On Poisson Realizations of Transitive Lie Algebroids

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Abstract

We show that every transitive Lie algebroid over a connected symplectic manifold comes from an intrinsic Lie algebroid of a symplectic leaf of a certain Poisson structure. The reconstruction of the corresponding Poisson structures from the Lie algebroid is given in terms of coupling tensors.

1 Introduction

The correspondence between Poisson structures and Lie algebroids plays an important role in various problems in Poisson geometry (see, for example, [1, 2, 3, 4]). As is well known [5], the cotangent bundle of an arbitrary Poisson manifold carries a natural Lie algebroid structure compatible with the symplectic foliation. Let \((M, \Psi)\) be a Poisson manifold with Poisson tensor \(\Psi\). Then the Poisson bracket on \(M\) admits the natural extension to the bracket for 1-forms on \(M\):

\[
\{\alpha, \beta\}_{T^*M} = \Psi^#(\alpha) \langle d\beta - \Psi^#(\beta) \rangle d\alpha - d\langle \alpha, \Psi^#(\beta) \rangle,
\]

here \(\Psi^# : T^*M \to TM\) is the vector bundle morphism associated with \(\Psi\). This structure makes the cotangent bundle \(T^*M\) into a Lie algebroid \((T^*M, \Psi^#, \langle \cdot, \cdot \rangle_{T^*M})\) called the Lie algebroid of the Poisson manifold \((M, \Psi)\). Given a symplectic leaf \((B, \omega)\) of \(M\) one can restrict the bracket \(\{ \cdot, \cdot \}_{T^*M}\) to a Lie bracket on smooth sections of the restricted cotangent bundle \(T^*_BM\) [3, 6, 7]. The result is a transitive Lie algebroid \((T^*_BM, \Psi^#_B, \langle \cdot, \cdot \rangle_{T^*_BM})\) over \(B\) called the Lie algebroid of the symplectic leaf \(B\).

So, every symplectic leaf of a Poisson manifold carries an intrinsic transitive Lie algebroid structure which controls the infinitesimal Poisson geometry around the leaf. One can ask the natural question: is there also a connection in the reverse direction between transitive Lie algebroids over a symplectic base and Poisson structures? In this paper we give an affirmative answer to this question. The reconstruction of the Poisson structure from a transitive Lie algebroid is based on the contravariant version [8] of the minimal

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coupling procedure due to Sternberg [9, 10]. The corresponding Poisson structure is called a coupling tensor [8] and represents the result of coupling the symplectic base structure and the fiberwise Lie-Poisson structure on a certain vector bundle via a connection of the transitive Lie algebroid [6, 7]. Remark that connection-dependent Poisson structures of such a type were studied in [11, 12] in the context of a Hamiltonian formulation of Wong’s equation. As an application, we discuss the passage between transitive Lie algebroids and coupling tensors from the viewpoint of Hamiltonian formalism.

2 Reconstruction of Poisson Structures

Recall that a Lie algebroid over a manifold B is a triple \((A, \rho, \{\ ,\ }_A)\) consisting of a vector bundle \(A \to B\) together with a bundle map \(\rho : A \to TB\), called the anchor, and a Lie algebra structure \(\{\ ,\ }_A\) on the space of smooth sections \(\Gamma(A)\) such that: \(\rho\) is a Lie algebra homomorphism and the Leibniz identity holds, \(\{a_1, fa_2\}_A = f\{a_1, a_2\}_A + (L_{\rho(a_1)}f)a_2\) for any \(a_1, a_2 \in \Gamma(A), f \in C^\infty(B)\). An isomorphism between two Lie algebroids is defined as vector bundle morphism compatible with the anchors and the Lie brackets in a natural way. The kernel \(g_B := \ker \rho \subset A\) of the anchor is called the isotropy of a Lie algebroid. If \(\rho\) is a fiberwise surjection, then the the Lie algebroid is said to be transitive [6, 7]. In this case, the isotropy \(g_B\) is a Lie algebra bundle. For every \(\xi \in B\) the fiber \(g_{\xi}\) carries a Lie bracket \([\ ,\]_\xi^{fib}\) induced from \(\{\ ,\ }_A\), which varies smoothly with \(\xi\). Then the dual \(g_{\xi}^*\) of \(g_{\xi}\) is equipped with the Lie-Poisson bracket which makes \(g_{\xi}^*\) into a bundle of Lie-Poisson manifolds (for more detail, see [1, 2]).

For example, in the case of the Lie algebroid \((TB^0, \Psi_B^\#, \{\ ,\ }_{TB^0})\) of a symplectic leaf \((B, \omega)\) of a Poisson manifold \((M, \Psi)\), the isotropy coincides with the annihilator \(TB^0 = \ker \Psi_B^\#\) of \(TB\) in \(T_B M\). The dual of the isotropy is identified with the normal bundle \(E = T_B M/TB\) to \(B\). The fiberwise Lie-Poisson structure of \(E\) is just the linearized transverse Poisson structure of \(\Psi\) at \(B\) [13].

Now we formulate our main result.

**Theorem 1.** Every transitive Lie algebroid \((A, \rho, \{\ ,\ }_A)\) over a connected symplectic manifold \((B, \omega)\) admits a Poisson realization in the following sense. In a neighborhood of the zero section \(B\) of the dual \(g_B^*\) of the isotropy there exists a Poisson tensor \(\Pi\) such that: (i) \((B, \omega)\) is a symplectic leaf of \(\Pi\), and (ii) the Lie algebroid of the leaf \((B, \omega)\) is isomorphic to \((A, \rho, \{\ ,\ }_A)\).

To prove this theorem we give an explicit description of the Poisson structure \(\Pi\) in terms of the algebroid \(A\). As we will see the main features of \(\Pi\) are completely different from the properties of the dual Lie-Poisson structure on \(A^*\) [14] uniquely determined by the homogeneity condition: the bracket of fiberwise linear functions on \(A^*\) is fiberwise linear.

Suppose we are given a transitive Lie algebroid \((A, \rho, \{\ ,\ }_A)\) over a connected symplectic manifold \((B, \omega)\). Then there is an exact sequence of vector bundles \(g_B \to A \to TB\). Fix a vector bundle morphism \(\gamma : TB \to A\) such that \(\rho \circ \gamma = \text{id}\). Such a mapping is called a connection of the transitive Lie algebroid [6, 7]. The curvature of \(\gamma\) is the \(g_B\)-valued valued 2-form \(R \in \Omega^2(B, g_B)\) determined by \(R(u_1, u_2) := \{\gamma(u_1), \gamma(u_2)\}_A - \gamma([u_1, u_2])\) for \(u_1, u_2 \in \mathcal{X}(B)\). One can associate to \(\gamma\) a linear connection \(\nabla\) on the isotropy \(g_B\), called
an adjoint connection, and defined as follows $\nabla_\omega \eta = \{\gamma(u), \eta\}_A$ for $u \in \mathcal{X}(B)$, $\eta \in \Gamma(\mathfrak{g}_B)$. From the Lie algebroid axioms we get the following information \[7\]. The adjoint connection $\nabla$ preserves the fiberwise Lie structure on $\mathfrak{g}_B$,

$$\nabla([\eta_1, \eta_2]^{fib}) = [\nabla \eta_1, \eta_2]^{fib} + [\eta_1, \nabla \eta_2]^{fib}. \quad (2.1)$$

and the curvature form $\text{Curv}^\nabla : TB \oplus TB \rightarrow \text{End}(\mathfrak{g}_B)$ of $\nabla$ is related with the 2-form $\mathcal{R}$ by the adjoint operator,

$$\text{Curv}^\nabla = \text{ad} \circ \mathcal{R}, \quad (2.2)$$

where $\text{ad} \circ \eta := [\eta, \cdot]^{fib}$ for $\eta \in \Gamma(\mathfrak{g}_B)$. Moreover, the modified Bianchi identity holds

$$\nabla \mathcal{R} = 0. \quad (2.3)$$

Using the symplectic structure $\omega$ on $B$ and $\mathcal{R}$, let us introduce the following 2-form on the total space $\mathfrak{g}_B^*$: $\mathcal{F} := \pi^* \omega - \ell \circ \pi^* \mathcal{R}$. Here $\pi : \mathfrak{g}_B^* \rightarrow B$ is the projection and $\ell : \Gamma(\mathfrak{g}_B) \rightarrow C^\infty(\mathfrak{g}_B)$ is the natural identification of $\Gamma(\mathfrak{g}_B)$ with the space $C^\infty(\mathfrak{g}_B)$ of the smooth fiberwise linear functions on $\mathfrak{g}_B^*$, $\ell(\eta)(x) = \langle x, \eta(x) \rangle$ (for $x \in \mathfrak{g}_B^*, \xi = \pi(x)$).

Fix a basis $\{\eta_\sigma\}$ of local sections of $\mathfrak{g}_B$. Let $x = (x^\sigma)$ be the associated coordinates on the fiber of $\mathfrak{g}_B^*$ and $\xi = (\xi^\lambda)$ are local coordinates on $B$. In coordinates, we have $\mathcal{F} = \frac{1}{2} \mathcal{F}_{ij}(\xi, x) d\xi^i \wedge d\xi^j$, where $\mathcal{F}_{ij}(\xi, x) = \omega_{ij}(\xi) - x^\sigma \mathcal{R}_{\sigma,ij}(\xi)$. It is clear that the 2-form $\mathcal{F}$ is nondegenerate in a neighborhood $\mathcal{N}_\gamma$ of $B$ in $\mathfrak{g}_B^*$. Let us define (local) vector fields on the total space $\mathfrak{g}_B^*$ by $\text{hor}_i := \frac{\partial}{\partial \xi^i} - \theta^\sigma_{ij}(\xi)x^\nu \frac{\partial}{\partial x^\sigma}$, where $\nabla_{\frac{\partial}{\partial \xi^i}} \eta_\nu = -\theta^\sigma_{ij}(\xi)\eta_\sigma$. The vector fields $\text{hor}_i$ form the horizontal distribution on $\mathfrak{g}_B^*$ of the dual connection $\nabla^*$. Next, the fiberwise Lie-Poisson structure on $\mathfrak{g}_B^*$ induces the Poisson tensor $\Lambda = \frac{1}{2} \chi_{\alpha\beta}^\sigma(\xi) x^\nu \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}$ on the total space, where $\chi^\sigma_{\alpha\beta}(\xi)$ are the structure constants of $\mathfrak{g}_\xi$ with respect to the basis $\{\eta_\sigma\}$. Finally, we define the following bivector field

$$\Pi_\gamma := -\frac{1}{2} \mathcal{F}^{ij} \text{hor}_i \wedge \text{hor}_j + \Lambda, \quad (2.4)$$

where $\mathcal{F}^{ij} \mathcal{F}_{sj} = \delta^i_j$. Although $\Pi_\gamma$ is defined in terms of coordinates and a basis, one can show that it is independent of these choices. Thus, $\Pi_\gamma$ is a well-defined bivector field on $\Pi_\gamma$ depending on the connection $\gamma$.

**Proposition 1.** Bi-vector field $\Pi_\gamma$ (2.4) is a Poisson tensor satisfying the properties (i),(ii) in Theorem 1.

The Jacobi identity for $\Pi_\gamma$ follows from the invariance property (2.1), the curvature identity (2.2) and the modified Bianchi identity (2.3), [8]. By construction ($B, \omega$) is a symplectic leaf of $\Pi_\gamma$. Taking into account splitting $A = \gamma(TB) \oplus \mathfrak{g}_B$, we identify $A$ with $T_B^*\mathfrak{g}_B = TB \oplus \mathfrak{g}_B$. Then, looking at the infinitesimal part of $\Pi_\gamma$ at $B$, we see that the property (ii) in Theorem 1 holds.

The Poisson tensor $\Pi_\gamma$ is called the coupling tensor [8] associated with a pair $(A, \gamma)$. Observe that $\Pi_\gamma$ is the sum of two bi-vector fields $\Pi_H^\gamma$ and $\Pi_V^\gamma$ called the horizontal and vertical parts, respectively. The vertical part $\Pi_V^\gamma = \Lambda$ is a Poisson tensor completely determined by the fiberwise Lie-Poisson structure of $\mathfrak{g}_B^*$. The horizontal part $\Pi_H^\gamma$ satisfies
the Jacobi identity if and only if the curvature $\mathcal{R}\gamma$ vanishes, or equivalently, the subspace $\pi^*C^\infty(B)$ is closed with respect to the Poisson bracket. In this case, $\Pi_{\gamma}^{H}$ is just the horizontal lift of the Poisson tensor on $(B, \omega)$.

For a given $A$ the Poisson tensor $\Pi_{\gamma}$ depends on the choice of $\gamma$. Suppose we are given other connection $\tilde{\gamma}$ of $A$ and let $\Pi_{\tilde{\gamma}}$ be the corresponding coupling tensor. We say that $\Pi_{\gamma}$ and $\Pi_{\tilde{\gamma}}$ are neighborhood equivalent if there exists a diffeomorphism $f : U \to \tilde{U}$ between two neighborhoods $U \subset N_{\gamma}$ and $\tilde{U} \subset N_{\tilde{\gamma}}$ of $B$ in $g_B^*$ such that $f^*\Pi_{\tilde{\gamma}} = \Pi_{\gamma}$.

**Proposition 2.** $\Pi_{\gamma}$ is independent of the choice of a connection $\gamma$ up to neighborhood equivalence.

So, one can speak on an intrinsic coupling tensor of the transitive Lie algebroid $A$. The proof of this Proposition is based on a contravariant version [8] of the Moser homotopy method. The key observation here is that there exists a smooth homotopy between any two connections of $A$ which induces a homotopy between corresponding coupling tensors.

### 3 Linear Hamiltonian Vector Fields

One can associate a Poisson algebra to a given a transitive Lie algebroid $A$ over a connected symplectic manifold $(B, \omega)$ in the following way. Let $C^\infty_{\aff}(g_B^*)$ be the space of smooth fiberwise affine functions on $g_B^*$. Then

$$C^\infty_{\aff}(g_B^*) \approx C^\infty(B) \oplus C^\infty_{\lin}(g_B^*)$$

and every fiberwise affine function $\phi$ is represented as $\phi = \pi^*f + \ell(\eta) \approx f \oplus \eta$, where $f \in C^\infty(B)$ and $\eta \in \Gamma(g_B)$. Fix a connection $\gamma$ of $A$ and consider the corresponding data $(\nabla, \mathcal{R})$. Then one can define a Lie bracket on $C^\infty_{\aff}(g_B^*)$ by

$$\{\phi_1, \phi_2\}_\aff := \{f_1, f_2\}_B \oplus \left(\nabla_{v_{f_1}} \eta_2 - \nabla_{v_{f_2}} \eta_1 + [\eta_1, \eta_2]_{\lin} + \mathcal{R}(v_{f_1}, v_{f_2})\right),$$

for $\phi_1 = f_1 \oplus \eta_1$ and $\phi_2 = f_2 \oplus \eta_2$. Here $\{,\}_B$ is the Poisson bracket and $v_f$ is the Hamiltonian vector field of $f$ on $(B, \omega)$. Define also the linearized pointwise product $\circ$ for affine functions by $\phi_1 \circ \phi_2 := (f_1 \cdot f_2) \oplus (f_1 \cdot \eta_2 + f_2 \cdot \eta_1)$. This makes $C^\infty_{\aff}(g_B^*)$ into a commutative associative algebra with unit $(1 \oplus 0)$. One can show that the bracket $\{,\}_\aff$ and the linearized pointwise product are compatible by the Leibniz identity and hence the triple $(C^\infty_{\aff}(g_B^*), \circ, \{,\}_\aff)$ defines a Poisson algebra associated with $(A, \gamma)$. This algebra is independent of the choice of $\gamma$ up to an isomorphism.

A vector field $\mathcal{V}$ on the total space $g_B^*$ is called linear if the Lie derivative $L_{\mathcal{V}} : C^\infty(g_B^*) \to C^\infty(g_B^*)$ along $\mathcal{V}$ sends $C^\infty_{\lin}(g_B^*)$ into $C^\infty_{\lin}(g_B^*)$. This implies that $\mathcal{V}$ descends to a vector field $v$ on $B$, $d\pi \circ \mathcal{V} = \pi \circ v$. The Lie algebra of linear vector fields is denoted by $\mathfrak{X}_{\lin}(g_B^*)$. By the analogy with Poisson manifolds, we say that a linear vector field $\mathcal{V}$ is Hamiltonian relative to bracket (3.2) if there exists a fiberwise affine function $\phi = f \oplus \eta$ such that

$$L_{\mathcal{V}} = \text{ad}_\phi \text{ on } C^\infty_{\aff}(g_B^*),$$

where $\text{ad}_\phi := \{\phi, \cdot\}_\aff$ is the adjoint operator of $\phi$. The Hamiltonian vector field of $\phi$ is denoted by $\mathcal{V}_\phi$. Clearly, $\mathcal{V}_\phi$ descends to $v_f$. In the contrast to the usual situation, one
can not say that every $\phi$ admits a linear Hamiltonian vector field. Indeed, condition (3.3) says that $\text{ad}_\phi$ as a derivation of the associative algebra $(C^\infty_\text{aff}(g^*_B), \circ)$ admits an extension to a derivation of $C^\infty_\text{lin}(g^*_B)$. But it is not true for an arbitrary $\phi$. To see that, for every $\phi = f \oplus \eta$ we define the torsion of $\text{ad}_\phi$ as a $R$-linear operator $T_\phi : C^\infty(B) \rightarrow C^\infty_\text{lin}(g^*_B)$ letting $T_\phi(g) := R(v_f, v_g) - \nabla_{v_g} \eta \forall g \in C^\infty(B)$. The space of all torsion free functions is denoted by $A_\gamma := \{ \phi \in C^\infty_\text{aff}(g^*_B) \mid T_\phi = 0 \}$. In fact, $A_\gamma$ is a Lie subalgebra in $(C^\infty_\text{aff}(g^*_B), \{ \} \text{aff})$. Since $C^\infty_\text{lin}(g^*_B)$ is an ideal in the Poisson algebra, splitting (3.1) is invariant with respect to $\text{ad}_\phi$ only if its torsion vanishes. Finally, we observe that $\phi \in C^\infty_\text{lin}(g^*_B)$ admits a linear Hamiltonian vector field in the sense of (3.3) if and only if $\phi \in A_\gamma$. Moreover, the correspondence

$$A_\gamma \ni \phi \mapsto \mathcal{V}_\phi \in X_\text{lin}(g^*_B)$$

is a Lie algebra homorphism.

**Proposition 3.** Let $\Pi_\gamma$ be the coupling tensor associated with $(A, \gamma)$. Then the Lie algebra of all linear vector fields on $g^*_B$ which are Hamiltonian relative to the Poisson structure $\Pi_\gamma$, coincides with the image of $A_\gamma$ under homorphism (3.4).

So, the complement $C^\infty_\text{aff}(g^*_B) \setminus A_\gamma$ consists of all affine functions $\phi$ whose Hamiltonian vector fields $\Pi_\gamma^* d\phi$ are not linear. One can show that $C^\infty_\text{aff}(g^*_B) \setminus A_\gamma$ is nonempty. This observation is related with the phenomenon: the linearization of Hamiltonian systems at a given symplectic leaf of a Poisson manifold may destroy the Hamiltonian property.

**References**


