array}{cc} \sqrt{v_{n-2}v_n} & \sqrt{v_{n-2}v_n} \\ -\sqrt{v_{n-2}v_{n-1}} & \sqrt{v_{n-2}v_{n-1}} \end{array} \right),
\]

\[
W = \frac{i}{2} \left( \begin{array}{cc} 0 & v_{n-1} - v_n \\ v_n - v_{n-1} & 0 \end{array} \right).
\]  

(6.4)

Equations (6.3) can be written in a Lax Pair form $\dot{L} = [L, B]$, where

\[
L = \begin{bmatrix}
0 & 0 & \cdots & 0 & \sqrt{v_1} & i\sqrt{v_1} \\
0 & O & X & O & \cdots & O \\
\vdots & X^t & O & X_{n-2} & \ddots & \vdots \\
0 & O & X_{n-2} & \ddots & \ddots & O \\
\sqrt{v_1} & \ddots & \ddots & O & X_2 \\
i\sqrt{v_1} & \ldots & O & X_2 & O
\end{bmatrix},
\]
The invariant polynomials of this system are given by the functions

\[
H_2, H_4, \ldots, H_{n-1} \quad \text{when } n \text{ is odd,} \\
H_2, H_4, \ldots, H_{n-2}, H_{n-1} \quad \text{when } n \text{ is even,}
\]

where \( H_k = \frac{1}{k} \text{Tr}(L^2 k). \)

As in the case of the BV system we use the variables \( b_j, 1 \leq j \leq n+1 \) of the equations (3.5) in order to find a cubic bracket \( \pi_3 \) of the BV system. The dynamical system (3.5) in the case of the Lie algebra of type \( D_{n+1} \) can be written in Hamiltonian form \( \dot{b}_j = \{b_j, H\} \), with Hamiltonian

\[
H = \log b_1 + 2 \sum_{j=2}^{n-1} \log b_j + \log b_n + \log b_{n+1},
\]

and Poisson bracket

\[
\{b_j, b_{j+1}\} = -\{b_{j+1}, b_j\} = 1 \quad \text{for } j = 1, 2, \ldots, n-1, \\
\{b_{n-1}, b_{n+1}\} = -\{b_{n+1}, b_{n-1}\} = 1.
\]

(6.7)

all other brackets are zero. In the new variables \( v_j \) (\( v_1 = b_1^{-1}b_2^{-1}, \ v_k = 2b_k^{-1}b_{k+1}^{-1}, \ k = 2, \ldots, n-2, \ v_{n-1} = b_{n-1}^{-1}, \ v_n = b_n^{-1}b_{n+1}^{-1} \)) the above skew-symmetric bracket, which we denote by \( \pi_3 \), is given by

\[
\{v_1, v_2\} = v_1 v_2 (2v_1 + v_2), \\
\{v_i, v_{i+1}\} = v_i v_{i+1} (v_i + v_{i+1}), \quad i = 2, \ldots, n-3, \\
\{v_{n-2}, v_{n-1}\} = v_{n-2} v_{n-1} (2v_{n-1} + v_{n-2}), \\
\{v_{n-1}, v_n\} = 2v_{n-1} v_n (v_n - v_{n-1}), \\
\{v_i, v_{i+2}\} = v_i v_{i+1} v_{i+2}, \quad i = 1, \ldots, n-3, \\
\{v_{n-2}, v_n\} = v_{n-2} v_n (v_{n-2} + 2v_n), \\
\{v_{n-3}, v_n\} = v_{n-3} v_n - 2v_n.
\]

(6.8)

All other brackets are zero. As in the case of KM system we suppose that \( n \) is odd \((n = 2l + 1)\) and we look again for a bracket \( \pi_1 \) which satisfies \( \pi_3 \nabla H_2 = \pi_1 \nabla H_4. \)
We define
\[ \tau_{ij} = -\tau_{ji} = v_{2i-1} \prod_{k=i}^{j-1} \frac{v_{2k+1}}{v_{2k}} \quad \text{for } i < j, \quad \tau_{ii} = v_{2i-1}, \] (6.9)
and we let \( \pi_1 \) be the bracket which is defined as follows:
\[ \{v_i, v_j\} = (-1)^{i+j-1} \frac{\tau_{ij}}{2} \quad \text{for } 1 \leq i < j \leq n - 2, \]
\[ \{v_i, v_{n-1}\} = \{v_i, v_n\} = \frac{(-1)^{i+n}}{2} \tau_{i+n-1, i+\frac{1}{2}} \quad \text{for } i = 1, \ldots, n - 2, \]
\[ \{v_{n-1}, v_n\} = -\{v_n, v_{n-1}\} = \frac{1}{2} (v_n - v_{n-1}). \] (6.10)

To illustrate, we give the Poisson matrix of the bracket \( \pi_1 \) in the case \( n = 7 \)
\[
\begin{bmatrix}
0 & \tau_{11} & -\tau_{12} & \tau_{12} & -\tau_{13} & \frac{1}{2} \tau_{13} & \frac{1}{2} \tau_{13} \\
-\tau_{11} & 0 & \tau_{22} & -\tau_{22} & \tau_{23} & -\frac{1}{2} \tau_{23} & -\frac{1}{2} \tau_{23} \\
\tau_{12} & -\tau_{22} & 0 & \tau_{22} & -\tau_{23} & \frac{1}{2} \tau_{23} & \frac{1}{2} \tau_{23} \\
-\tau_{12} & \tau_{22} & -\tau_{22} & 0 & \tau_{33} & -\frac{1}{2} \tau_{33} & -\frac{1}{2} \tau_{33} \\
\tau_{13} & -\tau_{23} & \tau_{23} & -\tau_{33} & 0 & \frac{1}{2} \tau_{33} & \frac{1}{2} \tau_{33} \\
-\frac{1}{2} \tau_{13} & \frac{1}{2} \tau_{23} & -\frac{1}{2} \tau_{23} & \frac{1}{2} \tau_{33} & -\frac{1}{2} \tau_{33} & 0 & \frac{1}{2} (v_7 - v_6) \\
-\frac{1}{2} \tau_{13} & \frac{1}{2} \tau_{23} & -\frac{1}{2} \tau_{23} & \frac{1}{2} \tau_{33} & -\frac{1}{2} \tau_{33} & -\frac{1}{2} (v_7 - v_6) & 0
\end{bmatrix}
\]

We obtain the following Theorem:

**Theorem 4.** (i) \( \pi_1, \pi_3 \) are Poisson.

(ii) The function
\[
\frac{1}{4} H_2 = \frac{1}{8} \text{Tr} \left( L^4 \right) = v_{n-2} v_n + 2 v_{n-1} v_n + \sum_{i=1}^{n-2} v_i v_{i+1} + \frac{1}{2} \sum_{i=2}^{n-2} v_i^2,
\]
is the Hamiltonian of the BV $D_{n+1}$ system with respect to the bracket $\pi_1$.

(iii) The function

$$H = (v_n - v_{n-1}) \prod_{i=1}^{n-2} v_i,$$

is the Casimir of the BV $D_{n+1}$ system in the bracket $\pi_1$.

(iv) $\pi_1, \pi_3$ are compatible.

Proof. (i) We denote $\{ \}$ the bracket $\pi_1$ of $D_{n+1}$ and $\{ \}$ the Poisson bracket $\pi_1$ of $B_n$ ($n = 2l + 1$). Then $\{ \}$ can be defined as follows:

$$\{ v_i, v_j \}_d = \{ v_i, v_j \}_b, \quad 1 \leq i, j \leq n - 2,$$

$$\{ v_i, v_{n-1} \}_d = \{ v_i, v_n \}_d = \frac{1}{2} \{ v_i, v_{n-1} \}_b, \quad 1 \leq i \leq n - 2,$$

$$\{ v_{n-1}, v_n \}_d = \frac{1}{2} (v_n - v_{n-1}).$$

We set

$$[v_i, v_j, v_k] = \{ v_i, \{ v_j, v_k \} \} + \{ v_j, \{ v_k, v_i \} \} + \{ v_k, \{ v_i, v_j \} \}.$$

For $i, j, k = 1, 2, \ldots, n - 2$

$$[v_i, v_j, v_k]_d = [v_i, v_j, v_k]_b = 0.$$

For $i, j = 1, 2, \ldots, n - 2$

$$[v_i, v_j, v_{n-1}]_d = [v_i, v_j, v_n]_d = \frac{1}{2} [v_i, v_j, v_{n-1}]_b = 0.$$

For $i = 1, 2, \ldots, n - 2$

$$[v_i, v_{n-1}, v_n]_d = \{ v_i, \{ v_{n-1}, v_n \}_d \} + \{ v_{n-1}, \{ v_i, v_n \}_d \} + \{ v_n, \{ v_i, v_{n-1} \}_d \}$$

$$= \frac{1}{2} \{ v_i, v_n - v_{n-1} \}_d + \frac{1}{2} \{ v_{n-1}, \{ v_i, v_n \}_d \} + \frac{1}{2} \{ v_n, \{ v_i, v_{n-1} \}_d \}$$

$$= \frac{1}{2} \{ v_i, v_n - v_{n-1} \}_b - \frac{1}{4} \{ v_{n-1}, \{ v_i, v_n \}_b \} + \frac{1}{4} \{ v_n, \{ v_i, v_{n-1} \}_b \}$$

$$= 0.$$

Therefore, $\{ \}$ is Poisson. Relation (6.7) implies that $\pi_3$ is Poisson as well.

(ii), (iii) follow from simple calculations.

(iv) The proof that the bracket $\pi_1 + \pi_3$ is Poisson is similar to the above proof that the $\{ \}$ is Poisson.

Remark. Since the functions $H_2, H_4, \ldots, H_{2l}, H$ are independent and in involution the BV $D_{2l+2}$ system is integrable.

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References


