Seiberg-Witten-like Equations on 7-Manifolds with $G_2$-Structure

Nedim DEĞIRMENCI and Nüfier ÖZDEMİR

Anadolu University, Science Faculty, Mathematics Department
26470 Eskisehir, Turkey
E-mail: ndegeirmenci@anadolu.edu.tr
E-mail: nozdemir@anadolu.edu.tr

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Abstract

The Seiberg-Witten equations are of great importance in the study of topology of smooth four-dimensional manifolds. In this work, we propose similar equations for 7-dimensional compact manifolds with $G_2$-structure.

1 Introduction

The Seiberg-Witten monopole equations are formulated for four dimensional compact $Spin^c$-manifolds. There are some analogues of these equations in higher dimensions (see [1, 9, 12, 13]). All of the higher dimensional equations are stated for even dimensional manifolds. The Seiberg-Witten monopole equations consist of two equations. The first equation is the harmonicity condition on the spinors; this condition is linear and can be stated for $Spin^c$-manifolds in any dimension. The second equation couples the self-dual part of the curvature 2-form with a spinor field and is non-linear. There is no natural generalization of the second equation to higher dimensions because self-duality of 2-forms in the sense of Hodge is meaningless if dimension $\neq 4$. Still, there are various definitions of self-duality in dimensions $> 4$ (see [2, 5, 6]).

Recently 7-dimensional manifolds with $G_2$-structure have become popular due to the works of Bryant, Joyce and Cleyton-Ivanov (see [3, 4, 7, 10]). In this work we also deal with 7-dimensional manifolds with $G_2$-structure. If $M$ has a $G_2$-structure, then $M$ is a $Spin$-manifold (see [11]). As a $Spin$-manifold, $M$ is then automatically a $Spin^c$-manifold. The existence of a $G_2$-structure on $M$ leads to a decompositions of the space of $k$-forms on $M$. The decomposition of 2-forms is especially crucial because it enables us to define an analog of self-duality of 2-forms on $M$.

Using this approach to self-duality and the standard spinor machinery, we suggest an analog of the Seiberg-Witten-like equations on 7-manifolds with $G_2$-structure and we show that these Seiberg-Witten-like equations possess non-trivial solutions.

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2 Preliminaries

Let us consider $\mathbb{R}^7$ with a basis $e_1, \ldots, e_7$. Endow $\mathbb{R}^7$ with a metric for which the basis $e_1, \ldots, e_7$ is orthonormal and choose the orientation given by $[e_1, \ldots, e_7]$. Set

$$\Phi = dx_{124} + dx_{235} + dx_{346} + dx_{457} + dx_{156} + dx_{267} + dx_{137}, \tag{2.1}$$

where $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$. The subgroup of $Gl(7, \mathbb{R})$ fixing $\Phi$ is the exceptional Lie group $G_2$; it is a compact, connected and simply-connected Lie subgroup of $SO(7)$ of dimension 14 (see [3, 4]).

A $G_2$-structure on a 7-manifold $M$ is a reduction of the structure group $SO(7)$ to $G_2$.

Let $M$ be a 7-manifold with a $G_2$-structure. The action of $G_2$ on the tangent bundle induces an action of $G_2$ on $\wedge^k(M)$. This action gives the following orthogonal decompositions of $\wedge^k(M)$:

$$\wedge^1(M) = \wedge^1_+,$$

$$\wedge^2(M) = \wedge^2_+ \oplus \wedge^2_{14},$$

$$\wedge^3(M) = \wedge^3_+ \oplus \wedge^3_{17} \oplus \wedge^3_{27},$$

where

$$\wedge^2_+ = \{ \alpha \in \wedge^2(M) : *(\alpha \wedge \Phi) = -2\alpha \},$$

$$\wedge^2_{14} = \{ \alpha \in \wedge^2(M) : *(\alpha \wedge \Phi) = \alpha \},$$

$$\wedge^3_+ = \{ t\Phi : t \in \mathbb{R} \},$$

$$\wedge^3_{17} = \{ *(\beta \wedge \Phi) : \beta \in \wedge^1(M) \},$$

$$\wedge^3_{27} = \{ \gamma \in \wedge^3(M) : \gamma \wedge \Phi = 0, \gamma \wedge \Phi = 0 \}$$

and $\wedge^k_l$ denotes an $l$-dimensional $G_2$-irreducible subspace of $\wedge^k(M)$.

Recall that $Spin(7)$ is the double cover of $SO(7)$ and a 7-dimensional manifold is called a Spin one if the structure group $SO(7)$ of $M$ (in the sense of $G$-structures) can be lifted to $Spin(7)$ (see [8, 11]).

Recall also that $Spin^c(7) = Spin(7) \times S^1/\mathbb{Z}_2$ with projection

$$Spin^c(7) \to SO(7)$$

$$[g, z] \mapsto \lambda g$$

where $\lambda : Spin(7) \to SO(7)$ is the covering map. A 7-dimensional manifold is called a $Spin^c(7)$ one if the structure group $SO(7)$ of $M$ can be lifted to $Spin^c(7)$ (see [8, 11]).

Let $M$ be a $Spin^c$-manifold of dimension $n$. By a complex spinor bundle for $M$ we mean a complex vector bundle $S$ associated to a representation of $Spin^c(n)$ by Clifford multiplication, i.e.,

$$S = P_{Spin^c(n)} \times_\kappa \Delta_n$$

where $\Delta_n \cong \mathbb{C}^{2^n}$ and $\kappa : Spin^c(n) \to End(\Delta_n)$ is given by restriction of the $\mathbb{Cl}_n$-representation to $Spin^c(n) \subseteq \mathbb{Cl}_n$. These spinor bundles are bundles of complex modules over the Clifford algebra bundle $\mathbb{Cl}(M)$. When $n$ is even the spinor bundle $S$ splits into a direct sum

$$S = S^+ \oplus S^-$$
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where $S^\pm = (1 \pm \omega_C) S$ and where $\omega_C = i^{n/2} e_1 e_2 \cdots e_n$ is the volume form (see [8, 11]).

It is known that a connection $A$ in the principal $U(1)$-bundle $P_1$ and Levi-Civita connection on $M$ determine a covariant derivative

$$\nabla^A : \Gamma (S) \longrightarrow \Gamma (T^* M \otimes S)$$

on the spinor bundle $S$. And it can be define a first-order differential operator $D_A : \Gamma (S) \longrightarrow \Gamma (S)$ called the Dirac operator of $S$ by setting

$$D_A \psi = \sum_{j=1}^{n} e_j \cdot \nabla^A_{e_j} \psi$$

where $e_1, e_2, \cdots, e_n$ is an orthonormal basis of $T_m M$, at $m \in M$, where $\nabla^A$ denotes the covariant derivative on $S$, "," denotes the complex Clifford module multiplication. If the dimension $n$ is even, the Dirac operator decomposes into the sum of two operators, $D^\pm_A : \Gamma (S^\pm) \longrightarrow \Gamma (S^\mp)$, since Clifford multiplication by vectors interchanges these summands (see [8, 11]).

### 3 Seiberg-Witten-like Equations in dimension 7

#### 3.1 The Seiberg-Witten Equations in dimension 4

Let $M$ be an oriented, compact 4-dimensional Riemannian manifold. It is known that every compact, orientable 4-dimensional manifold $M$ is a $Spin^c$-manifold (see [8]). Fix a $Spin^c$-structure and a connection $A$ in the principal $U(1)$-bundle $P_1$ associated to the $Spin^c$-structure. The Seiberg-Witten monopole equations on $M$ are

$$D_A \psi = 0, \quad \Omega^+_A = \sigma (\psi)$$

for $\psi \in \Gamma (S^+)$, where $S^+$ is the positive spinor bundle (see [8, 11]), and

$$\sigma : \Gamma (S^+) \longrightarrow \Lambda^{2+} M \quad \psi \mapsto -\frac{1}{4} \sum_{i<j} (e_i e_j \psi, \psi) e_i \wedge e_j$$

and $\Omega^+_A$ is the self dual part of the curvature 2-form $\Omega_A$. The self-dual part $\Omega^+_A$ of $\Omega_A$ can be expressed in terms of the Hodge star operator as $\Omega^+_A = \frac{1}{2} (\Omega_A + * \Omega_A)$. It is also possible to write $\Omega^+_A$ in terms of the basis elements $f_1, f_2, f_3 \in \Lambda^2 (M)$ as

$$\Omega^+_A = \sum_{i=1}^{3} (f_i, \Omega_A) f_i,$$

where

$$f_1 = e_1 \wedge e_2 + e_3 \wedge e_4, \quad f_2 = e_1 \wedge e_3 - e_2 \wedge e_4, \quad f_3 = e_1 \wedge e_4 + e_2 \wedge e_3.$$

Note that $\sigma$ can also be given by the formula

$$\sigma (\psi) = -\frac{1}{4} \sum_{i=1}^{3} (f_i \cdot \psi, \psi) f_i.$$
3.2 Seiberg-Witten-like Equations in dimension 7

Note that the first of Seiberg-Witten equations, \( D_A \psi = 0 \), is linear and meaningful for any \( \text{Spin}^c \)-manifolds whereas the second equation is non-linear and there is no natural generalization to higher dimensions, because self-duality of 2-forms in the Hodge sense is meaningful only for 4-manifolds. Our aim is to write similar equations on 7-dimensional manifolds with \( G_2 \)-structure. Since such manifolds are \( \text{Spin} \) and thus \( \text{Spin}^c \) ones, this enables us to construct the spinor bundle and Dirac operator on it. The \( G_2 \)-structure also provides us with a decomposition \( \bigwedge^2 7 \oplus \bigwedge^2 14 \) of the space of 2-forms \( \bigwedge^2 (M) \).

Let \( M \) be a 7-dimensional manifold with a \( G_2 \)-structure and \( A \) the 1-form of a connection in the principal \( U(1) \)-bundle \( P_1 \) associated to the \( \text{Spin}^c \) structure on \( M \); let \( S \) be the spinor bundle. Then we can define the Dirac operator \( D_A : \Gamma (S) \longrightarrow \Gamma (S) \).

For the second equation we need a kind of self-duality of 2-forms — the decomposition \( \bigwedge^2 (M) = \bigwedge^2 7 \oplus \bigwedge^2 14 \). Let \( \pi_7 : \bigwedge^2 (M) \longrightarrow \bigwedge^2 7 \) be the orthogonal projection. For \( \eta \in \bigwedge^2 (M) \), the self-dual part of \( \eta \) is, by definition, \( \pi_7 (\eta) \). If \( \pi_7 (\eta) = \eta \), then \( \eta \) is called self-dual. First, we define a quadratic map \( \sigma : \Gamma (S) \longrightarrow \bigwedge^2 7 \) by setting

\[
\sigma (\psi) = \sum_{i=1}^{7} \left( \frac{\langle f_i \psi, \psi \rangle}{\langle f_i, f_i \rangle} \right) f_i,
\]

where \( f_1, \ldots, f_7 \) is a basis of \( \bigwedge^2 7 \) associated to an orthonormal frame \( e_1, \ldots, e_7 \) of \( T_m M \) at any point \( m \in M \). Then the 7-dimensional version of the second Seiberg-Witten equation is

\[
\pi_7 (\Omega_A) = \sigma (\psi).
\]

Hence the Seiberg-Witten-like equations in 7-dimensions are

\[
D_A \psi = 0, \quad \pi_7 (\Omega_A) = \sigma (\psi) \tag{3.1}
\]

where \( A \) is the 1-form of the \( i \mathbb{R} \)-valued connection on \( P_1 \) and \( \psi \in \Gamma (S) \).

These equations admit non-trivial solutions. Consider, for example, the flat case \( M = \mathbb{R}^7 \) with the \( G_2 \)-structure given by (2.1). Then the spinor bundle is \( \mathbb{R}^7 \times \mathbb{C}^8 \). We use the spin representation emerging from the isomorphism \( \mathbb{C}l_7 \cong \text{End}(\mathbb{C}^8) \oplus \text{End}(\mathbb{C}^8) \), where \( \mathbb{C}l_7 \) is the complex Clifford algebra with 7 generators and \( \text{End}(\mathbb{C}^8) \) denotes the space of \( 8 \times 8 \) complex matrices. A direct verification shows that \( \psi = (\psi_1, i\psi_1, 0, 0, i\psi_1, \psi_1, 0, 0) \) with \( \psi_1 (x_1, x_2, \ldots, x_7) = e^{-\frac{i}{2} x_1^2 x_2} \) and \( A(x_1, x_2, \ldots, x_7) = (ix_1 x_2)dx_1 + (\frac{i}{2} x_1^2)dx_2 \) satisfy our Seiberg-Witten-like equations (3.1).

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References


