Poisson configuration spaces, von Neumann algebras, and harmonic forms

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Abstract
We give a short review of recent results on $L^2$-cohomology of infinite configuration spaces equipped with Poisson measures.

1 Introduction
Let $X$ be a complete, connected, oriented Riemannian manifold of infinite volume without boundary, with curvature bounded below. The configuration space $\Gamma_X$ over $X$ is defined as the set of all locally finite subsets (configurations) in $X$:

$$\Gamma_X := \{ \gamma \subset X : \gamma \cap \Lambda \text{ is finite for each compact } \Lambda \subset X \}. \quad (1.1)$$

The theory of configuration spaces has received a great interest in recent time. This can be explained by various important applications in statistical mechanics and quantum field theory, as well as by the rich and interesting intrinsic structure of $\Gamma_X$. $\Gamma_X$ represents a bright example of an infinite dimensional "manifold-like" space. However, it does not possess any proper structure of a Hilbert or Banach manifold. On the other hand, many important geometrical objects, like differential forms, connections and the de Rham complex over $\Gamma_X$ can be introduced in a specific way. A crucial role here is played by a probability measure on $\Gamma_X$ (in particular, a Poisson or Gibbs measure), which is quasi-invariant with respect to the action of a group of diffeomorphisms of $X$. This philosophy, inspired by the pioneering works [1] and [2], has been initiated and developed in [3], [4] and has lead to many interesting and important results in the field of stochastic analysis on configuration spaces and its applications, see also references in [5], [6].

In this note, we give a short review of recent results on the $L^2$-cohomology of $\Gamma_X$ equipped with the Poisson measure $\pi$. In Section 2, we define and study the de Rham complex of $\pi$-square-integrable differential forms over $\Gamma_X$ and the corresponding Laplacian acting in this complex. We describe the structure of the spaces of harmonic forms. In Section 3, we consider the case where $X$ is an infinite covering of a compact manifold, and compute the $L^2$-Betti numbers of $\Gamma_X$. That is, we introduce a natural von Neumann algebra containing projections onto the spaces of harmonic forms, and compute their
traces. Further, we introduce and compute a regularized index of the Dirac operator associated with the de Rham differential on $\Gamma_X$. For a more detailed exposition, see [5], [11], [7], [8] and references given there.

Let us remark that the situation changes dramatically if the Poisson measure $\pi$ is replaced by a different (for instance Gibbs) measure. From the physical point of view, this describes a passage from a system of particles without interaction (free gas) to an interacting particle system, see [4] and references within. For a wide class of measures, including Gibbs measures of Ruelle type and Gibbs measures in low activity-high temperature regime, the de Rham complex was introduced and studied in [6]. The structure of the corresponding Laplacian is much more complicated in this case, and the spaces of harmonic forms have not been studied yet.

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2 De Rham complex over a configuration space

Following [2], [3], we define the tangent space to $\Gamma_X$ at a point $\gamma$ as the Hilbert space

$$T_\gamma \Gamma_X = \bigoplus_{x \in \gamma} T_x X \quad \tag{2.1}$$

(we fix a Riemannian structure on $X$). Under a differential form $W$ of order $n$ over $\Gamma_X$, we will understand a mapping

$$\Gamma_X \ni \gamma \mapsto W(\gamma) \in (T_\gamma \Gamma_X)^{\wedge n}. \quad \tag{2.2}$$

Let $\gamma \in \Gamma_X$ and $x \in \gamma$. By $O_{\gamma,x}$ we will denote an arbitrary open neighborhood of $x$ in $X$ such that $O_{\gamma,x} \cap (\gamma \setminus \{x\}) = \emptyset$. We define the mapping

$$O_{\gamma,x} \ni y \mapsto W_x(\gamma, y) := W(\gamma_y) \in (T_{\gamma_y} \Gamma_X)^{\wedge n}, \quad \gamma_y := (\gamma \setminus \{x\}) \cup \{y\}. \quad \tag{2.3}$$

This is a section of the Hilbert bundle $(T_{\gamma_y} \Gamma_X)^{\wedge n} \hookrightarrow y \in O_{\gamma,x}$. The Levi–Civita connection on $TX$ generates in a natural way a connection on this bundle. We denote by $\nabla^{\Gamma}_{\gamma,x}$ the corresponding covariant derivative and use the notation

$$\nabla^{\Gamma} W(\gamma) := (\nabla^{X}_{\gamma,x} W_x(\gamma, x))_{x \in \gamma} \in T_\gamma \Gamma_X \otimes (T_\gamma \Gamma_X)^{\wedge n}. \quad \tag{2.4}$$

Let $\pi$ be the Poisson measure on $\Gamma_X$ with intensity given by the volume measure. We define the Hilbert space $L^2_\pi \Omega^n$ of complex $n$-forms which are $\pi$-square integrable.

We will now give an isomorphic description of the space $L^2_\pi \Omega^n$ via the space $L^2_\pi \Omega^0$ of $\pi$-square integrable functions on $\Gamma_X$ and spaces $L^2 \Omega^n(X^m)$ of square-integrable complex forms on $X^m$, $m = 1, \ldots, n$. We have

$$(T_\gamma \Gamma_X)^{\wedge n} = \bigoplus_{m=1}^n \bigoplus_{\eta \subset \gamma, |\eta| = m} T^{(n)}_{\eta} X^m, \quad \tag{2.5}$$
where \(|\eta|\) means the number of points in \(\eta\), and \(\mathbb{T}_\eta^{(n)}X^m := \bigoplus_{k_1+...+k_m=m} (T_{x_1}X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m}X)^{\wedge k_m}\) for any finite configuration \(\eta = \{x_1, \ldots, x_m\}\). We denote by \(W_m(\gamma; \eta)\) the projection of \(W(\gamma) \in (T_\gamma X)^{\wedge m}\) onto the subspace \(\mathbb{T}_\eta^{(n)}X^m\).

**Theorem 1.** [5] Setting for \(m = 1, \ldots, n\)

\[
(I_n W)(\gamma; x_1, \ldots, x_m) := \frac{1}{\sqrt{m!}} W_m(\gamma \cup \{x_1, \ldots, x_m\}; \{x_1, \ldots, x_m\}),
\]

we obtain the isometry

\[
I_n : L^2_\pi^0 \Omega^n \rightarrow L^2_\pi^0 \bigotimes_{m=1}^{n} L^2_\pi^0(\Omega^m)
\]

(2.6)

We define the de Rham differential \(d_n: L^2_\pi^0 \Omega^n \rightarrow L^2_\pi^0 \Omega^{n+1}\) by the formula

\[
(d_n W)(\gamma) := (n+1)^{1/2} AS_{n+1} (\nabla^T W(\gamma)),
\]

(2.7)

where \(AS_{n+1}: (T_\gamma \Gamma X)^{\wedge(n+1)} \rightarrow (T_\gamma \Gamma X)^{\wedge(n+1)}\) is the anti-symmetrization. Let \(d_n^*\) be the adjoint operator.

**Theorem 2.** [5] \(d_n^*: L^2_\pi^0 \Omega^{n+1} \rightarrow L^2_\pi^0 \Omega^n\) is a densely defined operator.

Thus the operator \(d_n\) is closable. We denote its closure by \(\bar{d}_n\) and introduce an infinite Hilbert complex

\[
L^2_\pi^0 \xrightarrow{d_0} \cdots \xrightarrow{d_{n-1}} L^2_\pi^0 \xrightarrow{d_n} L^2_\pi^0 \xrightarrow{d_{n+1}} \cdots
\]

(2.9)

Define in the standard way \(H_\pi^{(n)} := \bar{d}_{n-1}d_n^* + d_n^*\bar{d}_n\), which is a self-adjoint operator in \(L^2_\pi^0\). It will be called the Hodge-de Rham Laplacian of the Poisson measure \(\pi\). Standard operator arguments show that the Hilbert spaces \(H_\pi^{(n)} := \text{Ker} \bar{d}_n / \text{clo}\{\text{Im} d_{n-1}\}\) and \(\text{Ker} H_\pi^{(n)}\) are canonically isomorphic, and we identify them.

**Theorem 3.** [5] 1) Let \(H_\pi^{(n)}\) be the Hodge-de Rham Laplacian in \(L^2 \Omega^n(X^m)\). Then:

\[
I_n H_\pi^{(n)} = \left( H_\pi^{(0)} \otimes 1 + 1 \otimes \bigoplus_{m=1}^{n} H_\pi^{(n)} \right) I_n,
\]

(2.10)

2) The isometry \(I_n\) generates the unitary isomorphism of Hilbert spaces

\[
\mathcal{H}_\pi^m = \text{Ker} H_\pi^{(n)} \simeq \bigoplus_{s_1, \ldots, s_d = 0, 1, 2, \ldots} (\mathcal{H}_X^{(1)})^{s_1} \otimes \cdots \otimes (\mathcal{H}_X^{(d)})^{s_d}
\]

(2.11)

where \(\mathcal{H}_X^{(m)} := \text{Ker} H_X^{(m)}, m = 1, 2, \ldots, d, d = \text{dim} X - 1\), and \(s\) means the symmetric tensor power when \(m\) is even and skew-symmetric tensor power when \(m\) is odd.

**Remark 1.** \(H_\pi^{(0)}\) is the Dirichlet operator of the Poisson measure \(\pi\) introduced in [3]. In particular, it was shown there that \(\text{Ker} H_\pi^{(0)}\) consists of constant functions, which together with (2.10) implies (2.11).
Remark 2. Formula (2.11) holds also for spaces of finite configurations, see [8]. In fact, it is a “symmetrized” version of the Künneth formula.

We see from (2.11) that all spaces $H^n_{\pi}$, $n \in \mathbb{N}$, are finite dimensional provided the spaces $H^m(X)$, $m = 1, \ldots, d$, are so. In this case,

$$\dim H^n_{\pi} = \sum_{s_1, \ldots, s_d = 0, 1, 2, \ldots}^{s_1 + 2s_2 + \ldots + ds_d = n} \beta_1^{(s_1)} \cdots \beta_d^{(s_d)},$$

(2.12)

where $\beta_k^{(s)} := \left( \frac{\beta_k}{s} \right)$, when $k$ is odd, $\beta_k^{(s)} := \left( \frac{\beta_k + s - 1}{s} \right)$, when $k$ is even, and $\beta_k^{(0)} := 1$. Here $\beta_k := \dim H^k(X)$.

3 $L^2$ Betti numbers of configuration spaces of coverings

An important example of a manifold $X$ with infinite dimensional spaces $H^p(X)$ is given by an infinite covering of a compact Riemannian manifold (say $M$). In this case, an infinite discrete group $G$ acts freely by isometries on $X$ and consequently on all spaces $L^2\Omega^p(X)$, and $X/G = M$. The orthogonal projection

$$P_p : L^2\Omega^p(X) \to H^p(X)$$

(3.1)

commutes with the action of $G$ and thus belongs to the commutant $A_p$ of this action which is a semifinite von Neumann algebra. The corresponding von Neumann trace $b_p := \text{Tr}_{A_p} P_p$ gives a regularized dimension of the space $H^p(X)$ and is called the $L^2$-Betti number of $X$ (or $M$). $L^2$-Betti numbers were introduced in [9] and have been studied by many authors (see e.g. [10] and references given there). It is known [9] that $b_p < \infty$.

It is natural to ask whether the notion of $L^2$-Betti numbers can be extended to configuration spaces over infinite coverings. It particular, is formula (2.12) valid in this case (with $\beta_k$ replaced by $b_k$)? In what follows, we construct a natural von Neumann algebra containing the orthogonal projection

$$P_n : L^2\Omega^n \to H^n_{\pi},$$

(3.2)

and compute its von Neumann trace. We assume that $G$ is an ICC group (that is, all nontrivial classes of conjugate elements are infinite). Under this condition, the von Neumann algebra $A_p$ is a $II_\infty$ factor. In the sequel, $\text{Tr}_N$ denotes the faithful normal semifinite trace on a $II_\infty$ factor $N$.

The following general statement holds. Let $\mathcal{M}$ be a $II_\infty$ factor and $H$ be a separable $\mathcal{M}$-module. Let $S_m \ni g \mapsto U_g$ be the natural action of the symmetric group $S_m$ in $H^\otimes m$ by permutations, and let $P_s = \frac{1}{m!} \sum_{g \in S_m} U_g$ and $P_a = \frac{1}{m!} \sum_{g \in S_m} (-1)^{\text{sign}(g)} U_g$ be projections in $H^\otimes m$ onto symmetric tensor product $H^\otimes m$ and antisymmetric tensor powers $H^\wedge m$ respectively. Denote $\mathcal{M}_s^{(m)} := \{\mathcal{M}^\otimes m, P_s\}^\sigma$, $\mathcal{M}_a^{(m)} := \{\mathcal{M}^\otimes m, P_a\}^\sigma$. Let $W^*(\mathcal{M}_s^{(m)}, S_m)$ be the cross-product of $\mathcal{M}_s^{(m)}$ and $S_m$. 
Theorem 4. [8] $\mathcal{M}_s^{(m)}$ and $\mathcal{M}_a^{(m)}$ are $\text{II}_\infty$ factors isomorphic to $W^*(\mathcal{M}^\otimes m, S_m)$, and for any $A \in \mathcal{M}$

$$\text{Tr}_{\mathcal{M}_s^{(m)}}(A^\otimes m P_s) = \text{Tr}_{\mathcal{M}_a^{(m)}}(A^\otimes m P_a) = \frac{(\text{Tr}_M A)^m}{m!}. \quad (3.3)$$

Let us consider the orthogonal projection $P_s^{(m)} : (L^2\Omega^p(X))^\otimes m \to (\mathcal{H}^p(X))^\otimes m$. Theorem 4 shows that $A_s^{(m)} := \{A^\otimes m P_s^{(m)}\}' = W^*(A^\otimes m, S_m)$, and $\text{Tr}_{\mathcal{A}_s^{(m)}}(A^\otimes m P_s^{(m)}) = \frac{(b_s)^m}{m!}$.

Next, we introduce the von Neumann algebra

$$A_n := \prod_{s_1, \ldots, s_d = 0, 1, 2, \ldots} \mathcal{A}_1^{(s_1)} \otimes \cdots \otimes \mathcal{A}_d^{(s_d)}. \quad (3.4)$$

Since all algebras $\mathcal{A}_k^{(s_k)}$ are $\text{II}_\infty$ factors, so is $A_n$, with the trace defined by the traces in $\mathcal{A}_k^{(s_k)}$.

Theorem 5. [11] Let $P_n = I_n P_n I_n^{-1}$. Then: $P_n \in A_n$ and

$$\text{Tr}_{A_n} P_n = \sum_{s_1, \ldots, s_d = 0, 1, 2, \ldots} \frac{(b_1)^{s_1}}{s_1!} \cdots \frac{(b_d)^{s_d}}{s_d!}, \quad (3.5)$$

where $b_1, \ldots, b_d$ are the $L^2$-Betti numbers of $X$.

Theorem 4 implies that $A_n$ is the minimal natural von Neumann algebra containing $P_n$. We will use the notation $b_n := \text{Tr}_{A_n} P_n$ and call $b_n$ the $n$-th $L^2$-Betti number of $\Gamma_X$.

Example. Let $X = \mathbb{H}^d$, the hyperbolic space of dimension $d$. It is known that the only non-zero $L^2$-Betti number of $\mathbb{H}^d$ is $b_{d/2}$ (provided $d$ is even). Then

$$b_n = \left\{ \begin{array}{ll} \frac{(b_{d/2})^k}{k!}, & n = \frac{k d}{2}, \ k \in \mathbb{N} \\ 0, & n \neq \frac{k d}{2}, \ k \in \mathbb{N} \end{array} \right. \quad (3.6)$$

Let us introduce a regularized index $\text{ind}_\Gamma X (d + d^*)$ of the Dirac operator associated with the de Rham differential $d : L^2\Omega^\text{even}_\pi \to L^2\Omega^\text{odd}_\pi$ setting

$$\text{ind}_\Gamma X (d + d^*) := \sum_{k=0}^\infty (-1)^k b_k. \quad (3.7)$$

Theorem 6. [11] The series on the right hand side of (3.7) converges absolutely, and

$$\text{ind}_\Gamma X (d + d^*) = e^{\chi(M)}, \quad (3.8)$$

where $\chi(M)$ is the Euler characteristic of $M$.

Corollary. The $L^2$-cohomology of $\Gamma_X$ is infinite provided $\chi(M) \neq 0$. 
References


