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A Nonlinearly Dispersive Fifth Order Integrable Equation and its Hierarchy

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Abstract

In this paper, we study the properties of a nonlinearly dispersive integrable system of fifth order and its associated hierarchy. We describe a Lax representation for such a system which leads to two infinite series of conserved charges and two hierarchies of equations that share the same conserved charges. We construct two compatible Hamiltonian structures as well as their Casimir functionals. One of the structures has a single Casimir functional while the other has two. This allows us to extend the flows into negative order and clarifies the meaning of two different hierarchies of positive flows. We study the behavior of these systems under a hodograph transformation and show that they are related to the Kaup-Kupershmidt and the Sawada-Kotera equations under appropriate Miura transformations. We also discuss briefly some properties associated with the generalization of second, third and fourth order Lax operators.

1 Introduction

There has been a lot of interest in recent years in the study of the Harry Dym [1, 3, 4, 5] and the Hunter-Zheng equations [6, 7] which define an integrable hierarchy. Besides having applications in various physical systems, these equations represent a class of equations where the hierarchy can be extended to both positive as well as negative integer flows. Furthermore, in these systems of equations, the dispersive term in the equation is nonlinear, unlike the KdV equation where the dispersive term is linear in the dynamical variable. This is manifest in the structure of the Lax operator and the equation for the Harry Dym equation

$$L = u^2\partial^2, \quad \frac{\partial L}{\partial t} = \left[\left(L^3\right)_{\geq 2}, L\right], \quad u_t = u^3 u_{xxx} \ . \tag{1.1}$$

In trying to understand the properties of other systems that belong to this class of nonlinearly dispersive equations, we have chosen to consider the generalized Lax operator for the Harry Dym equation, introduced in [2] and study the resulting systems. This leads us to a very interesting integrable hierarchy which was proposed a few years ago [8], but whose properties were never studied in detail. However, having a Lax representation for such a system, we are able to analyze various aspects of this system of equations. In particular, we construct the conserved charges for the positive flows directly from the Lax operator itself and show that the system, in fact, contains two infinite families of conserved charges. Correspondingly, the Lax equation really gives rise to two hierarchies of equations. Such a behavior has been noted earlier in connection with dispersionless polytropic gas dynamics [9], but to the best of our knowledge, this is the first example of such a behavior in systems with dispersive terms. We obtain the two compatible Hamiltonian structures thereby showing that the two hierarchies are bi-Hamiltonian. We determine the Casimir functionals of both the Hamiltonian structures which allows us to extend the flows to negative orders (which are nonlocal). We construct the recursion operator as well as its inverse which allows us to construct the non-local charges and the nonlocal flows of the system recursively. We also study the behavior of these systems under a hodograph transformation and make connection with the Kaup-Kupershmidt and Sawada-Kotera equations.

The paper is organized as follows. In section **2**, we discuss the Lax representation and the two associated hierarchies of integrable equations. In section **3** we present the conserved charges following from the Lax operator as well as the two compatible Hamiltonian structures which also guarantees integrability of the system. The Casimir functionals of the system are obtained which allows us to extend the flows to negative orders. We construct explicitly the first few conserved nonlocal charges through the recursion operator. In section **4**, we study the behavior of these systems under a hodograph transformation and make connection with the Kaup-Kupershmidt and the Sawada-Kotera equations. We point out various interesting properties of the systems under the hodograph transformation. We discuss briefly, in sec **5**, some features associated with a generalized Lax operator of second, third and the fourth order and conclude with a brief summary in section **6**.

2 Lax Representation

Let us consider the Lax operator introduced by Konopelchenko and Oevel [2]

$$L = u^3 \partial^3 \,. \tag{2.1}$$

It can be checked with some algebra that this Lax operator leads to two consistent hierarchies of equations of the form

$$\frac{\partial L}{\partial t_n^{(\mp)}} = \left[\left(L^{2n \mp \frac{1}{3}} \right)_{\geq 2}, L \right] , \tag{2.2}$$

where $n = 1, 2, 3 \ldots$. Let us note that the first few equations of the two hierarchies have the forms

$$
\begin{aligned}
u_{t_1^{(-)}} &= u^4 \left(u^2\right)_{5x} \\
u_{t_2^{(-)}} &= -\,\frac{1}{81}\,u^4 \left[30u^7 u_{6x} + 270u^6 u_x u_{5x} \cdots\right]_{5x}, \\
u_{t_1^{(+)}} &= \frac{1}{3} u^4 \left(2u^3 u_{xx} - u^2 u_x^2\right)_{5x}, \\
u_{t_2^{(+)}} &= -\,\frac{1}{243}\,u^4 \left[42u^9 u_{8x} + 840u^8 u_x u_{7x} \cdots\right]_{5x}.
\end{aligned}
\tag{2.3}
$$

These are even more nonlinearly dispersive than the Harry Dym equation (as we had anticipated earlier) and we note that in terms of the variable $\varphi = u^{-3}$, the two lowest order equations take the forms

$$
\begin{aligned}
\varphi_{t_1^{(-)}} &= \left(\varphi^{-2/3}\right)_{5x} \\
\varphi_{t_1^{(+)}} &= -\,2\left(\varphi^{-1}\left(\left(\varphi^{-1/3}\right)_{xx} - 2\left(\varphi^{-1/6}\right)_x^2\right)\right)_{5x}.
\end{aligned}
\tag{2.4}
$$

These equations indeed coincide with the equations considered by Holm and Qiao [8]. However, having a Lax representation for these two hierarchies, we can study their properties systematically in the following. In particular, the Lax representation shows that the two equations belong to two distinct hierarchies (this will become more clear in the next section) and guarantees that the two hierarchies are integrable.

We recall here that the Lax operator for the Harry Dym equation is unique. In fact, it can be checked that the only generalization of the Lax operator (1.1) that leads to a consistent equation has the form

$$
\begin{aligned}
L &= u^2\partial^2 + uu_x\partial = (u\partial)^2, \\
\frac{\partial L}{\partial t} &= \left[\left(L^{3/2}\right)_{\geq 2}, L\right],
\end{aligned}
\tag{2.5}
$$

and leads to the equation

$$
u_t = \left(u^3 u_{xx}\right)_x, \tag{2.6}
$$

which is different from (1.1). However, in the present case, we find that the Lax operator is not unique. For example, it can be checked that the Lax operator and the equation

$$
L = u^3\partial^3 + \frac{3}{2}u^2 u_x \partial^2, \quad \frac{\partial L}{\partial t_1^{(-)}} = \left[\left(L^{5/3}\right)_{\geq 2}, L\right], \tag{2.7}
$$

also lead to the fifth order equation in (2.3). We will comment on the general third order Lax operator in one dynamical variable in sec **5**.

Finally, we note that one can try to generalize such Lax operators to even higher orders. However, a simple Lax operator and equation such as

$$
L = u^m\partial^m, \quad m \geq 4, \quad \frac{\partial L}{\partial t} = \left[\left(L^{n/m}\right)_{\geq 2}, L\right], \tag{2.8}
$$

where n is not a multiple of m, does not lead to consistent equations.

3 Bi-Hamiltonian Structure

In this section we will show that the equations (2.3) are Hamiltonian equations. In fact, as we will show, they represent a bi-Hamiltonian system of equations. From the structure of the Lax operator, we note that we can define two series of conserved charges for the system corresponding to (up to multiplicative constants)

$$H_n^{(-)} = \int \mathrm{d}x \,\mathrm{Tr}\, L^{2n-\frac{1}{3}}, \quad H_n^{(+)} = \int \mathrm{d}x \,\mathrm{Tr}\, L^{2n+\frac{1}{3}}, \quad n = 0, 1, 2, \dots , \tag{3.1}$$

where "Tr" represents the Adler trace over pseudo-differential operators. The other fractional powers of the Lax operator can be easily shown to give trivial charges. Even though the charges in (3.1) come from the same Lax operator, we have written them as two distinct series for reasons that will become clear shortly. We note here that the first few charges of the series can be written explicitly as

$$\begin{aligned}
H_0^{(-)} &= -\int \mathrm{d}x \,\frac{1}{u},\\
H_1^{(-)} &= \frac{1}{81}\int \mathrm{d}x \,\big(- 6u_{6x}u^4 - 54u_{5x}u_xu^3 - 102u_{4x}u_{xx}u^3 - 84u_{4x}u_x^2u^2 - 57u_{3x}^2u^3\\
&\quad - 204u_{3x}u_{2x}u_xu^2 + 12u_{3x}u_x^3u - 40u_{2x}^3u^2 + 60u_{xx}^2u - 30u_{xx}u_x^4 + 5u_x^6u^{-1}\big)\,,\\
H_0^{(+)} &= -\frac{1}{3}\int \mathrm{d}x \,\frac{u_x^2}{u}\,,\\
H_1^{(+)} &= -\frac{1}{243}\int \mathrm{d}x \Big[6u^6u_{8x} + 120u^5u_xu_{7x}\\
&\quad + 264u^5u_{xx}u_{6x} + 372u^5u_{xxx}u_{5x} + 213u^5u_{4x}^2\\
&\quad + 708u^4u_x^2u_{6x} + 2376u^4u_xu_{xx}u_{5x}2712u^4u_xu_{xxx}u_{4x} + 1488u^4u_{xx}^2u_{4x}\\
&\quad + 1308u^4u_{xx}u_{xxx}^2 + 1332u^3u_x^3u_{5x} + 4452u^3u_x^2u_{xx}u_{4x} + 2058u^3u_x^2u_{xxx}^2\\
&\quad + 2976u^3u_xu_{xx}^2u_{xxx} + 112u^3u_{xx}^4 + 552u^2u_x^4u_{4x} + 984u^2u_x^2u_{xxx}^2\\
&\quad - 224u^2u_x^2u_{xx}^3 + 24uu_x^5u_{xxx} + 168uu_x^4u_{xx}^2 - 56u_x^6u_{xx} + 7u^{-1}u_x^8\Big]\,.
\end{aligned} \tag{3.2}$$

In addition, we have found another charge (not following from the Lax operator) that is also conserved,

$$H_{-1} = \int \mathrm{d}x \,\frac{1}{u^3}\,. \tag{3.3}$$

Given the conserved charges in (3.2) and (3.3), it is now easy to check that the two hierarchies of nonlinearly dispersive equations in (2.3) can be written in the bi-Hamiltonian form

$$\begin{aligned}
u_{t_n^{(-)}} &= \mathcal{D}_1 \frac{\delta H_{n-1}^{(-)}}{\delta u} = \mathcal{D}_2 \frac{\delta H_n^{(-)}}{\delta u}\,,\\
u_{t_n^{(+)}} &= \mathcal{D}_1 \frac{\delta H_{n-1}^{(+)}}{\delta u} = \mathcal{D}_2 \frac{\delta H_n^{(+)}}{\delta u}, \quad n = 1, 2, \dots ,
\end{aligned} \tag{3.4}$$

where

$$\mathcal{D}_1 = u^4\partial^5 u^4$$
$$\mathcal{D}_2 = u^2\partial u^{-1}\partial^{-3}u^{-1}\partial u^2\,. \tag{3.5}$$

The Hamiltonian structures in (3.5) are manifestly anti-symmetric. It can also be checked using the method of prolongation [10] that they satisfy Jacobi identity. For example, for the first structure, we have

$$\mathcal{D}_1\theta = u^4\left(u^4\theta\right)_{5x}\,,$$
$$\Theta_{\mathcal{D}_1} = \frac{1}{2}\int \mathrm{d}x\ \left(u^4\theta\right)\wedge\left(u^4\theta\right)_{5x}\,,$$
$$\mathbf{pr}\,V_{\mathcal{D}_1\theta}\left(\Theta_{\mathcal{D}_1}\right) = 4\int \mathrm{d}x\ \left(u^4\theta\right)_{5x}\wedge\left(u^7\theta\right)\wedge\left(u^4\theta\right)_{5x} = 0\,. \tag{3.6}$$

Similarly, the vanishing of the prolongation for the second structure can also be easily checked. The compatibility of the two Hamiltonian structures in (3.5) can also be checked through the method of prolongation [10] with a little bit of algebra. This shows that the two hierarchies of equations (2.3) are indeed bi-Hamiltonian. As a result, they are integrable.

We note that because of the simplicity in the structure in (3.5), we can construct the recursion operator for the system in the closed form as

$$\mathcal{R} = \mathcal{D}_2^{-1}\mathcal{D}_1 = u^{-2}\partial^{-1}u\partial^3 u\partial^{-1}u^2\partial^5 u^4\,, \tag{3.7}$$

and it can be checked that the conserved charges (3.2) are related recursively as

$$\frac{\delta H_{n+1}^{(\mp)}}{\delta u} = \mathcal{R}\,\frac{\delta H_n^{(\mp)}}{\delta u}, \quad n = 0,1,2,\ldots\,. \tag{3.8}$$

We note also that both of the Hamiltonian structures in (3.5) have Casimir functionals corresponding to conserved quantities whose gradients are annihilated by the appropriate Hamiltonian structure. Thus, for example, $\mathcal{D}_1$ has the Casimir functional H_{-1} defined in (3.3) such that

$$\mathcal{D}_1\,\frac{\delta H_{-1}}{\delta u} = 0\,. \tag{3.9}$$

Similarly, $\mathcal{D}_2$ has two Casimir functionals, namely, $H_0^{(-)}, H_0^{(+)}$ such that

$$\mathcal{D}_2\,\frac{\delta H_0^{(\mp)}}{\delta u} = 0\,. \tag{3.10}$$

This as well as (3.8) clarifies why we grouped the conserved charges into two infinite series even though they arise from the same Lax operator.

Because of the existence of Casimir functionals, it is possible to extend the flows into negative orders. In fact, let us note that the recursion operator (3.7) can be inverted in a closed form to give

$$\mathcal{R}^{-1} = \mathcal{D}_1^{-1}\mathcal{D}_2 = u^{-4}\partial^{-5}u^{-2}\partial u^{-1}\partial^{-3}u^{-1}\partial u^2\,, \tag{3.11}$$

and we can define negative flows for negative values of n through the recursion relation

$$\frac{\delta H_{-n-1}}{\delta u} = \mathcal{R}^{-1}\,\frac{\delta H_{-n}}{\delta u}, \quad n = 1, 2, \ldots . \tag{3.12}$$

Using these one can construct the charges associated with the negative flows recursively and the first few take the forms

$$\begin{aligned}
H_{-1} &= \int \mathrm{d}x\,\frac{1}{u^3}\,,\\
H_{-2} &= \frac{1}{18}\int \mathrm{d}x\,\left(\partial^{-2}u^{-3}\right)^3\,,\\
H_{-3} &= \int \mathrm{d}x\left(9\left(\partial^{-2}u^{-3}\right)\left(\partial^{-1}\left(\partial^{-2}u^{-3}\right)^2\right)^2 + 4\left(\partial^{-2}u^{-3}\right)^3\left(\partial^{-2}\left(\partial^{-2}u^{-3}\right)^2\right)\right)\,,
\end{aligned} \tag{3.13}$$

and so on. These nonlocal charges are conserved under the positive flows by construction, which can also be explicitly checked. However, at this time we do not know how to obtain them from the Lax operator.

The nonlocal Hamiltonians (3.13) lead to nonlocal flows of the form

$$u_{t_{-n}} = \mathcal{D}_1\,\frac{\delta H_{-n-1}}{\delta u} = \mathcal{D}_2\,\frac{\delta H_{-n}}{\delta u}, \quad n = 1, 2, \cdots . \tag{3.14}$$

The first few flows of the negative order have the explicit forms

$$\begin{aligned}
u_{t_{-1}} &= -9u^4\left(\left(\partial^{-2}u^{-3}\right)^2\right)_{xxx}\,,\\
u_{t_{-2}} &= u^4\left[\frac{3}{2}\left(\partial^{-2}u^{-3}\right)^2\partial^{-1}\left(\partial^{-2}u^{-3}\right)^2 + \left(\left(\partial^{-2}u^{-3}\right)^2\partial^{-2}\left(\partial^{-2}u^{-3}\right)^2\right)_x\right.\\
&\quad + \frac{2}{3}\left(\left(\partial^{-2}u^{-3}\right)\partial^{-2}\left(\partial^{-2}u^{-3}\right)^3\right)_x\\
&\quad \left. - 3\left(\left(\partial^{-2}u^{-3}\right)\partial^{-1}\left(\partial^{-2}u^{-3}\right)\partial^{-1}\left(\partial^{-2}u^{-3}\right)^2\right)_x\right]_{xx}\,,
\end{aligned} \tag{3.15}$$

and so on. We note here that if we introduce the variable $v_{xx} = u^{-3}$, then the first equation in (3.15) can also be rewritten as

$$v_{t_{-2}} = 54vv_x\,, \tag{3.16}$$

which is the Riemann equation.

The reason for the existence of two hierarchies of positive flows (2.3) is now clear. As observed earlier [11], because $\mathcal{D}_2$ has two Casimir functionals (as opposed to one for $\mathcal{D}_1$), a single hierarchy of negative flows splits up into two branches of positive flows. This in itself is an extremely interesting point. For, it may suggest a mechanism for a Lax description for several negative flows associated with other models.

4 Hodograph Transformation

In this section, we will study the behavior of the fifth order dispersive equation in (2.3) under a hodograph transformation [12]. Let us consider the change of variables

$$x = p(y, \tau), \quad u = p_y\,, \quad \tau = t\,. \tag{4.1}$$

Under such a transformation we have

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial y}{\partial x}\frac{\partial}{\partial y} = \frac{1}{p_y}\frac{\partial}{\partial y} = \frac{1}{u}\,\partial_y\,, \\ \frac{\partial}{\partial \tau} &= \frac{\partial}{\partial t} + \frac{p_\tau}{p_y}\frac{\partial}{\partial y}\,. \end{aligned} \tag{4.2}$$

Using this as well as the identification $u = p_y$, it is straightforward to see that the fifth order equation in (2.3) goes into

$$p_\tau = p_{5y} - 5\,\frac{p_{4y}p_{yy}}{p_y} + 5\,\frac{p_{3y}p_{yy}^2}{p_y^2}\,. \tag{4.3}$$

Introducing the variable $f = \frac{p_{yy}}{p_y}$, we can write (4.3) as

$$f_\tau = \left(f_{4y} + 5f_y f_{yy} - 5f^2 f_{yy} - f f_y^2 + f^5\right)_y\,. \tag{4.4}$$

This is exactly the equation considered by Fordy and Gibbons [13] who also showed that by a proper choice of the Miura transformation, it is possible to transform this equation to Kaup-Kupershmidt [14, 15] or Sawada-Kotera [16] equations. Indeed, if we choose the Miura transformation

$$g = f_y - \frac{1}{2}f^2\,, \tag{4.5}$$

then (4.3) takes the form

$$g_\tau = \frac{1}{6}\left(6g_{4y} + 60gg_{yy} + 45g_y^2 + 40g^3\right)_y\,, \tag{4.6}$$

which is the Kaup-Kupershmidt equation. On the other hand, a slightly modified Miura transformation of the form

$$g = -f_y + f^2\,, \tag{4.7}$$

takes (4.3) to

$$g_\tau = (g_{4y} + 5gg_{yy} + \frac{5}{3}g^3)_y\,. \tag{4.8}$$

This is, in fact, the Sawada-Kotera equation. This shows that the fifth order dispersive equation in (2.3) can be mapped into both Kaup-Kupershmidt and Sawada-Kotera equations under a hodograph transformation depending on the Miura transformation used.

This can be partly seen at the level of the Lax operator also. For example, we note that under the transformations (4.1), the Lax operator in (2.1) transforms to

$$\hat{L} = \partial_y^3 - 3f\partial_y^2 - (f_y - 2f^2)\partial_y\,, \tag{4.9}$$

where we have used the identification $f = \frac{p_{yy}}{p_y}$. The transformed Lax operator generates the following fifth order equation

$$\frac{\partial \hat{L}}{\partial t} = 9\left[\left(\hat{L}^{5/3}\right)_{\geq 1}, L\right]$$
$$\Longrightarrow\; f_t = (-f_{4y} - 5f_{yy}f_y + 5f_{yy}f^2 + 5f_y^2 f - f^5)_y\,. \tag{4.10}$$

A gauge transformation takes us to a new Lax operator of the form

$$\bar{L} = e^{-\int f}\hat{L}e^{\int f} = \partial_y^3 + 2g\partial_y + g_y\,, \tag{4.11}$$

where $g = f_y - \frac{1}{2}f^2$ defines the Miura transformation and we note that this is indeed the Lax operator considered in [15]. In terms of this transformed Lax operator, the equation

$$\frac{\partial \bar{L}}{\partial t} = 9\left[\left(L^{5/3}\right)_{+}, \bar{L}\right]\,, \tag{4.12}$$

leads to

$$g_t = \frac{1}{6}\left(6g_{4y} + 60gg_{yy} + 45g_y^2 + 40g^3\right)_y\,. \tag{4.13}$$

We recognize this to be the Kaup-Kupershmidt equation (4.6). However, starting from (2.1), we have been unable to obtain the Lax operator for the Sawada-Kotera equation using the hodograph transformation. We would like to point out that a Lax representation for the Sawada-Kotera equation is already known [15] and has the form

$$L = \partial^3 + g\partial\,. \tag{4.14}$$

Interestingly enough, under a hodograph transformation the seventh order equation in (2.3) goes over to the next higher equation of the Kaup-Kupershmidt or the Sawada-Kotera equation (depending on the Miura transformation used). This is interesting in that two distinct hierarchies associated with the system of equations (2.3) map onto a single hierarchy of equations under a hodograph transformation. Finally, to end this section, we note that one can carry out the hodograph transformation for the negative order flows in (3.15) as well and we simply point out that the first of these transforms under a hodograph transformation to the Liouville equation while ythe second has the form

$$p_\tau = 6v\partial^{-1}p_y\partial^{-1}p_yv^2 + 2\partial^{-1}p_y\partial^{-1}p_yv^3 - 9\partial^{-1}p_yv\partial^{-1}p_yv^2\,, \tag{4.15}$$

where $v = \partial^{-1}p_y\partial^{-1}p_y^{-2}$.

Finally let us mention that Kawamoto [17] has considered quite different fifth order equation then our equation

$$r_t = r^5 r_{xxxxx} + 5r^4 r_x r_{xxxx} + \frac{5}{2}r^4 r_{xx}r_{xxx} + \frac{15}{4}r^3 r_x^2 r_{xxx} \tag{4.16}$$

and used different transformation of dependent and nodependent variables

$$\frac{\partial}{\partial x} \to \frac{1}{r}\frac{\partial}{\partial y}\,, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial \tau} - r_x \frac{\partial}{\partial y}\,, \tag{4.17}$$

in order to put forward the connection of the equation (4.16) with the Kaup-Kupershmidt or Sawada-Kotera equation. However his transformation cannot be used to our equations (2.3) because the Kawamoto transformation breaks the time transformation. Indeed notice that our function r is zero dimensional and our time transformation (4.2) preserve weight in contrast to the Kawamoto transformation.

5 Generalization of the Lax Operator

We have already commented in section **2** on the fact that the second order Lax operator for the system of equations in (1.1) can be generalized and leads to a new third order equation (2.6). Under a hodograph transformation (4.1), it can be shown that this new equation (2.6) maps into a linear equation of the form

$$p_\tau = p_{yyy}\,. \tag{5.1}$$

In this section, we will study further generalizations of such Lax operators. First, let us consider the most general Lax operator of third order parametrized by only one zero-dimensional function u of the form

$$L = u^3\partial_x^3 + k_1 u^2 u_x \partial_x^2 + (k_2 u^2 u_{xx} + k_3 u u_x^2)\partial_x + (k_4 u^2 u_{xxx} + k_5 u u_x u_{xx} + k_6 u_x^3)\,, \tag{5.2}$$

where $k_i, i = 1..6$ are arbitrary constant coefficients. It can be checked that, at the level of fifth order equations, the Lax equation

$$\frac{\partial L}{\partial t} = \left[\left(L^{5/3}\right)_{\geq 2}, L\right]\,, \tag{5.3}$$

leads to consistent equations only for four choices of the coefficients k_i. For the two cases where $k_1 = 0$ or $k_1 = \frac{3}{2}$ and all other coefficients vanishing, we obtain the same fifth order equation as in (2.3), as we have already pointed out in section **2**. On the other hand, the Lax operator with $k_1 = 3$ with all other coefficients vanishing,

$$L = u^3\partial_x^3 + 3u^2 u_x \partial_x^2\,, \tag{5.4}$$

leads to the equation

$$u_t = u^5 u_{5x} + 5 u u_x u_{4x} + \frac{5}{2} u^4 u_{xx} u_{xxx} + \frac{15}{4} u^3 u_x^2 u_{xxx}\,. \tag{5.5}$$

(Parenthetically we remark here that under a hodograph transformation this Lax operator goes over to the one for the Sawada-Kotera equation.) Finally, for $k_1 = 3, k_2 = k_3 = 1$, with all other coefficients vanishing, the Lax operator

$$L = u^3\partial_x^3 + 3u^2 u_x \partial_x^2 + (u^2 u_{xx} + u u_x^2)\partial_x = (u\partial)^3\,, \tag{5.6}$$

yields the equation

$$u_t = \left(u^5 u_{4x} + 5u^4 u_x u_{3x} + 5u^4 u_{xx}^2 + 5u^3 u_x^2 u_{xx}\right)_x . \tag{5.7}$$

Under a hodograph transformation, it can be shown that with appropriate Miura transformations, equation (5.5) can be mapped to either Kaup-Kupershmidt or Sawada-Kotera equations, much like the earlier case discussed. However, under a hodograph transformation, interestingly enough, the highly nonlinear equation (5.7) maps into a linear equation of the form,

$$p_\tau = p_{5y} . \tag{5.8}$$

In a similar manner, one can carry out the general analysis of the fourth order Lax operator parameterized by only one variable. It is, of course, much more involved and we simply quote the results here. There are four such Lax operators that lead to consistent seventh order equations following from

$$\frac{\partial L}{\partial t} = \left[\left(L^{7/4}\right)_{\geq 2}, L\right] . \tag{5.9}$$

Three of these with

$$\begin{aligned} L_1 &= u^4 \partial_x^4 + 6u^3 u_x \partial_x^3 + \left(6u^2 u_x^2 + 2u^3 u_{xx}\right) \partial_{xx} , \\ L_2 &= u^4 \partial_x^4 + 4u^3 u_x \partial_x^3 + 2\left(u^2 u_x^2 + u^3 u_{xx}\right) \partial_x^2 = \left(u^2 \partial_x^2\right)^2 , \\ L_3 &= u^4 \partial_x^4 + 2u^3 u_x \partial_x^3 , \end{aligned} \tag{5.10}$$

lead to the same dynamical equation (whose form is complicated and we do not write it here). On the other hand,

$$\begin{aligned} L_4 &= u^4 \partial_x^4 + 6u^3 u_x \partial_x^3 + \left(4u^3 u_{xx} + 7u^2 u_x^2\right) \partial_x^2 \\ &+ \left(u^3 u_{xxx} + 4u^2 u_x u_{xx} + u u_x^3\right) \partial_x = (u\partial)^4 , \end{aligned} \tag{5.11}$$

leads to a very different equation that is highly nonlinear

$$\begin{aligned} u_t = \big(&u_{6x} u^7 + 14 u_{5x} u_x u^6 + 28 u_{4x} u_{xx} u^6 + 56 u_{4x} u_x^2 u^5 \\ &+ 14 u_{3x}^2 u^6 + 168 u_{3x} u_{xx} u_x u^5 + 70 u_{3x} u_x^3 u^4 + 42 u_{xx}^3 u^5 \\ &+ 126 u_{xx}^2 u_x^2 u^4 + 21 u_{xx} u_x^4 u^3\big)_x v . \end{aligned} \tag{5.12}$$

However, under a hodograph transformation (4.1), this equation goes over to a seventh order linear equation of the form

$$p_\tau = p_{7y} . \tag{5.13}$$

Equations (5.1), (5.8) and (5.13) are quite puzzling and even more so because of the fact that under the hodograph transformation, the Lax operators go over respectively to

$$L^{(2)} \to \partial_y^2, \quad L_4^{(3)} \to \partial_y^3, \quad L_4^{(4)} \to \partial_y^4 . \tag{5.14}$$

All these equations interestingly follows from the same Lax operator $L = u\partial$. Indeed this Lax operator constitute the following hierarchy of equations

$$\frac{\partial L}{\partial t_n} = [L^{n+1}_{\geq 2}, L] \,, \tag{5.15}$$

where $L = u\partial$ and $n = 1, 2, 3, ...$ For $n = 1$ we obtain

$$u_{t_1} = u^2 u_{xx} \,, \tag{5.16}$$

while for n=2 we obtain the equation (2.6). For $n = 4$ and for $n = 6$ we have equations (5.7) and (5.12) respectively. Interestingly all these equations belongs to the lineralizable hierarchy of nonlinear partial differential equations considered by M. Euler, N. Euler and N. Petersson [18, 19]. It is possible to write these equations in terms of the following recursion operator [18]

$$u_{t_n} = R^{n-1} u_{t_1} \,, \tag{5.17}$$

where $n = 1, 2, 3, ...$ and

$$R = u\partial + u^2 u_{xx} \partial^{-1} u^{-2} \,. \tag{5.18}$$

In order to prove that this recursion operator generates the whole hierarchy of equations (5.15) we extract this operator directly from the Lax operator using a generalization of the method by Gurses, Karasu and Sokolov [20]. Let us note that

$$L^{n+1}_{\geq 2} = (L(L^{(n-1)+1})_{\geq 2})_{\geq 2} + (L(L^n)_{<2})_{\geq 2} = L(L^{(n-1)+1})_{\geq 2} + S \,, \tag{5.19}$$

where $L = u\partial$ in the present case and it follows that $S = a\partial^2$ with a an arbitray function to be determined. Substituting this into (5.15) we obtain

$$\frac{\partial L}{\partial t_n} = L \frac{\partial L}{\partial t_{n-1}} + a u_{xx} \partial + (2a u_x - u a_x) \partial^2 \,, \tag{5.20}$$

which determines

$$a = u^2 \partial^{-1} \frac{u_{t_{n-1}}}{u^2} \,. \tag{5.21}$$

With this equation (5.20) leads to

$$u_{t_n} = R u_{t_{n-1}} \,, \tag{5.22}$$

where R coincides with (5.18) for $n > 1$.

6 Conclusion

In this paper, we have studied the properties of a nonlinearly dispersive integrable system of fifth order and its associated hierarchy. We have described a Lax representation for such a system which leads to two infinite series of conserved charges and two hierarchies of equations that share the same conserved charges. We have constructed two compatible

Hamiltonian structures as well as their Casimir functionals. One of the structures has a single Casimir functional while the other has two. This allows us to extend the flows into negative order and clarifies the meaning of two different hierarchies of positive flows. We have studied the behavior of these systems under a hodograph transformation and have shown that they are related to the Kaup-Kupershmidt and the Sawada-Kotera equations under appropriate Miura transformations. We have also discussed briefly some properties associated with the generalization of second, third and fourth order Lax operators.

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