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Subgroup Type Coordinates and the Separation of Variables in Hamilton-Jacobi and Schrödinger Equations

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Abstract

Separable coordinate systems are introduced in complex and real four-dimensional flat spaces. We use maximal Abelian subgroups to generate coordinate systems with a maximal number of ignorable variables. The results are presented (also graphically) in terms of subgroup chains. Finally the explicit solutions of the Schrödinger equation in the separable coordinate systems are computed.

1 Introduction

The bicentennial of both Carl Gustav Jacob Jacobi and William Rowan Hamilton provides us with an excellent opportunity to take a new look at the equation associated with both of their names, as well as its twentieth-century descendant, the Schrödinger equation. The integrability, or superintegrability [11, 14, 15, 16, 18, 31, 33], of these equations, i.e. the existence of n, respectively k (with $n+1 \leq k \leq 2n-1$) integrals of motion, belongs to the fundamental problems concerning any classical, or quantal, Hamiltonian system. Of course much has transpired since the days of Hamilton and Jacobi. In particular Lie group theory has been created and its power applied to classical and quantum mechanics.

Among Hamiltonian systems that are integrable a special class consists of those that allow the separation of variables in the Hamilton-Jacobi and Schrödinger equations. This occurs typically when the integrals of motion are polynomials quadratic in the momenta in classical mechanics or second-order differential operators in quantum mechanics. Superintegrable systems with more than n second-order integrals of motion are typically multiseparable, i.e. separable in more than one coordinate system.

An extensive literature exists on Lie theory and the separation of variables [1, 3, 5, 6, 9, 10, 11, 12, 13, 14, 15, 22, 26, 27, 30]. Experts on the separation of variables immediately think of ellipsoidal coordinates and degenerate cases thereof. Most "practitioners" think of the simplest types of coordinates in Euclidean three-space, cartesian, cylindrical and spherical coordinates. These coordinates have been called "subgroup type coordinates" because they are related to different subgroup chains of the Lie group G, the isometry group of the space under consideration [15, 19, 22, 30, 31].

The purpose of this article is to analyze further these subgroup type coordinates. They exist in any space with a nontrivial isometry group. We consider complex and real Euclidean spaces, as well as pseudo-Euclidean real spaces and their isometry groups $E(n,\mathbb{C})$, $E(n)$ and $E(p,q)$, respectively. We see that a much greater variety of such coordinates exists for the $E(n,\mathbb{C})$ and $E(p,q)$ groups than for real Euclidean ones. Moreover some of them have new and interesting properties. The coordinates are not necessarily orthogonal. The separated ordinary differential equations are not necessarily of second order; quite often they are first-order equations and the solutions then involve elementary functions rather than special ones.

Here we consider free motion only, that is, there is no potential in the Hamiltonian. Once separable coordinates and the corresponding integrals are established, it is an easy task to add a potential to the Hamiltonian and modify the integrals of motion in such a manner as to preserve separability [5, 6, 11, 15].

From the mathematical point of view this article is an application of a research programme, the aim of which is to classify the maximal Abelian subalgebras (MASAs) of all classical Lie algebras. In particular in earlier articles [17, 24, 25] we presented a classification of MASAs of the $e(n,\mathbb{C})$ and $e(p,q)$ algebras. Here we apply this classification to the problem at hand.

The problem we are considering can be posed as follows. Consider a complex, or real, n-dimensional Riemannian or pseudo-Riemannian space S with metric

$$ds^2 = \sum_{i,k=1}^{n} g_{ik}(x)dx^i dx^k \tag{1.1}$$

and an isometry group G of dimension $N \geq n$. We wish to construct all coordinate systems that satisfy the following requirements.

1. They allow the separation of variables in the time-independent free Schrödinger equation

$$H\Psi = E\Psi, \quad H = -\frac{1}{2}\frac{1}{\sqrt{g}}\sum_{i,k=1}^{n}\frac{\partial}{\partial x^i}\sqrt{g}g^{ik}\frac{\partial}{\partial x^k}, \quad g_{ik}g^{kl} = \delta_{il}, \quad g = det(g_{ik}) \tag{1.2}$$

and also in the Hamilton-Jacobi equation

$$\sum_{i,k=1}^{n} g^{ik}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_k} = E. \tag{1.3}$$

Thus for the Schrödinger equation (1.2) and Hamilton-Jacobi equation (1.3) we require

$$\Psi = \prod_{i=1}^{n} \Psi_i(x^i, \lambda_1, \ldots, \lambda_n), \tag{1.4}$$

$$W = \sum_{i=1}^{n} W_i(x^i, \lambda_1, \ldots, \lambda_n), \tag{1.5}$$

respectively, where $\lambda_1, \ldots, \lambda_n$ are constants of separation.

2. All coordinates x_i are coordinates of subgroup type. By this we mean that a general element of the isometry group G is written as a product of one-dimensional subgroups $g = g_1 g_2 \ldots g_N$, $N = dim\ G$ and the coordinates x_i are generated by the action of G on some chosen origin $|o\rangle$

$$|x\rangle = g|o\rangle. \tag{1.6}$$

3. Each coordinate system contains a maximal number of ignorable coordinates (i.e. variables not figuring in the metric tensor $g_{ik}(x)$) generated by a maximal Abelian subgroup $G_M \subset G$.

The above conditions are satisfied by parametrizing the group element $g \in G$ as follows

$$g = g_M(a_1, \ldots, a_k) h(s_1, \ldots, s_l) g_0(u_1, \ldots, u_m) \qquad k + l + m = \mathrm{dim} G, \quad k + l = \mathrm{dim} M. \tag{1.7}$$

In eq (1.7) g_M is a maximal Abelian subgroup, h is a product of one-parameter subgroups (not necessarily a subgroup itself) and g_0 is the isotropy group of the origin $|0\rangle$. The group parameters a_i provide ignorable variables while the parameters s_i are the "essential" variables (that do figure in $g_{ik}(x) \equiv g_{ik}(s_1, \ldots, s_l)$).

Not every maximal Abelian subgroup is suitable for this purpose. The requirement is that g_M, when acting on a generic point in space S, should sweep out orbits of dimension k (not lower-dimensional ones).

In the process we also solve a more general problem, namely that of constructing all coordinates of subgroup type, also those not involving a maximal number of ignorable variables.

We actually work with the Schrödinger equation (1.2) only and obtain separable coordinates $(x_1, \ldots, x_n)$ in the sense of eq (1.4). The additive separation (1.5) for the Hamilton-Jacobi equation in the same coordinate system follows automatically. The separated wave functions (1.4) are eigenfunctions of a complete set of commuting operators

$$\{X_1, X_2, \ldots, X_n\}. \tag{1.8}$$

Among them k operators are basis elements of a maximal Abelian subalgebra (MASA) of the Lie algebra of the isometry group. One operator is the Hamiltonian H and the others are all second-order Casimir operators of subgroups in a subgroup chain

$$G \supset G_1 \supset G_2 \supset \ldots \supset G_M, \tag{1.9}$$

where G_M is a maximal Abelian subgroup of G.

More specifically we consider a flat space M with a complex Euclidean, real Euclidean or pseudo-Euclidean isometry group. We realize its Lie algebra, $e(n,\mathbb{C})$, $e(n)$ or $e(p,q)$, by matrices

$$E = \begin{pmatrix} X & \alpha \\ 0 & 0 \end{pmatrix}, \qquad X \in F^{n\times n}, \qquad \alpha \in F^{n\times 1}, \tag{1.10}$$
$$XK + KX^T = 0, \qquad K = K^T, \qquad \det K \neq 0,$$

where we have $F = \mathbb{R}$ or $F = \mathbb{C}$. For $F = \mathbb{R}$ the matrix K determining the metric has a signature

$$\mathrm{sgn} K = (p,q), \qquad p \geq q \geq 0, \tag{1.11}$$

(i.e. p positive and q negative eigenvalues).

We use several different choices of the matrix K. Different choices of K and X are related by the transformation

$$GK_1G^T = K_2, \qquad GX_1G^{-1} = X_2, \qquad G \subset GL(N,F). \tag{1.12}$$

Coordinates in the space S have the form

$$|y\rangle = \begin{pmatrix} |x\rangle \\ 1 \end{pmatrix}, \quad x \in F^{n\times 1}, \quad |0\rangle = \begin{pmatrix} |o\rangle \\ 1 \end{pmatrix}, \tag{1.13}$$

where $|o\rangle$ is chosen to be the origin of the space M. The isotropy group of the origin is $G_0 \sim O(n,\mathbb{C})$ for $F = \mathbb{C}$, $G_0 \sim O(p,q)$, $p+q=n$ for $F = \mathbb{R}$.

We give some decomposition theorems in Section 2 for arbitrary n and then concentrate on the case $n = 4$. This is the most interesting case from the point of view of physical applications, mainly in the context of special and general relativity [3, 29, 30] but also in that of integrable and superintegrable systems [16, 18, 21]. Separation of variables also plays a role in the study of Huygens' principle [2], where dimension 4 is again the one of physical importance.

From the mathematical point of view $\dim M = 4$ is sufficiently simple that it can be treated in a complete and detailed manner. On the other hand it is rich enough to demonstrate most of the phenomena that occur for any value of n.

2 Decompositions of spaces and algebras

2.1 Chains of subgroups and decompositions of $M(n)$

The role of subgroup chains in the study of coordinate separation has been emphasized in many articles [9, 19, 22, 28]. For real Euclidean groups $E(n)$ and their Lie algebras $e(n)$ the situation is quite simple. Only two types of maximal subgroup exist, namely the following

$$E(n) \supset E(n_1)\otimes E(n_2), \quad n_1+n_2=n, \quad n_1 \geq n_2 \geq 1 \tag{2.1}$$
$$E(n) \supset O(n), \qquad n \geq 2. \tag{2.2}$$

Maximal subgroups of $O(n)$ can be embedded in the defining representation of $O(n)$ reducibly or irreducibly. The reducibly embedded ones leave a vector subspace of the Euclidean space $M(n)$ invariant. The corresponding maximal subgroups are one of the following:

$$O(n) \supset O(n_1) \otimes O(n_2), \quad n_1 + n_2 = n, \quad n_1 \geq n_2 \geq 2 \tag{2.3}$$
$$O(n) \supset O(n-1), \qquad n \geq 3. \tag{2.4}$$

The subgroup link (2.1) leads to the decomposition of the Euclidean space $M(n)$ into the direct sum of two Euclidean subspaces

$$M(n) = M(n_1) \oplus M(n_2), \quad n_1 + n_2 = n, \quad n_1 \geq n_2 \geq n. \tag{2.5}$$

Separable coordinates can then be introduced separately on $M(n_1)$ and $M(n_2)$; a lower-dimensional task. The subgroup chain (2.2) leads to the embedding of a sphere S_{n-1} into $M(n)$. The links (2.3) and (2.4) lead to various types of spherical and polyspherical coordinates on this sphere [9, 26, 27].

Irreducibly embedded subgroups of $O(n)$, like $U(n) \subset O(2n)$, or $G_2 \subset O(7)$, have not been used to generate separable coordinates.

For complex Euclidean groups, $E(n, \mathbb{C})$, and pseudo-Euclidean groups, $E(p, q)$, the first subgroup links are essentially the same as (2.1) and (2.2) (mutatis mutandis), e.g. for $E(p, q)$ we have two possibilities:

$$E(p,q) \supset E(p_1,q_1) \otimes E(p_2,q_2), \quad p_1 + p_2 = p, \quad q_1 + q_2 = q \tag{2.6}$$
$$E(p,q) \supset O(p,q), \qquad n = p + q \geq 2. \tag{2.7}$$

However, the subgroup links of the type (2.3) and (2.4) are not the only ones for $O(n, \mathbb{C})$ or $O(p, q)$ with $p \geq q \geq 1$. In particular the possibilities for $O(p, 1)$ were discussed in Ref. [22] and are

$$O(p,1) \supset O(p), \qquad O(p,1) \supset O(p-1,1) \tag{2.8}$$
$$O(p,1) \supset O(p_1,1) \otimes O(p_2), \quad p_1 + p_2 = p, \quad p_1 \geq 1, p_2 \geq 2 \tag{2.9}$$

and also

$$O(p,1) \supset E(p-1). \tag{2.10}$$

For $O(p, q)$, $p \geq q \geq 2$, the possibilities are even richer as we see below in the case of $O(2, 2)$.

In any case we are not interested in subgroup links of the type (2.1) or (2.6) since they lead to a decomposition of the space M considered and we assume that the problem is already solved in lower dimensions. In other words we are only interested in "indecomposable" coordinate systems in the space M.

2.2 Decomposition of MASAs

Maximal Abelian subalgebras (MASAs) of Euclidean Lie algebras can contain generators of translations. In the real case, $e(n)$, these translations can only be space-like and their presence in a subgroup chain leads to the decomposition of the space $M(n)$. For $E(n,\mathbb{C})$ there can be up to $[\frac{n}{2}]$ lightlike (isotropic) translations in a MASA and for $e(p,q)$, $p \geq q \geq 1$, up to q lightlike translations. These do not lead to a decomposition of $M(n)$, or $M(p,q)$ respectively and must hence be considered. The algebras $o(n,\mathbb{C})$ allow three types of MASAs [7]:

1. Orthogonally decomposable (OD) MASAs
2. MASAs that are decomposable but not orthogonally (OID but D)
3. Indecomposable MASAs (OID and ID).

The MASAs of $o(n,\mathbb{C})$ have been classified elsewhere [7]. The algebras $o(p,q)$ have up to six types of MASAs, depending on their decomposability properties over the real numbers and their behavior under complexification [8]. All types of MASAs occur in $e(4,\mathbb{C})$ and $e(2,2)$, respectively.

3 Separable coordinates in the complex space $M(4,\mathbb{C})$

3.1 MASAs of $e(n,\mathbb{C})$.

The MASAs of $e(n,\mathbb{C})$ have been classified into conjugacy classes under the action of the group $E(n,\mathbb{C})$ in an earlier publication [17]. Each class of MASAs is represented by one "canonical" MASA. In any basis a MASA of $e(n,\mathbb{C})$ contains k_0+k_1 mutually orthogonal translations, k_0 of them isotropic, k_1 anisotropic. We have

$$0 \leq k_0 \leq \left[\frac{n}{2}\right], \qquad 0 \leq k_1 \leq n, \qquad 0 \leq k_0+k_1 \leq n. \tag{3.1}$$

We are mainly interested in MASAs with $k_1 = 0$ since the presence of anisotropic translations leads to a decomposition of the space into $M(n,\mathbb{C}) \equiv M(k_1,\mathbb{C}) \oplus M(n-k_1,\mathbb{C})$ and hence to decomposable coordinate systems.

We now consider the four-dimensional space $M(4,\mathbb{C})$. We denote the MASAs of $e(4,\mathbb{C})$ as $M_{4,j}(k_0)$. We run through all inequivalent MASAs and construct the separable coordinates. In each case we represent the MASA by a matrix $X \in \mathbb{C}^{5\times 5}$ and also the corresponding metric $K = K^T \in \mathbb{C}^{4\times 4}$, $\det K \neq 0$ (see eq.(1.10)). The coordinates are generated by a group action as in eq.(1.7). More specifically we write

$$|y\rangle = \begin{pmatrix} |x\rangle \\ 1 \end{pmatrix} = g \begin{pmatrix} |o\rangle \\ 1 \end{pmatrix}, \qquad |x\rangle = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad |o\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.2}$$

The group action is specified by giving $g = g_1g_2g_3g_4$, where each g_i is a one-dimensional subgroup of $E(4,\mathbb{C})$. The MASAs are either two- or three-dimensional (for indecomposable coordinate systems). They generate g_1 and g_2 for two-dimensional MASAs and g_1, g_2 and g_3 for three-dimensional ones.

We also give the complete sets of commuting operators (1.8) related to each MASA. For three-dimensional MASAs the set consists of $\{\square_4, X_1, X_2, X_4\}$, where $\square_4$ is the Laplace operator and $\{X_1, X_2, X_3\}$ generates the MASA. For two-dimensional MASAs the set consists of $\square_4, X_1, X_2$ and Y, where Y is a second-order Casimir operator of a subalgebra L satisfying

$$\{X_1, X_2\} \subset L \subset e(4, \mathbb{C}). \tag{3.3}$$

In each case we give L and its Casimir operator.

The notation E_{ik} denotes the matrices satisfying $(E_{ik})_{ab} = \delta_{ia}\delta_{kb}$, always in a basis corresponding to the metric K.

3.2 MASAs of $e(4, \mathbb{C})$ with $k_0 = 0$

There are two MASAs with $k_0 = 0$.

1. The Cartan subalgebra $M_{4,1}(0) \sim o(2, \mathbb{C}) \oplus o(2, \mathbb{C})$

$$M_{4,1}(0) = \begin{pmatrix} 0 & -a & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}. \tag{3.4}$$

The coordinates on the complex Euclidean space $M_4(\mathbb{C})$ are generated by the following group action on the origin:

$$|y\rangle = e^{-a(E_{12}-E_{21})} e^{-b(E_{34}-E_{43})} e^{-c(E_{13}-E_{31})} e^{rE_{15}} |0\rangle. \tag{3.5}$$

The variables then are:

$$\begin{aligned} x_1 &= r\cos c\cos a \qquad & x_3 &= r\sin c\cos b \\ x_2 &= r\cos c\sin a \qquad & x_4 &= r\sin c\sin b. \end{aligned} \tag{3.6}$$

Here r, c, a and b are all complex. For r constant we have a complex sphere on which a and b provide cylindrical coordinates. The subgroup chain that provides these coordinates and the complete set of commuting operators is

$$E(4, \mathbb{C}) \supset O(4, \mathbb{C}) \supset O(2, \mathbb{C}) \otimes O(2, \mathbb{C}). \tag{3.7}$$

The Laplace operator $\square_{4\mathbb{C}}$ in this case is

$$\square_{4\mathbb{C}} = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_{LB}, \tag{3.8}$$

where

$$\triangle_{LB} = \frac{\partial^2}{\partial c^2} + 2\cot 2c\frac{\partial}{\partial c} + \frac{1}{\cos^2 c}\frac{\partial^2}{\partial a^2} + \frac{1}{\sin^2 c}\frac{\partial^2}{\partial b^2} \tag{3.9}$$

is the Laplace-Beltrami operator on the sphere $S_4(\mathbb{C})$ (the Casimir operator of $O(4,\mathbb{C})$ that does not vanish on the sphere). The complete set of commuting operators in this case consists of

$$\{\Box_{4\mathbb{C}},\, \triangle_{LB},\, L_{12}=E_{12}-E_{21},\, L_{34}=E_{34}-E_{43}\}. \tag{3.10}$$

The one parameter subgroups, L_{12}, L_{34} and $L_{13}=E_{13}-E_{31}$, figuring in the chain (3.5) generate all of $O(4,\mathbb{C})$.

2. The orthogonally indecomposable but decomposable (OID but D) MASA of $o(4,\mathbb{C})$

$$M_{4,2}(0)=\begin{pmatrix} a & b & 0 & 0 & 0\\ 0 & a & 0 & 0 & 0\\ 0 & 0 & -a & 0 & 0\\ 0 & 0 & -b & -a & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}, \quad K=\begin{pmatrix} 0 & I_2\\ I_2 & 0\end{pmatrix}. \tag{3.11}$$

The coordinates on $M_4(\mathbb{C})$ are generated by the following group action:

$$|y\rangle = e^{a(E_{11}+E_{22}-E_{33}-E_{44})}e^{b(E_{12}-E_{43})}e^{c(E_{11}-E_{22}-E_{33}+E_{44})}e^{r(E_{15}+E_{25}+E_{35}+E_{45})}|0\rangle. \tag{3.12}$$

We get

$$\begin{array}{ll} x_1=\frac{1}{2}re^{a}(e^{c}+be^{-c}) & x_3=\frac{1}{2}re^{-a}e^{-c}\\ x_2=\frac{1}{2}re^{a}e^{-c} & x_4=\frac{1}{2}re^{-a}(e^{c}-be^{-c}). \end{array} \tag{3.13}$$

The subgroup chain is

$$E(4,\mathbb{C})\supset O(4,\mathbb{C})\supset \exp(M_{4,2}(0)) \tag{3.14}$$

and the complete set of commuting operators is

$$\{\Box_{4\mathbb{C}},\, \triangle_{LB},\, X_1,\, X_2\}, \tag{3.15}$$

where $\Box_{4\mathbb{C}}$ is the operator in (3.8) and $\triangle_{LB}$ is

$$\triangle_{LB}=-\frac{\partial^2}{\partial c^2}+2\frac{\partial}{\partial c}+4e^{4c}\frac{\partial}{\partial b^2}-4e^{2c}\frac{\partial^2}{\partial a\partial b}. \tag{3.16}$$

The first three subgroups in (3.12) are all contained in a subgroup $GL(2,\mathbb{C})\subset O(4,\mathbb{C})$. However, the Casimir operator of this subgroup coincides with that of $O(4,\mathbb{C})$.

The coordinates (3.13) lead to a nonorthogonal separation of variables in $\triangle_{LB}$ and hence in $\Box_{4\mathbb{C}}$.

3.3 MASAs of $e(4,\mathbb{C})$ with $k_0 = 1$

All MASAs of $e(4,\mathbb{C})$ containing one isotropic translation can be written in the form [17, 25]

$$X = \begin{pmatrix} 0 & \alpha & 0 & z \\ 0 & 0 & -K_0\alpha^T & BK_0\alpha^T \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & K_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.17}$$

$$\begin{array}{lll} \alpha = (a_1, a_2), & K_0 = K_0^T \in \mathbb{C}^{2\times 2}, & \det K_0 \neq 0, \\ BK_0 = K_0 B^T, & B \in \mathbb{C}^{2\times 2}. & \end{array}$$

The pair of matrices (B, K_0) can be transformed into one of the following standard forms [4]

$$B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad K_0 = I_2 \tag{3.18}$$

$$B_2 = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}, \quad \begin{array}{lll} \beta = |\beta| e^{i\phi}, & 0 \le |\beta| < 1, & 0 \le \phi < 2\pi \\ |\beta| = 1, & 0 \le \phi < \pi, & K_0 = I_2 \end{array} \tag{3.19}$$

$$B_3 = \begin{pmatrix} \kappa & 0 \\ 1 & \kappa \end{pmatrix}, \quad \kappa = \begin{cases} 0 \\ 1 \end{cases}, \quad K_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.20}$$

We mention that the MASAs (3.17) are maximal Abelian nilpotent subalgebras of $e(4,\mathbb{C})$ (MANS)[23, 32]. This implies that they are represented by nilpotent matrices in any finite-dimensional representation.

The corresponding MASAs are all three-dimensional and directly provide three commuting operators and three ignorable variables. We continue our list of MASAs and coordinates:

1.

$$M_{4,3}(1) = \begin{pmatrix} 0 & -a_1 & -a_2 & 0 & z \\ 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.21}$$

$$|y\rangle = e^{a_1(-E_{12}+E_{24})} e^{a_2(-E_{13}+E_{34})} e^{zE_{15}} e^{rE_{45}} |0\rangle \tag{3.22}$$

$$\begin{array}{ll} x_1 = z - \frac{1}{2} r(a_1^2 + a_2^2) & x_3 = ra_2 \\ x_2 = ra_1 & x_3 = ra_2. \end{array} \tag{3.23}$$

The $M(4,\mathbb{C})$ Laplace operator with these new variables is

$$\square_{4C} = 2\frac{\partial^2}{\partial z \partial r} + \frac{2}{r}\frac{\partial}{\partial z} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial a_1^2} + \frac{\partial^2}{\partial a_2^2}\right). \tag{3.24}$$

2.

$$M_{4,4}(1) = \begin{pmatrix} 0 & -a_1 & -a_2 & 0 & z \\ 0 & 0 & 0 & a_1 & a_1 \\ 0 & 0 & 0 & a_2 & a_2\beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.25}$$

$$|y\rangle = e^{a_1(-E_{12}+E_{24}+E_{25})} e^{a_2(-E_{13}+E_{34}+\beta E_{35})} e^{zE_{15}} e^{rE_{45}} |0\rangle \tag{3.26}$$

$$\begin{array}{lcl} x_1 & = & z - \frac{r}{2}(a_1^2 + a_2^2) - \frac{1}{2}(a_1^2 + \beta a_2^2) \\ x_2 & = & (r+1)a_1 \\ x_3 & = & (r+\beta)a_2 \\ x_4 & = & r. \end{array} \qquad \begin{array}{l} \beta = |\beta| e^{i\phi} \\ 0 \le |\beta| < 1, \qquad 0 \le \phi < 2\pi \\ |\beta| = 1, \quad 0 \le \phi < \pi \end{array} \tag{3.27}$$

The Laplace operator is

$$\Box_{4C} = 2\frac{\partial^2}{\partial r \partial z} + \frac{2r+\beta+1}{(r+1)(r+\beta)}\frac{\partial}{\partial z} + \frac{1}{(r+1)^2}\frac{\partial^2}{\partial a_1^2} + \frac{1}{(r+\beta)^2}\frac{\partial^2}{\partial a_2^2}. \tag{3.28}$$

3.

$$M_{4,5}(1) = \begin{pmatrix} 0 & -a_1 & -a_2 & 0 & z \\ 0 & 0 & 0 & a_2 & \kappa a_2 \\ 0 & 0 & 0 & a_1 & \kappa a_1 + a_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.29}$$

$$|y\rangle = e^{a_1(-E_{12}+E_{34}+\kappa E_{35})} e^{a_2(-E_{13}+E_{24}+E_{35}+\kappa E_{25})} e^{zE_{15}} e^{rE_{45}} |0\rangle \tag{3.30}$$

$$\begin{array}{ll} x_1 = z - (r+\kappa)a_1 a_2 - \frac{1}{2}a_2^2 & x_3 = (r+\kappa)a_1 + a_2 \\ x_2 = (r+\kappa)a_2 & x_4 = r \qquad\qquad \kappa = 0 \quad \text{or} \quad 1. \end{array} \tag{3.31}$$

The Laplace operator in these coordinates is

$$\Box_{4C} = 2\frac{\partial^2}{\partial r \partial z} + \frac{2}{(r+\kappa)}\frac{\partial}{\partial z} - \frac{2}{(r+\kappa)^3}\frac{\partial^2}{\partial a_1^2} + \frac{2}{(r+\kappa)^2}\frac{\partial^2}{\partial a_1 \partial a_2}. \tag{3.32}$$

3.4 MASAs of $e(4,\mathbb{C})$ with $k_0 = 2$

There are two mutually inequivalent MASAs of $e(4,\mathbb{C})$ with two isotropic translations [17, 25]. Both of them are three-dimensional. One of them has three-dimensional orbits in $M(4,\mathbb{C})$ and the other only two-dimensional ones. We consider the two of them separately.

$$M_{4,6}(2) = \begin{pmatrix} 0 & 0 & 0 & z & a_1 \\ 0 & 0 & -z & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{3.33}$$

Coordinates are generated by the action:

$$|y\rangle = e^{z(E_{14}-E_{23}+E_{45})} e^{a_1 E_{15}} e^{a_2 E_{25}} e^{r E_{35}} |0\rangle \tag{3.34}$$

$$\begin{aligned} x_1 &= a_1 + \tfrac{1}{2} z^2 \qquad & x_3 &= r \\ x_2 &= a_2 - rz \qquad & x_4 &= z. \end{aligned} \tag{3.35}$$

The Laplace operator is

$$\Box_{4C} = 2\left(\frac{\partial^2}{\partial a_1 \partial r} + r\frac{\partial^2}{\partial a_2^2} + \frac{\partial^2}{\partial a_2 \partial z}\right). \tag{3.36}$$

The other MASA is represented by

$$M_{4,7}(2) = \begin{pmatrix} 0 & 0 & 0 & z & a_1 \\ 0 & 0 & -z & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{3.37}$$

The orbits that the three-parameter subgroup sweeps out are only two-dimensional. This is reflected in the fact that the generators of the subalgebra when acting on $M(4,\mathbb{C})$ are represented, in the metric considered, by the commuting operators

$$P_1 = \partial_{s_1}, \quad P_2 = \partial_{s_2}, \quad B = s_4 \partial_{s_1} - s_3 \partial_{s_2}. \tag{3.38}$$

These are linearly connected, i.e., they span a two-dimensional subspace of the tangent space rather than a three-dimensional one. The operator B can hence not be straightened, i.e. $s_1 = a_1$ and $s_2 = a_2$ are already ignorable variables. Hence B must be dropped from the MASA. The algebra, $\{P_1, P_2\}$, itself generates coordinates, equivalent to Cartesian ones in $M(4,\mathbb{C})$ in which all four translations are simultaneously diagonalized and the space $M(4,\mathbb{C})$ is decomposed.

$$\begin{aligned} x_1 &= a_1 \qquad & x_3 &= s_3 \\ x_2 &= a_2 \qquad & x_4 &= s_4. \end{aligned} \tag{3.39}$$

The Laplace operator is

$$\Box_{4C} = 2\left(\frac{\partial^2}{\partial a_1 \partial s_3} + \frac{\partial^2}{\partial a_2 \partial s_4}\right). \tag{3.40}$$

3.5 Decompositions of $M(4,\mathbb{C})$

Possible decompositions of $M(4,\mathbb{C})$ are

$$\begin{array}{ll} M(4,\mathbb{C}) = M(3,\mathbb{C}) \oplus M(1,\mathbb{C}), & M(4,\mathbb{C}) = 2M(2,\mathbb{C}) \\ M(4,\mathbb{C}) = M(2,\mathbb{C}) \oplus 2M(1,\mathbb{C}), & M(4,\mathbb{C}) = 4M(1,\mathbb{C}). \end{array} \tag{3.41}$$

In order to obtain a complete list of subgroup-type coordinates on $M(4,\mathbb{C})$ we must also take these decompositions into account and introduce indecomposable coordinate systems on them. Each $M(1,\mathbb{C})$ space corresponds to a Cartesian coordinate. Each $M(2,\mathbb{C})$ corresponds to complex polar coordinates.

Two types of indecomposable coordinate systems exist on $M(3,\mathbb{C})$, corresponding to two different MASAs, both of them two-dimensional. The two MASAs of $e(3,\mathbb{C})$ can be written as

$$X = \begin{pmatrix} 0 & a & 0 & z \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -\kappa a \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.42}$$

with $\kappa = 0$ for the first and $\kappa = 1$ for the second. We consider the two cases separately.

Case I: $\kappa = 0$

The coordinates are induced by the action

$$|y\rangle = e^{a(E_{12}-E_{23})} e^{zE_{14}} e^{rE_{34}} |0\rangle \tag{3.43}$$

with $|0\rangle = (0\,0\,0\,1)^T$. The coordinates on $M(3,\mathbb{C})$ are

$$x_1 = z - \frac{1}{2} r a^2, \qquad x_2 = -ar, \qquad x_3 = r. \tag{3.44}$$

The Laplace operator on $M(3,\mathbb{C})$ is

$$\Box_{3C} = 2\frac{\partial^2}{\partial r \partial z} + \frac{1}{r^2}\frac{\partial^2}{\partial a^2} + \frac{1}{r}\frac{\partial}{\partial z}. \tag{3.45}$$

We mention that these coordinates are conformally equivalent to Cartesian ones on $M(3,\mathbb{C})$ [2, 24].

Case II: $\kappa = 1$

The coordinates are induced by the following action

$$|y\rangle = e^{a(E_{12}-E_{23}-E_{34})} e^{zE_{14}} e^{rE_{24}} |0\rangle \tag{3.46}$$

with $|0\rangle = (0\,0\,0\,1)^T$. The coordinates on $M(3,\mathbb{C})$ are

$$x_1 = z + ar + \tfrac{1}{6}a^3, \qquad x_2 = r + \tfrac{1}{2}a^2, \qquad x_3 = -a. \tag{3.47}$$

The Laplace operator on $M(3,\mathbb{C})$ is

$$\Box_{3C} = \frac{\partial^2}{\partial r^2} - 2\frac{\partial^2}{\partial a \partial z} + 2r\frac{\partial^2}{\partial z^2}. \tag{3.48}$$

3.6 Subgroup type coordinates with fewer ignorable variables

For completeness we consider indecomposable subgroup type coordinates that are not related to maximal Abelian subgroups of $E(n,\mathbb{C})$. We still parametrize a group element as in eq. (1.7), but replace the subgroup g_M by g_A, where $g_A \subset g_M$, i.e. g_A is Abelian, but not maximal. Indecomposability requires that the subgroup chain (1.9) start as $E(n,\mathbb{C}) \supset O(n,\mathbb{C})$. After that links of the type $O(n,\mathbb{C}) \supset O(n-1,\mathbb{C})$, $O(n,\mathbb{C}) \supset O(n_1,\mathbb{C}) \otimes O(n_2,\mathbb{C})$ and $O(n,\mathbb{C}) \supset E(n-2,\mathbb{C})$ are allowed.

In the metric $K = I_n$ the $e(n-2)$ subalgebra of $o(n,\mathbb{C})$ is represented by the matrices

$$X = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha^T & A & -i\alpha^T \\ 0 & i\alpha & 0 \end{pmatrix}, \quad \alpha \in \mathbb{C}^{1\times(n-2)}, \quad K = I_n, \tag{3.49}$$

$$A = -A^T \in \mathbb{C}^{(n-2)\times(n-2)}. \tag{3.50}$$

For $M(n,\mathbb{C})$ we use the metric $K = I_n$ and induce the coordinates by the action

$$|y\rangle = g_1 g_2 \dots g_n |0\rangle \tag{3.51}$$

with

$$g_n = e^{rE_{1,n+1}}, \qquad g_{n-1} = e^{c(-E_{1n}+E_{n1})}. \tag{3.52}$$

The other one-parameter subgroups depend upon the subgroup chain considered. The action of g_n takes us onto a complex sphere of radius r. The one-parameter subgroups, $g_1, \dots g_{n-1}$, introduce coordinates on this sphere.

We now specialize to the case of $M(4,\mathbb{C})$. We use the metric $K = I_4$. A basis for the algebra $o(4,\mathbb{C})$ can be chosen to be

$$L_{ik} = -E_{ik} + E_{ki}, \qquad 1 \le i \le k \le 4. \tag{3.53}$$

An alternative basis, to be used when the $e(2,\mathbb{C})$ subalgebra is important, is

$$\begin{array}{lll} L_{23}, & X_1 = L_{12} - iL_{24}, & X_2 = L_{13} - iL_{34} \\ L_{14}, & Y_1 = L_{12} + iL_{24}, & Y_2 = L_{13} + iL_{34}. \end{array} \tag{3.54}$$

The complete set of commuting operators is

$$\{\square_{4\mathbb{C}}, \triangle_{LB}(4), R_2, R_1\}, \tag{3.55}$$

where R_2 and R_1 must be specified in each case and R_1 is an element of $o(4,\mathbb{C})$. Similarly g_4 and g_3 are as in eq (3.52), but the one-parameter subgroups g_2 and g_1 are specified in each case.

Four subgroup chains and four types of separable coordinates occur.

1. $E(4,\mathbb{C}) \supset O(4,\mathbb{C}) \supset O(3,\mathbb{C}) \supset O(2,\mathbb{C})$
   We take $g_1 = e^{aL_{12}}$, $g_2 = e^{bL_{13}}$ and we obtain complex spherical coordinates

$$\begin{array}{ll} x_1 = r\cos c\cos b\cos a & x_3 = r\cos c\sin b \\ x_2 = r\cos c\cos b\sin a & x_4 = r\sin c. \end{array} \tag{3.56}$$

We have

$$R_2 = \triangle_{LB}(3), \qquad R_1 = L_{12} \tag{3.57}$$

and

$$\Box_{4C} = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_{LB}(4) \tag{3.58}$$

with

$$\triangle_{LB}(4) = \frac{\partial^2}{\partial c^2} - 2\tan c\frac{\partial}{\partial c} + \frac{1}{\cos^2 c}\frac{\partial^2}{\partial b^2} + \frac{1}{\cos^2 c\cos^2 b}\frac{\partial^2}{\partial a^2} - \frac{\tan b}{\cos^2 c}\frac{\partial}{\partial b}. \tag{3.59}$$

2. $E(4,\mathbb{C}) \supset O(4,\mathbb{C}) \supset O(3,\mathbb{C}) \supset E(1,\mathbb{C})$
We choose $g_1 = e^{aX_1}$ and $g_2 = e^{bL_{13}}$. The coordinates are

$$\begin{array}{ll} x_1 = r\cos c(\cos b - \frac{1}{2}a^2e^{ib}) & x_3 = r\cos c(\sin b - \frac{1}{2}ia^2e^{ib}) \\ x_2 = r\cos c\, a\, e^{ib} & x_4 = r\sin c \end{array} \tag{3.60}$$

and we have

$$R_2 = \triangle_{LB}(3), \qquad R_1 = X_1. \tag{3.61}$$

The d'Alambertian $\Box_{4C}$ is as in eq. (3.58) with

$$\triangle_{LB}(4) = \frac{\partial^2}{\partial c^2} - 2\tan c\frac{\partial}{\partial c} + \frac{1}{\cos^2 c}\left(\frac{\partial^2}{\partial b^2} + i\frac{\partial}{\partial b} + e^{-2ib}\frac{\partial^2}{\partial a^2}\right), \tag{3.62}$$

where

$$\triangle_{LB}(3) = \frac{\partial^2}{\partial b^2} + i\frac{\partial}{\partial b} + e^{-2ib}\frac{\partial^2}{\partial a^2}. \tag{3.63}$$

3. $E(4,\mathbb{C}) \supset O(4,\mathbb{C}) \supset E(2,\mathbb{C}) \supset O(2,\mathbb{C})$
We choose $g_1 = e^{aL_{23}}$ and $g_2 = e^{bX_1}$. The separable coordinates are

$$\begin{array}{ll} x_1 = r(\cos c - \frac{1}{2}b^2e^{ic}) & x_3 = rbe^{ic}\sin a \\ x_2 = rbe^{ic}\cos a & x_4 = r(\sin c - \frac{1}{2}ib^2e^{ic}). \end{array} \tag{3.64}$$

We have

$$R_2 = \triangle_{LB}(2), \qquad R_1 = L_{23}. \tag{3.65}$$

Again $\Box_{4C}$ is as in (3.58) with

$$\triangle_{LB}(4) = \frac{\partial^2}{\partial c^2} + 2i\frac{\partial}{\partial c} + e^{-2ic}\triangle_{LB}(3), \tag{3.66}$$

where

$$\triangle_{LB}(3) = \frac{\partial^2}{\partial b^2} + \frac{1}{b}\frac{\partial}{\partial b} + \frac{1}{b^2}\frac{\partial^2}{\partial a^2}. \tag{3.67}$$

4. Chain $E(4,\mathbb{C}) \supset O(4,\mathbb{C}) \supset E(2,\mathbb{C}) \supset E(1,\mathbb{C}) \otimes E(1,\mathbb{C})$
We take $g_1 = e^{a_1 X_1}$ and $g_2 = e^{a_2 X_2}$. The coordinates are

$$\begin{aligned} x_1 &= r(\cos c - \tfrac{1}{2}(a_1^2 + a_2^2)e^{ic}) \qquad & x_3 &= r a_2 e^{ic} \\ x_2 &= r a_1 e^{ic} & x_4 &= r \end{aligned} \tag{3.68}$$

and we have

$$R_2 = X_2, \qquad R_1 = X_1. \tag{3.69}$$

The operator $\Box_{4C}$ is again as in eq. (3.58) with

$$\triangle_{LB}(4) = \frac{\partial^2}{\partial c^2} + 2i\frac{\partial}{\partial c} + e^{-2ic}\left(\frac{\partial^2}{\partial a_1^2} + \frac{\partial^2}{\partial a_2^2}\right). \tag{3.70}$$

Both variables a_1 and a_2 are ignorable. However, X_1, X_2 is only a MASA of $o(4,\mathbb{C})$ and not of $e(4,\mathbb{C})$.

3.7 A graphical formalism

All separable subgroup type coordinates on $M(4,\mathbb{C})$ can be summarized using subgroup diagrams similar to those of the real groups $O(n)$ and $O(n,1)$ [9, 22, 29, 30]. They are directly related to "tree" diagrams introduced in Reff [27, 28] and discussed in Reff [9, 22].

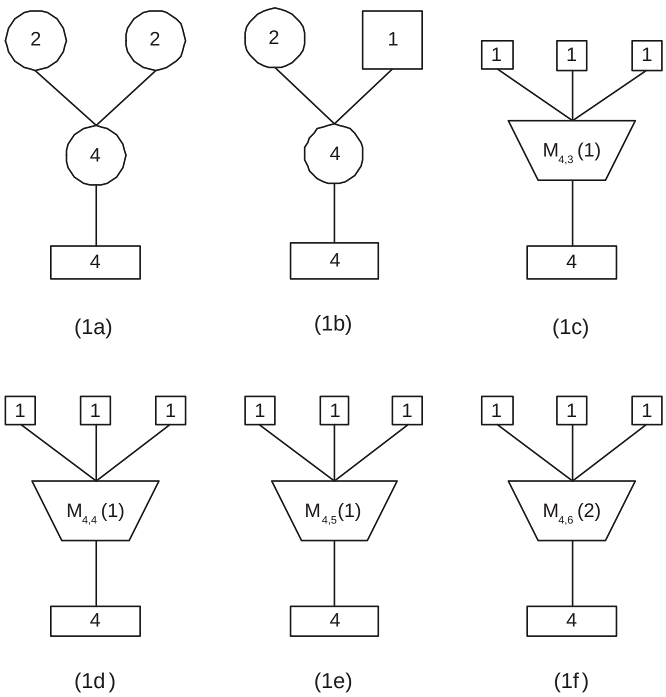

Figure 1. Subgroup chains for $E(4,\mathbb{C})$ ending in maximal Abelian subgroups.

On Figs 1 and 2 we use rectangles to denote Euclidean groups $E(n, \mathbb{C})$ and circles to denote $O(n, \mathbb{C})$. In both cases the value of n is indicated inside the rectangle, or circle. Trapezia are used to denote maximal Abelian subgroups. Inside the trapezium we indicate which MASA is involved. A rectangle with $n = 1$ denotes a one-dimensional unipotent subgroup. The corresponding algebra is represented by a nilpotent matrix. Figs (1a), ..., (1f) correspond to the coordinate systems (3.6), (3.13), (3.23), (3.27), (3.31) and (3.35). Similarly Figs (2a), ..., (2d) correspond to the coordinate systems (3.56), (3.60), (3.64) and (3.68).

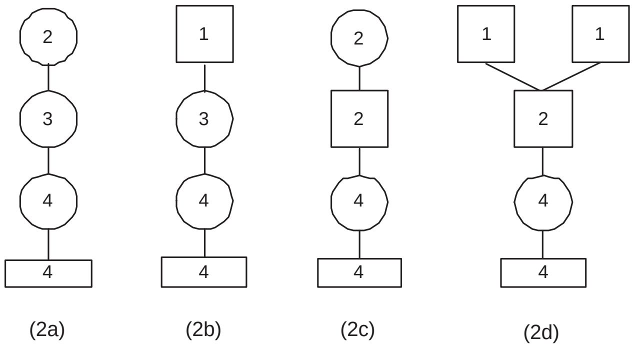

Figure 2. Subgroup chains for $E(4, \mathbb{C})$ ending in nonmaximal Abelian subgroups.

4 Subgroup type coordinates in the real spaces $M(4)$ and $M(3, 1)$

When one passes from the complex Euclidean space $M(n, \mathbb{C})$ to the real Euclidean or pseudo-Euclidean spaces $M(p, q)$, $p \geq q \geq 0$, two different phenomena must be taken into account. Firstly some subgroups, in particular Abelian ones, that exist for $E(n, \mathbb{C})$ may have counterparts only for certain signatures (p, q). Secondly in real spaces with $q \geq 1$ vectors can have positive, negative and zero length. The existence of two types of anisotropic vectors leads to a proliferation of subgroup chains and of coordinate systems.

4.1 The real Euclidean space $M(4)$

Only two of the subgroup chains illustrated on Fig 1 and Fig 2 are realized in this case, namely those of Fig (1a) and Fig (2a). The corresponding systems of separable coordinates are (3.6) with $0 \leq r \leq \infty$, $0 \leq c \leq \frac{\pi}{2}$, $0 \leq a \leq 2\pi$, $0 \leq b \leq 2\pi$ and (3.56) with $0 \leq r \leq \infty$, $0 \leq c \leq \pi$, $0 \leq b \leq \pi$, $0 \leq a < 2\pi$.

All other subgroup chains lead to decomposable coordinate systems. The decomposition patterns are $4 = 3 + 1$, $4 = 2 + 2$, $4 = 2 + 1 + 1$ and $4 = 1 + 1 + 1 + 1$.

4.2 The real Minkowski space $M(3, 1)$

This case is somewhat richer than the previous one. Indeed a subgroup $O(n, \mathbb{C}) \subset E(n, \mathbb{C})$ can correspond to $O(n) \subset E(n)$ or $O(n - 1, 1) \subset E(n)$ in the real case. We denote $O(n)$ by a circle and $O(n - 1, 1)$ by a semicircle ("hyperbola") on the corresponding subgroup diagrams.

Among the six coordinates of subgroup type in $M(4,\mathbb{C})$ related to MASAs only three are represented on $M(3,1)$, see Fig. 3.

The Cartan subalgebra (3.4) goes into $o(2) \oplus (1,1) \subset o(3,1)$ and the corresponding cylindrical coordinates (3.6) go into

$$\begin{aligned} x_1 &= r\sinh c\cos a & 0 \le a < 2\pi \\ x_2 &= r\sinh c\sin a & 0 \le b < \infty \\ x_3 &= r\cosh c\sinh b & 0 \le c < \infty \\ x_4 &= r\cosh c\cosh b. \end{aligned} \tag{4.1}$$

The MASA $M_{4,2}(0)$ has no analogue in $o(3,1)$. Among the $k_0 = 1$ MASAs $M_{4,3}(1)$ and $M_{4,4}(1)$ have analogs in $M(3,1)$ while $M_{4,5}(1)$ does not. Finally $k_0 = 2$ is not allowed. The coordinate system corresponding to $M_{4,3}(1)$ in the space $M(3,1)$ is as in (3.23) with $0 \le r \le \infty$, $-\infty < a_i < \infty$ $(i = 1,2)$, $-\infty < z < \infty$. For $M_{4,4}(1)$ the coordinates are as in (3.27) with $-1 \le \beta \le 1$ and r, a_1, a_2, z as for the case of $M_{4,3}(1)$. Thus the only three subgroup diagrams of Fig 1 that give rise to subgroup diagrams and separable coordinates on $M(3,1)$ are (1a)→(3a), (1c)→(3b) and (1d)→(3c).

The four subgroup chains of Fig 2 give rise to the six chains of Fig 4. More specifically we have (2a)→(4a), (4b) and (4c), (2b)→(4d), (2c)→(4e) and (2d)→(4f). We do not detail the six different types of coordinates systems on the upper sheet of the two-sheeted hyperboloid $x_1^2 + x_2^2 + x_3^2 - x_0^2 = -1$ here. They can be found in e.g., [29], [30] and [22].

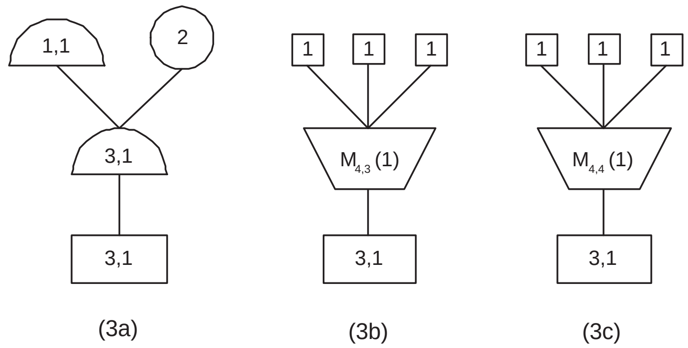

Figure 3. Subgroup chains for $E(3,1)$ ending in maximal Abelian subgroups.

5 Subgroup type coordinates in the real space $M(2,2)$

5.1 Structure of MASAs of $e(2,2)$

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A complete classification of MASAs of $e(p,2)$, in particular $e(2,2)$, was performed earlier [25]. It is somewhat more complicated than that of $e(4,\mathbb{C})$ although similar. A MASA of $e(p,q)$ contains $k = k_0 + k_+ + k_-$ mutually orthogonal translations, where k_0, k_+ and k_- are the numbers of isotropic, positive length and negative length translations, respectively. For $p \ge q$ we have

$$0 \le k_0 \le q, \quad 0 \le k_+ \le p, \quad 0 \le k_- \le q, \quad 0 \le k_0 + k_+ + k_- \le p + q. \tag{5.1}$$

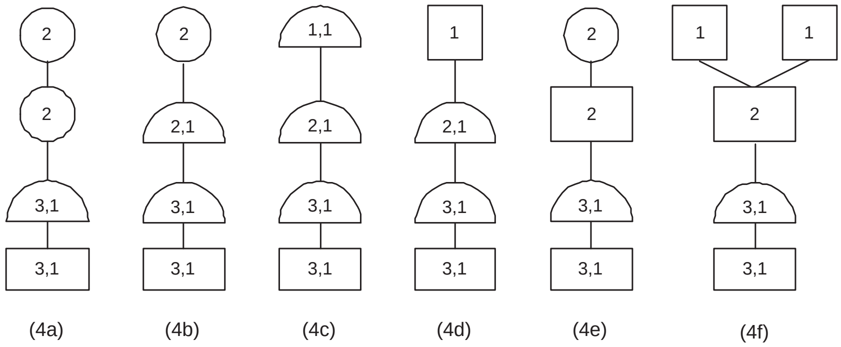

Figure 4. Subgroup chains for $E(3,1)$ ending in nonmaximal Abelian subgroups.

We are mainly interested in MASAs with $k_+ = k_- = 0$ since the presence of any anisotropic translations leads to a decomposition of the space $M(p,q)$ and to decomposable coordinate systems.

We proceed as in the case of $e(4,\mathbb{C})$. The algebra $e(2,2)$ is realized by 5×5 matrices as in eq (1.10) and the metric $K = K^T \in \mathbb{R}^{4\times 4}$ has the signature $(2,2)$.

As in the case of $e(4,\mathbb{C})$ we consider $k_0 = 0, 1$ and 2 separately. Each subgroup chain and the corresponding coordinate system on $M(4,\mathbb{C})$ gives rise to at least one system on $M(2,2)$.

5.2 MASAs of $e(2,2)$ with $k_0 = 0$

We are dealing here with MASAs of $o(2,2)$ that are also maximal in $e(2,2)$. The algebra $o(2,2)$ has three inequivalent Cartan subalgebras and so three different coordinate systems correspond to the cylindrical coordinates (3.6). In each case we give the subgroup chain, the group action on the origin, the coordinates and the complete set of commuting operators.

1. The compact Cartan subalgebra $o(2)\oplus o(2)$.
   The subgroup chain is

$$E(2,2) \supset O(2,2) \supset O(2)\otimes O(2). \tag{5.2}$$

   We use the diagonal metric $K = \mathrm{diag}(1,1,-1,-1)$ and put

$$|y\rangle = e^{-a(E_{12}-E_{21})}e^{-b(E_{34}-E_{43})}e^{c(E_{13}+E_{31})}e^{rE_{15}}|0\rangle. \tag{5.3}$$

   The coordinates are

$$\begin{aligned} x_1 &= r\cosh c\cos a & & 0\le r<\infty \\ x_2 &= r\cosh c\sin a & & 0\le a<2\pi, \qquad 0\le b<2\pi \\ x_3 &= r\sinh c\cos b & & 0\le c<\infty \\ x_4 &= r\sinh c\sin b. \end{aligned} \tag{5.4}$$

The Laplace operator and complete set of commuting operators are

$$\begin{aligned} \Box_{2,2} &= \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_{LB} \\ \triangle_{LB} &= \frac{4e^{2c}}{(e^{2c}+1)^2}\frac{\partial^2}{\partial a^2} - \frac{4e^{2c}}{(e^{2c}-1)^2}\frac{\partial^2}{\partial b^2} - \frac{\partial^2}{\partial c^2} - \frac{2(e^{4c}+1)}{e^{4c}-1}\frac{\partial}{\partial c} \end{aligned} \tag{5.5}$$

$$\{\Box_{2,2}, \triangle_{LB}(2,2), L_{12} = E_{12} - E_{21}, L_{34} = E_{34} - E_{43}\}. \tag{5.6}$$

2. The noncompact Cartan subalgebra $o(1,1) \oplus o(1,1)$.
The subgroup chain is

$$E(2,2) \supset O(2,2) \supset O(1,1) \otimes O(1,1). \tag{5.7}$$

Again in the diagonal metric $K = \mathrm{diag}(1,1,-1,-1)$ we have

$$|y\rangle = e^{a(E_{13}+E_{31})}e^{b(E_{24}+E_{42})}e^{c(E_{14}+E_{41})}e^{rE_{15}}|0\rangle. \tag{5.8}$$

The coordinates are

$$\begin{array}{ll} x_1 = r\cosh c\cosh a & x_3 = r\cosh c\sinh a \\ x_2 = r\sinh c\sinh b & x_4 = r\sinh c\cosh b. \end{array} \tag{5.9}$$

The Laplace operator is as in eq (5.5) with

$$\triangle_{LB}(2,2) = -\frac{4e^{2c}}{(e^{2c}+1)^2}\frac{\partial^2}{\partial a^2} + \frac{4e^{2c}}{(e^{2c}-1)^2}\frac{\partial^2}{\partial b^2} - \frac{\partial^2}{\partial c^2} - \frac{2(e^{4c}+1)}{e^{4c}-1}\frac{\partial}{\partial c} \tag{5.10}$$

and the commuting set is

$$\{\Box_{2,2}, \triangle_{LB}(2,2), L_{13} = E_{13} + E_{31}, L_{24} = E_{24} + E_{42}\}. \tag{5.11}$$

3. The "semicompact" Cartan subalgebra $o(2) \oplus o(1,1)$.
The algebra $o(2,2)$ has a third Cartan subalgebra, with one compact and one noncompact element. The subgroup chain is

$$E(2,2) \supset O(2,2) \supset O(2) \otimes O(1,1). \tag{5.12}$$

We represent this cartan subalgebra by the matrices

$$X = \begin{pmatrix} a & -b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a & -b & 0 \\ 0 & 0 & b & -a & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{5.13}$$

where we are using a nondiagonal metric. As a MASA (5.12) is decomposable but not absolutely orthogonally indecomposable (D but NAOID) [8, 25] (i.e., after complexification the matrix X can be diagonalized).

The appropriate group action is

$$|y\rangle = e^{a(E_{11}+E_{22}-E_{33}-E_{44})}e^{b(-E_{21}+E_{12}-E_{34}+E_{43})}e^{c(E_{12}+E_{21}-E_{34}-E_{43})}e^{r(E_{15}+E_{35})/\sqrt{2}}|0\rangle \tag{5.14}$$

so that we have

$$\begin{aligned} x_1 &= \frac{r}{\sqrt{2}}e^{a}(\cosh c\cos b-\sinh c\sin b)\\ x_2 &= \frac{r}{\sqrt{2}}e^{a}(\cosh c\sin b+\sinh c\cos b)\\ x_3 &= \frac{r}{\sqrt{2}}e^{-a}(\cosh c\cos b+\sinh c\sin b)\\ x_4 &= \frac{r}{\sqrt{2}}e^{-a}(\cosh c\sin b-\sinh c\cos b). \end{aligned} \tag{5.15}$$

The Laplace operator in these coordinates is as in eq (5.5) with

$$\triangle_{LB}(2,2) = -\frac{\partial^2}{\partial c^2} - 2\tanh 2c\frac{\partial}{\partial c} + 2\frac{\sinh 2c}{(\cosh 2c)^2}\frac{\partial^2}{\partial a\partial b} + \frac{1}{(\cosh 2c)^2}\left(\frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial b^2}\right) \tag{5.16}$$

and the complete set of commuting operators is

$$\{\Box_{2,2}, \triangle_{LB}(2,2), E_{11}+E_{22}-E_{33}-E_{44}, -E_{21}+E_{12}-E_{34}+E_{43}\}. \tag{5.17}$$

The relation between the real coordinates (5.15) and the complex cylindrical coordinates (3.6) is best seen if we transform (3.4), (3.5) and (3.6) to the antidiagonal metric K of (5.13). The transformation $gK_Dg^T = K$ and $gX_Dg^{-1} = X$, where $K_D = I_4$, is realized by

$$g = \frac{1}{2}\begin{pmatrix} 1 & i & 1 & i\\ -i & 1 & i & -1\\ 1 & -i & 1 & -i\\ i & 1 & -i & -1 \end{pmatrix}. \tag{5.18}$$

Putting $|x\rangle = g|x_D\rangle$ with $|y_D\rangle$ as in eq (3.6) we obtain

$$\begin{aligned} x_1 &= \frac{1}{2}re^{i\frac{\tilde a+\tilde b}{2}}\left((\cos\tilde c+\sin\tilde c)\cos\frac{\tilde a-\tilde b}{2} + i((\cos\tilde c-\sin\tilde c)\sin\frac{\tilde a-\tilde b}{2})\right)\\ x_2 &= \frac{1}{2}re^{i\frac{\tilde a+\tilde b}{2}}\left((\cos\tilde c+\sin\tilde c)\sin\frac{\tilde a-\tilde b}{2} - i((\cos\tilde c-\sin\tilde c)\cos\frac{\tilde a-\tilde b}{2})\right)\\ x_3 &= \frac{1}{2}re^{-i\frac{\tilde a+\tilde b}{2}}\left((\cos\tilde c+\sin\tilde c)\cos\frac{\tilde a-\tilde b}{2} - i((\cos\tilde c-\sin\tilde c)\sin\frac{\tilde a-\tilde b}{2})\right)\\ x_4 &= \frac{1}{2}re^{-i\frac{\tilde a+\tilde b}{2}}\left((\cos\tilde c+\sin\tilde c)\sin\frac{\tilde a-\tilde b}{2} + i((\cos\tilde c-\sin\tilde c)\cos\frac{\tilde a-\tilde b}{2})\right). \end{aligned} \tag{5.19}$$

Putting

$$i\frac{\tilde{a}+\tilde{b}}{2}=a,\qquad \frac{\tilde{a}-\tilde{b}}{2}=b,\qquad \cos\tilde{c}+\sin\tilde{c}=\sqrt{2}\cosh c,\qquad \cos\tilde{c}-\sin\tilde{c}=i\sqrt{2}\sinh c$$

and restricting to $a,b,c\in\mathbb{R}$, we obtain the coordinates (5.15).

The complex algebra $e(4,\mathbb{C})$ has only one further MASA with $k_0=0$, namely that of eq (3.11). It, however, has two distinct real forms. One of them, $M_1(0)$, coincides with that of eq (3.11) with a and $b\in\mathbb{R}$. In $e(2,2)$ this MASA is absolutely orthogonally indecomposable but decomposable (AOID but D). The subgroup chain is

$$E(2,2)\supset O(2,2)\supset \exp(M_1(0)). \tag{5.20}$$

The coordinates, the Laplace operator and commuting operators are as in (3.13), (3.15) and (3.16) with all entries real.

The other MASA of $e(2,2)$, $M_2(0)$, is absolutely orthogonally indecomposable, indecomposable but not absolutely indecomposable (AOID, ID but NAID). We represent $M_2(0)$ by the matrices:

$$M_2(0)=\begin{pmatrix} 0 & a & 0 & b & 0\\ -a & 0 & -b & 0 & 0\\ 0 & 0 & 0 & a & 0\\ 0 & 0 & -a & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix},\qquad K=\begin{pmatrix} 0 & I_2\\ I_2 & 0\end{pmatrix}. \tag{5.21}$$

The subgroup chain is

$$E(2,2)\supset O(2,2)\supset \exp M_2(0). \tag{5.22}$$

The MASA $M_{4,2}(0)\subset e(4)$ (see eq (3.11)) can be transformed into the form of (5.21) by a similarity transformation with

$$G=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1\\ i & 0 & 0 & -i\\ 0 & 1 & 1 & 0\\ 0 & i & -i & 0\end{pmatrix} \tag{5.23}$$

which preserves the metric K. To obtain the real space $M(2,2)$ we put

$$a\to -ia,\qquad b\to ib,\qquad a,b\in\mathbb{R}. \tag{5.24}$$

The corresponding coordinates in $M(2,2)$ are obtained from (3.13) by the transformation $|x\rangle\to G|x\rangle$ and we obtain

$$\begin{aligned} x_1&=\tfrac{1}{\sqrt{2}}r(e^{c}\cos a+be^{-c}\sin a) & x_3&=\tfrac{1}{\sqrt{2}}re^{-c}\cos a\\ x_2&=\tfrac{1}{\sqrt{2}}r(e^{c}\sin a-be^{-c}\cos a) & x_4&=\tfrac{1}{\sqrt{2}}re^{-c}\sin a \end{aligned} \tag{5.25}$$

with $0\le r<\infty, 0\le a<2\pi, -\infty<c<\infty$ and $0\le b<\infty$. These coordinates correspond to the group action

$$e^{a(-E_{12}+E_{21}-E_{34}+E_{34})}e^{b(E_{14}-E_{23})}e^{c(-E_{11}-E_{22}+E_{33}+E_{44})}e^{r(E_{15}+E_{35})}. \tag{5.26}$$

The Laplace operator is

$$\Box_{2,2} = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(-\frac{\partial^2}{\partial c^2} + \frac{\partial}{\partial c} - 4e^{4c}\frac{\partial^2}{\partial b^2} - 4e^{2c}\frac{\partial^2}{\partial a \partial b}\right). \tag{5.27}$$

This corresponds to the case (3.8) plus (3.16) for $E(4,\mathbb{C})$. The commuting operators are

$$\{\Box_{2,2}, \triangle_{LB}(2,2), E_{12} - E_{21} + E_{34} - E_{43}, E_{14} - E_{23}\}. \tag{5.28}$$

5.3 MASAs of $e(2,2)$ with $k_0 = 1$

The maximal Abelian subalgebras of $e(2,2)$ with $k_0 = 1$ can be written exactly in the same form as those of $e(4,\mathbb{C})$ (see Section 3.3). All parameters and variables are real. As for $e(4,\mathbb{C})$ there are three different cases to consider. All can be written as in eq (3.17). In eq (3.19) the parameter β is real and satisfies $-1 \le \beta \le 1$, $\beta \ne 0$.

The three separable coordinate systems are (3.23), (3.27) and (3.31). The Laplace operators and commuting operators are the same as in the complex case with all variables restricted to being real.

5.4 MASAs of $e(2,2)$ with $k_0 = 2$

For $k_0 = 2$ the MASAs of $e(2,2)$ have the same form as those of $e(4,\mathbb{C})$. There are just two of them. The first leads to the coordinates (3.35) with a_1, a_2, z and r real. The Laplace operator is (3.36). The second MASA has the form (3.37) with real entries and, like (3.37), does not provide a coordinate system.

5.5 Decompositions of the space $M(2,2)$

In view of the existence of a signature in $M(p,q)$ spaces, the four decompositions of $M(4,\mathbb{C})$ of eq (3.41) give rise to six inequivalent decompositions of $M(2,2)$. They are:

1. $M(2,2) = M(2,1) \oplus M(0,1)$, or $M(1,2) \oplus M(1,0)$
2. $M(2,2) = M(2,0) \oplus M(0,2)$
3. $M(2,2) = M(1,1) \oplus M(1,1)$
4. $M(2,2) = M(2,0) \oplus 2M(0,1)$, or $M(0,2) \oplus 2M(1,0)$
5. $M(2,2) = M(1,1) \oplus M(1,0) \oplus M(0,1)$
6. $M(2,2) = 2M(1,0) \oplus 2M(0,1)$.

Each one-dimensional space corresponds to a Cartesian coordinate. The spaces $M(2,0)$ and $M(0,2)$ correspond to polar coordinates (trigonometric ones), the space $M(1,1)$ to "hyperbolic polar coordinates", e.g. $x_1 = \cosh\alpha$ and $x_2 = \sinh\alpha$. All subgroup type coordinates with a maximal number of ignorable variables on $M(2,1)$ (or $M(1,2)$) spaces were given earlier [24].

5.6 Subgroup type coordinates on $M(2,2)$ with fewer ignorable variables

Each of the subgroup chains in Section 3.6 has at least one analog for the group $E(2,2)$. Altogether that leads to five types of subgroup coordinates. Here we just give the subgroup chains and the corresponding coordinates. The Laplacians and complete sets of commuting operators are easy to calculate. In all case we take the metric to be $K = \mathrm{diag}(1,1,-1,-1)$.

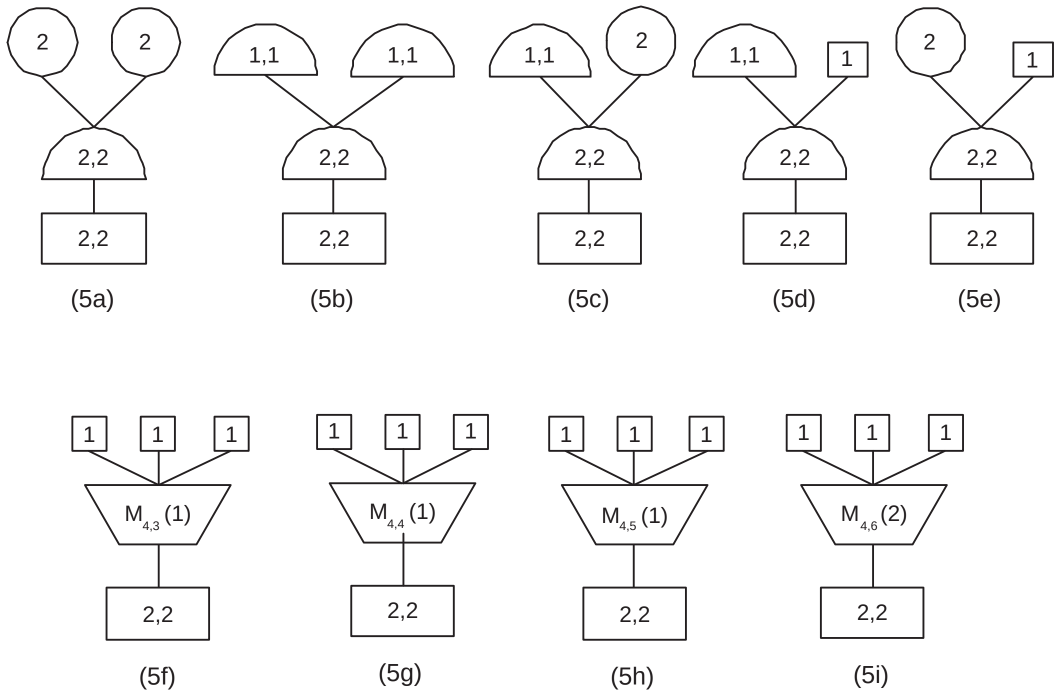

Figure 5. Subgroup diagrams for $E(2,2)$ ending in maximal Abelian subgroups.

i) $E(2,2) \supset O(2,2) \supset O(2,1) \supset O(2)$

$$\begin{aligned} x_1 &= r\cosh c \cosh b \cos a \qquad & x_3 &= r\cosh c \sinh b \\ x_2 &= r\cosh c \cosh b \sin a \qquad & x_4 &= r \sinh c. \end{aligned} \tag{5.29}$$

ii) $E(2,2) \supset O(2,2) \supset O(2,1) \supset O(1,1)$

$$\begin{aligned} x_1 &= r\cosh c \cos b \cosh a \qquad & x_3 &= r\cosh c \sin b \\ x_2 &= r\cosh c \cos b \sinh a \qquad & x_4 &= r \sinh c. \end{aligned} \tag{5.30}$$

iii) $E(2,2) \supset O(2,2) \supset O(2,1) \supset E(1)$

$$\begin{aligned} x_1 &= r\cosh c(\cosh b - \tfrac{1}{2}a^2 e^b) \qquad & x_3 &= r\cosh c(\sinh b + \tfrac{1}{2}a^2 e^b) \\ x_2 &= r\cosh c\, a\, e^b \qquad & x_4 &= r \sinh c. \end{aligned} \tag{5.31}$$

iv) $E(2,2) \supset O(2,2) \supset E(1,1) \supset O(1,1)$

$$\begin{aligned} x_1 &= r(\cosh a - \tfrac{1}{2}b^2e^c) & x_3 &= rbe^c \sinh a \\ x_2 &= rbe^c \cosh a & x_4 &= r(\sinh c + \tfrac{1}{2}b^2)e^c. \end{aligned} \tag{5.32}$$

v) $E(2,2) \supset O(2,2) \supset E(1,1) \supset E(1) \otimes E(1)$

$$\begin{aligned} x_1 &= r(\cosh c - \tfrac{1}{2}(a^2 + b^2)e^c) & x_3 &= rbe^c \\ x_2 &= rae^c & x_4 &= r(\sinh c + \tfrac{1}{2}(a^2 + b^2)e^c). \end{aligned} \tag{5.33}$$

All indecomposable coordinate systems on $M(2,2)$ and the corresponding subgroups chains are illustrated on Fig 5 and Fig 6.

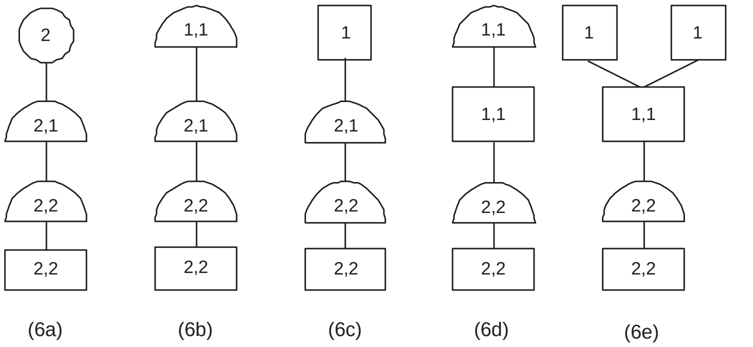

Figure 6. Subgroup diagrams for $E(2,2)$ ending in nonmaximal Abelian subgroups.

We see that Fig (1a), corresponding to the Cartan subalgebra of $O(4,\mathbb{C})$, gives rise to Figs (5a), (5b) and (5c) and thus to the coordinate systems (5.4), (5.9) and (5.15), respectively. Fig (1b) gives rise to Figs (5d) and (5e). The corresponding coordinates on the space $M(2,2)$ are (3.13) (with real entries) and (5.25). Figs (1c), (1d), (1e) and (1f) give rise to Figs (5f), (5g), (5h) and (5i), respectively. The coordinates on $M(2,2)$ are the same as on $M(4,\mathbb{C})$ with real entries, i.e., (3.23), (3.27), (3.31) and (3.35), respectively. Similarly Fig (2a) gives rise to Figs (6a) and (6b) and to the coordinates (5.29) and (5.30), respectively. Figs (2b), (2c) and (2d) go to Figs (6c), (6d) and (6e) and coordinates are (5.31), (5.32) and (5.33), respectively.

6 Solutions of the separated Schrődinger equations in complex spaces

In this Section we give the explicit solutions of the Schrődinger equation (1.2) in the separable coordinate systems introduced above. We do this for the complex spaces $M(4,\mathbb{C})$ and $M(3,\mathbb{C})$. The results for the real spaces $M(n)$ [20] and $M(n,1)$ are known [24]. Those for $M(2,2)$ can be obtained from the results of this section by imposing suitable reality conditions.

We follow the diagrams on Figs 1 and 2.

6.1 Subgroup chains involving link $E(4,\mathbb{C}) \supset O(4,\mathbb{C})$

The subgroup chains of Figs (1a), (1b) and Figs (2a), …, (2d) all involve the subgroup link $E(4,\mathbb{C}) \supset O(4,\mathbb{C})$. The corresponding coordinate systems contain a four-dimensional (complex) radius r. This is a nonignorable variable. The coordinate systems corresponding to Figs (1a), (1b) and (2d) contain one more nonignorable variable denoted c. Those corresponding to Figs (2a), (2b) and (2c) contain three nonignorable variables, r, c and b.

We write the separated wave function for the cases illustrated in Figs. (1a), (1b) and (2d) as

$$\Psi(r,c,a,b) = R(r)C(c)e^{i(\alpha a+\beta b)}, \tag{6.1}$$

where α and β are (complex) constants of separation. The function $R(r)$ is always the same. It satisfies the equation

$$R'' + \frac{3}{r}R' + \left(\frac{\lambda}{r^2} - E\right) R = 0 \tag{6.2}$$

and is hence expressed in terms of cylindrical functions

$$R(r) = \frac{1}{r}J_\nu\left(\sqrt{-E}r\right) \qquad \nu = -5 - \lambda. \tag{6.3}$$

The function $C(c)$ in eq (6.1) is different in each case.

For Fig (1a), i.e. coordinates (3.6), we have

$$C'' + 2\cot 2c\, C' - \left(\frac{\alpha^2}{\cos^2 c} + \frac{\beta^2}{\sin^2 c} + \lambda\right) C = 0. \tag{6.4}$$

Eq (6.4) can be solved in terms of Jacobi functions as

$$C(c) = (\cos c)^\alpha (\sin c)^\beta P_n^{(\alpha,\beta)}(-\cos 2c), \quad n = -\frac{1}{2}(\alpha+\beta+1 \pm \sqrt{1-\lambda}). \tag{6.5}$$

$P_n^{(\alpha,\beta)}(z)$ is a polynomial for $n \in Z_0^+$.

For Fig (1b), i.e. coordinates (3.13), the equation for $C(c)$ is

$$C'' - 2C' - (4e^{4c}\beta^2 - 4e^{2c}\alpha\beta + \lambda)C = 0. \tag{6.6}$$

Eq (6.6) is solved in terms of Whittaker functions and we have

$$C(c) = W_{\frac{i\alpha}{2},\frac{1}{2}\sqrt{1-\lambda}}(2i\beta e^{2c}). \tag{6.7}$$

For Fig (2d), i.e. coordinates (3.68), the equation for $C(c)$ is

$$C'' + 2iC' - (k^2 e^{-2ic} + \lambda)C = 0 \tag{6.8}$$

and is solved in terms of cylindrical functions

$$C(c) = e^{-ic} Z_\nu(ke^{-ic}), \quad \nu = \pm\sqrt{1-\lambda}. \tag{6.9}$$

Solutions corresponding to the subgroup chains on Figs (2a), (2b) and (2c) have the form

$$\Psi(r,c,b,a) = R(r)C(c)B(b)e^{i\alpha a}, \tag{6.10}$$

where α is a constant of separation. In all cases we have $R(r)$ as in eqs (6.2) and (6.3).

In both Fig (2a) and Fig (2b) we have an $O(4,\mathbb{C}) \supset O(3,\mathbb{C})$ link corresponding to the variable c (in eqs (3.56) and (3.60)). The function $C(c)$ in both cases satisfies

$$C'' - 2\tan c\, C' + \left(\frac{k}{\cos^2 c} - \lambda\right) C = 0 \tag{6.11}$$

and hence

$$C(c) = \frac{1}{\cos c} P_\nu^\mu(i\tan c), \quad \nu = \tfrac{1}{2}(-1 \pm \sqrt{1-4k}), \quad \mu = \sqrt{1-\lambda}, \tag{6.12}$$

where $P_\nu^\mu(z)$ is an associated Legendre function.

The function $B(b)$ is different in the two cases.

For Fig (2a) we have an $O(3,\mathbb{C}) \supset O(2,\mathbb{C})$ link and correspondingly

$$B'' - \tan b\, B' - \left(\frac{\alpha^2}{\cos^2 b} + k\right) B = 0 \tag{6.13}$$

$$B(b) = \frac{1}{\sqrt{\cos b}} P_\nu^\mu(i\tan b), \quad \nu = -\tfrac{1}{2} \pm \alpha, \quad \mu = \sqrt{\tfrac{1}{4} - k}. \tag{6.14}$$

The corresponding subgroup link on Fig (2b) is $O(3,\mathbb{C}) \supset E(1,\mathbb{C})$ and we have

$$B'' + iB' - (\alpha^2 e^{-2ib} + k)B = 0 \tag{6.15}$$

$$B(b) = \exp(-\tfrac{1}{2}ib) Z_\nu(\pm\alpha e^{-ib}), \quad \nu = \sqrt{\tfrac{1}{4} - k}, \tag{6.16}$$

where $Z_\nu(z)$ is a cylindrical function.

The subgroup chain on Fig (2c) involves the link $O(4,\mathbb{C}) \supset E(2,\mathbb{C})$ which also figures on Fig (2d). Correspondingly the function $C(c)$ satisfies eq. (6.8) and is given by eq (6.9). The final link on Fig (2c) corresponds to $E(2,\mathbb{C}) \supset O(2,\mathbb{C})$ and the function $B(b)$ satisfies

$$B'' + \frac{1}{b} B - \left(\frac{\alpha^2}{b^2} + k\right) B = 0 \tag{6.17}$$

$$B(b) = Z_\nu(\sqrt{-k}\, b), \quad \nu = \pm\alpha, \tag{6.18}$$

where $Z_\nu(z)$ is a cylindrical function.

6.2 Subgroup chains leading directly to maximal Abelian subgroups of $E(4,\mathbb{C})$

The coordinate systems corresponding to Fig (1c), ..., Fig (1f) all have three ignorable variables and so we can write the corresponding wave functions as

$$\Psi = R(r) e^{\zeta z} e^{\alpha_1 a_1} e^{\alpha_2 a_2}, \tag{6.19}$$

where ζ, α_1 and α_2 are constants.

The coordinates are all nonorthogonal and the differential equations for $R(r)$ are all linear first-order equations. The functions $R(r)$ are in all cases elementary ones. We do not give the equations satisfied by the functions $R(r)$, but only give the corresponding four solutions.

Fig (1c) and coordinates (3.23):

$$R(r) = \frac{1}{r}\exp\left(\frac{1}{2\zeta}\left(\frac{\alpha_1^2+\alpha_2^2}{r} - Er\right)\right). \tag{6.20}$$

Fig (1d) and coordinates (3.27):

$$R(r) = ((r+1)(r+\beta))^{-\frac{1}{2}}\exp\left(\frac{1}{2\zeta}\left(\frac{\alpha_1^2}{r+1} + \frac{\alpha_2^2}{r+\beta} + Er\right)\right). \tag{6.21}$$

Fig (1e) and coordinates (3.31):

$$R(r) = \frac{1}{r+\kappa}\exp\left(\frac{1}{2\zeta}\left(-\frac{\alpha_1^2}{(r+\kappa)^2} + \frac{2\alpha_1\alpha_2}{(r+\kappa)} + Er\right)\right). \tag{6.22}$$

Fig (1f) and coordinates (3.35):

$$R(r) = \exp\left(\frac{1}{2\alpha_1}\left(-\alpha_2^2 r^2 + (E - 2\alpha_2\zeta)r\right)\right). \tag{6.23}$$

6.3 Indecomposable coordinate systems in the space $M(3,\mathbb{C})$

As we have seen in Section 3.5, two indecomposable separable coordinate systems of subgroup type exist in $M(3,\mathbb{C})$, namely (3.44) and (3.47).

In both cases we write the separated wave function as

$$\Psi(r,a,z) = R(r)e^{\alpha a + \zeta z}. \tag{6.24}$$

The two cases are quite different.

In coordinates (3.44) the equation for $R(r)$ is of first order:

$$2\zeta R' + \left(\frac{1}{r^2}\alpha^2 + \frac{1}{r}\zeta - E\right)R = 0 \tag{6.25}$$

and its solution is

$$R(r) = \frac{1}{\sqrt{r}}\exp\left(\frac{1}{2\zeta}\left(\frac{\alpha^2}{r} + Er\right)\right). \tag{6.26}$$

In the system (3.47) we obtain a second-order equation for $R(r)$, namely

$$R'' + (2r\zeta^2 - 2\alpha\zeta - E)R = 0. \tag{6.27}$$

Equation (6.27) can be solved in terms of Airy functions (related to Bessel functions with index $\nu = 1/3$)

$$R(r) = Ai(x), \quad x = \left(r - \frac{\alpha}{\zeta} - \frac{E}{2\zeta^2}\right)\left(2\zeta^2\right)^{\frac{1}{3}}. \tag{6.28}$$

7 Conclusions

We have provided a complete classification of all subgroup-type coordinates in complex and real four-dimensional flat spaces in which the free Schrödinger and Hamilton-Jacobi equations allow separation of variables. The results are best summarised in terms of the subgroup diagrams that are presented in this article.

For the complex Euclidean group $E(4,\mathbb{C})$ and the corresponding complex space $M(4,\mathbb{C})$ there are six subgroup chains on Fig 1 leading to maximal Abelian subgroups. In four cases these Abelian subgroups are three-dimensional. The corresponding maximal Abelian subalgebras, $M_{4,3}(1)$, $M_{4,4}(1)$, $M_{4,5}(1)$ and $M_{4,6}(2)$, each provides three first-order operators in the Lie algebra $e(4,\mathbb{C})$ which together with the Laplace operator $\Box_{4\mathbb{C}}$ form complete sets of commuting operators. The subgroup chains in these four cases lead directly from $E(4,\mathbb{C})$ to a maximal Abelian subgroup. Other subgroups could be included between $E(4,\mathbb{C})$ and the Abelian subgroups. However, they do not have second-order Casimir operators and play no role in the problem of separating variables. The remaining two MASAs, the Cartan subalgebra $M_{4,1}(0)$ and the orthogonally indecomposable MASA $M_{4,2}(0)$, are two-dimensional. The missing operator in the complete set is provided by a Casimir operator of $O(4,\mathbb{C})$, i.e. the Laplace-Beltrami operator on the complex sphere $S_4(\mathbb{C})$.

Four more subgroup diagrams are given on Fig 2. The corresponding subgroup chains end in Abelian subgroups that are not maximal. In three cases they are just one-dimensional (Figs (2a), (2b) and (2c)). Two missing commuting operators are provided by the Casimir operators of the intermediate groups in the chains. Fig (2d) corresponds to a two-dimensional Abelian subgroup at the end of the chain. It again is not a maximal Abelian subgroup.

For the real Euclidean group $E(4,\mathbb{R})$ and the corresponding Euclidean space $M(4,\mathbb{R})$ the situation is much simpler. Only two subgroups chains leading to indecomposable separable coordinate systems exist, namely those of Fig (1a) and Fig (2a). The corresponding separable coordinates are cylindrical and spherical, respectively.

For the inhomogeneous Lorentz group $E(3,1)$ and the corresponding Minkowski space $M(3,1)$ the situation is much richer and is illustrated on Fig 3 and Fig 4.

The full richness of the problem manifests itself in the case of a balanced signature, i.e. the group $E(2,2)$ and the space $M(2,2)$. As shown in Fig 5, five subgroup chains lead through $O(2,2)$ to two-dimensional maximal Abelian subgroups. Those on Figs (5a), (5b) and (5c) correspond to three different Cartan subalgebras. Figs (5f), (5g), (5h) and (5i) correspond to different three-dimensional maximal Abelian subgroups. Finally the five diagrams of Fig 6 correspond to chains ending in Abelian subgroups that are not maximal. The corresponding coordinate systems have less than the maximal possible number of ignorable coordinates.

The results presented above for the complex space, $M(4,\mathbb{C})$, and the corresponding complex Euclidean group, $E(4,\mathbb{C})$, are in complete agreement with the algebraic theory of separation of variables in four-dimensional Riemannian spaces presented elsewhere [3, 12, 13]. Here we have added a group theoretical background to the theory. Moreover we have shown how coordinates of subgroup type on a homogeneous space are directly generated by the action of the isometry group G of the space. To do this we needed a classification of the subgroups of G, in particular the Abelian ones.

More specifically for $M(4,\mathbb{C})$ the coordinates (3.6) corresponding to the Cartan subalgebra are orthogonal. All other MASAs lead to nonorthogonal coordinates. The orthogonally indecomposable but decomposable MASA, $M_{4,2}(0)$, contains both semisimple and nilpotent elements. It leads to a nonorthogonal separation; the ignorable coordinates a and b are the nonorthogonal ones (see eq.(3.16)). The remaining MASAs, $M_{4,3}(1),\ldots,M_{4,6}(2)$, are all maximal Abelian nilpotent subalgebras (MANS). They all lead to nonorthogonal coordinate systems with three ignorable variables. Moreover the nonignorable variable r is of "first order". Only first-order derivatives with respect to r figure in all corresponding Laplace operators and hence the separated solutions are expressed in terms of elementary functions.

All the coordinates of subgroup type of Section 3.6, involving nonmaximal Abelian subalgebras, are orthogonal.

In the case of the $M(2,2)$ space the situation is similar. Two of the Cartan subalgebras, namely the one isomorphic to $o(2)\oplus o(2)$ and the one isomorphic to $o(1,1)\oplus o(2)$, are orthogonally decomposable and lead to orthogonal coordinates. The third Cartan subalgebra, isomorphic to $o(2)\oplus o(1,1)$, is decomposable but orthogonally indecomposable. It is, however, not absolutely orthogonally indecomposable. It leads to nonorthogonal coordinates; the nonorthogonality concerns the ignorable variables only (see eq (5.16)). For all other coordinate systems the situation is the same as in the complex case.

The subgroup diagrams of Fig 1, …, Fig 6 introduced in this article are closely related to the tree diagrams used for compact groups [26, 27, 28] and $O(n,1)$ [22, 30]. We have used the subgroup diagrams for $O(n,\mathbb{C})$ in Section 6 when constructing the separated wave functions in the different coordinate systems. Very briefly the rules are as follows.

i) A link from a rectangle to a circle, i.e. $E(n,\mathbb{C})\supset O(n,\mathbb{C})$, corresponds to a cylindrical function (as in eq (6.3)).

ii) A link from a circle to a rectangle, i.e. $O(n,\mathbb{C})\supset E(n-2,\mathbb{C})$, also leads to cylindrical functions (as in eqs (6.16) and (6.18)).

iii) Links from circles to circles, i.e. $O(n,\mathbb{C})\supset O(n-1,\mathbb{C})$ or $O(n,\mathbb{C})\supset O(n_1,\mathbb{C})\otimes O(n_2,\mathbb{C})$, lead to Jacobi and Legendre functions.

iv) Diagrams involving trapezia, i.e. subgroup chains including only Abelian subgroups, lead to a wave-function expressed in terms of elementary functions.

These rules can be further elaborated and specified, but we leave this for a future study.

One of the possible applications of the present results is to study integrable and superintegrable systems in the spaces considered. Indeed for each separable coordinate system in $M(4,\mathbb{C})$, $M(3,1)$, $M(2,2)$ or $M(4,\mathbb{R})$ we can find a potential, $V(x_1,\ldots,x_4)$, to be added to the kinetic energy term in eq (1.2) such that the Schrödinger equation still allows the separation of variables. The obtained Hamiltonians are integrable by construction: the Hamiltonian commutes with three linearly independent second-order differential operators. The separable potentials involve four arbitrary functions of one variable each. Among the integrable and separable systems obtained we can search for superintegrable and multiseparable systems. This, however, goes beyond the scope of the present article.

A generalization of the present results to $M(n,\mathbb{C})$ and $M(p,q)$ spaces is under consideration. The tools are available since MASAs of $e(n,\mathbb{C})$ and $e(p,q)$ have already been studied [17, 24, 25]. Separable coordinate systems for the real Euclidean space $M(n)$ are already known (not only the subgroup type ones) [10].

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