Universal Solitonic Hierarchy

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Abstract

We describe recent results on the construction of hierarchies of nonlinear evolution equations associated to generalized second order spectral problems. The first results in this subject had been presented by Francesco Calogero.

Introduction

In order to introduce the G-model considered recently in joint with L. Martinez Alonso papers, one can exploit an invariance of this model under hodograph type transformations. Namely, let us begin with the hierarchy of times $t_i, i \in \mathbb{Z}$ and the corresponding set of commuting differentiations $D_i = \partial_{t_i}$. Choose $t_0$ as a basic independent variable and change variables $t = (\ldots, t_{-1}, t_0, t_1, \ldots)$ into $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$ by

$$x_0 = q(t), \quad x_j = t_j, j \neq 0 \Rightarrow b_0 dx_0 = dt_0 + \sum_{1}^{\infty} g_k dt_k - \sum_{1}^{\infty} b_k dt_{-k}. \quad (0.1)$$

Then

$$D_0(q) = \frac{1}{b_0}, \quad D_j(q) = \begin{cases} \frac{g_j}{b_0} & \text{for } j > 0, \\ -\frac{b_j}{b_0} & \text{for } j < 0 \end{cases} \quad (0.2)$$

and we see that

$$D_1 \left( \frac{D_1 q}{D_0 q} \right) = D_0 \left( \frac{D_2 q}{D_0 q} \right) \Leftrightarrow D_1(g_1) = D_0(g_2). \quad (0.3)$$

This three-dimensional nonlinear PDE for the function $q(t)$ represents one of multi-dimensional versions of equations of $G$-model considered in the last paper of ref. [3].

The full set of equations of the $G$-model can be introduced by

$$D_n(G) = < A_n, G >, \quad < A, B > \overset{\text{def}}{=} AD_0(B) - BD_0(A), \quad (0.4)$$

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where $G$ is an auxiliary function and, for any $n > 0$, we set

$$A_n \equiv \lambda^n + g_1 \lambda^{n-1} + \cdots + g_n, \quad A_{-n} \equiv \frac{b_0}{\lambda^n} + \frac{b_1}{\lambda^{n-1}} + \cdots + \frac{b_{n-1}}{\lambda}.$$  

(0.5)

In §1 we reformulate some results obtained by [3].

In §§2 and 3 we discuss applications of the $G$-model to the generalized second order spectral problems

$$\psi_{xx} = U(x, \lambda) \psi, \quad D_x \equiv D_0$$

with potential

$$U = U(x, \lambda) = u_0(x) \lambda^n + u_1(x) \lambda^{n-1} + \cdots + u_m(x) + \sum_{k=1}^{\infty} \lambda^{-k} v_k(x), \quad x = t_0.$$  

(0.6)

The main point is that this generalization allows us to prove (Theorem 3) that there exist well-defined mappings $U \leftrightarrow G$ of potentials (0.6) into solutions of the equations (0.4) and vice versa.

All solitonic hierarchies are defined by the equations (0.4) up to this mapping and do not depend from the particular spectral problem.

1 Integrability of $G$-models

Excluding an auxiliary function $G$ from equations (0.4) by cross differentiation we get

$$D_n(A_m) - D_m(A_n) = <A_n, A_m>, \quad \text{for any} \quad n, m.$$  

(1.1)

Replacing the wronskian in the right hand side by the commutator,

$$[AD_0, BD_0] = <A, B > D_0, \quad <A, B > = AD_0(B) - BD_0(A),$$

we get the zero-curvature representation of the $G$-model:

$$[D_n - A_n D_0, D_m - A_m D_0] = 0, \quad \text{for any} \quad n, m.$$  

(1.2)

Due formulae (0.5) one obtains for the infinite set of dynamical variables

$$\ldots, b_j, b_{j-1}, \ldots, b_0, g_1, g_2, \ldots, g_j, \ldots$$  

(1.3)

the infinite set of equations which we take here for a definition of the $G$-model. It is important to notice that the change of variables (0.1) gives rise to an analogous to (1.2) set of commuting operators $\hat{D}_j = \partial_{x_j}$:

$$\hat{D}_{-n} = D_{-n} + b_n D_0, \quad \hat{D}_0 = b_0 D_0, \quad \hat{D}_n = D_n - g_n D_0.$$  

(1.4)

It can be proved (Cf [3]) that equations (1.1) are equivalent to equations (0.4) if we insert in the latter the asymptotic expansions

$$G = \begin{cases} 1 + \frac{g_1}{\lambda} + \cdots + \frac{g_n}{\lambda^n} + \cdots & \text{as} \quad \lambda \rightarrow \infty, \\ b_0 + \lambda b_1 + \cdots + \lambda^n b_n + \cdots & \text{as} \quad \lambda \rightarrow 0. \end{cases}$$  

(1.5)
Thus, the function $G$ can be used as the *generating function* of the hierarchy (1.2) and the infinite set of equations of the $G$-model can be rewritten in very compact form

$$D_n(G) = \langle \lambda^n G, G \rangle, \quad n \in \mathbb{Z},$$

where, for any $n > 0$,

$$(\lambda^n G)_+ = \lambda^n + g_1 \lambda^{n-1} + \cdots + g_n = A_n, \quad (\lambda^{-n} G)_+ = \frac{b_0}{\lambda^n} + \frac{b_1}{\lambda^{n-1}} + \cdots + \frac{b_{n-1}}{\lambda} = A_{-n}.$$

Summing up we formulate for future references:

**Theorem 1** ([3]) The nonlinear partial differential equations (1.2), (0.5) for the infinite set of dynamical variables (1.3) are equivalent to either equation of the following triad

$$D_n(G) = \langle A_n, G \rangle, \quad D_n(H) = D_0(A_n H), \quad \hat{D}_n(b_0 H) = \lambda b_0 D_0(A_{n-1} H), \quad (1.6)$$

where the operators $\hat{D}_n$ are defined by (1.4), $H = G^{-1}$ and $G$ is the generating function (1.5).

The substituting formal expansions (1.5) into equation $D_n(G) = \langle A_n, G \rangle$ define the action of $D_n$ on the dynamical variables $g_n$ and $b_n$, respectively. For example, for $n = 1$, we have $A_1 = \lambda + g_1$ and obtain in this case an infinite system of $1 + 1$-dimensional equations

$$D_1(g_n) = D_0(g_{n+1}) + \langle g_1, g_n \rangle, \quad D_1(b_n) = D_0(b_{n-1}) + \langle g_1, b_n \rangle. \quad (1.7)$$

Analogous formal expansions of $H = G^{-1}$ as $\lambda \to \infty$ and $\lambda \to 0$ yield the two series of conservation laws for the equations $D_n(G) = \langle A_n, G \rangle$. The coefficients of these formal series for $H$ are defined by inverting the series (1.5) as functions of dynamical variables (1.3).

Coming back to the change of variables (0.1), which was our starting point and will be used later on, we can now demonstrate the *invariance* of equations (1.2) by proving that

$$\hat{D}_n(H) = \hat{D}_1(A_{n-1} H), \quad \forall n \neq 0. \quad (1.8)$$

Thanks to (1.6) this equation means that the transformation $x_0 = q(t)$ does not change the form of equations yet results in the shift of the numeration $n \mapsto n - 1$ of independent variables. Since

$$D_1(A_j) = D_0(A_{j+1}) + \langle g_1, A_j \rangle,$$

the equation (1.8) is a corollary of the last equation in (1.6):

$$\hat{D}_n(H) - \hat{D}_1(A_{n-1} H) = D_0(A_n) H + \lambda A_{n-1} D_0(H) - (D_1 - g_1 D_0)(A_{n-1} H) = 0.$$

**Integration**

It is most natural to ask what kind of restrictions will arise if the dynamics in the $G$-model is defined by a finite number of independent variables $t_n$. 
Theorem 2. Let $G(\lambda)$ be generating function from Theorem 1 and, for fixed $N > 0$, there exist constants $\varepsilon_1, \ldots, \varepsilon_N$ such that
\[
D_N + \varepsilon_1 D_{N-1} + \varepsilon_2 D_{N-2} + \cdots + \varepsilon_N D_0 = 0.
\]

Then there exist formal series with constant coefficients
\[
\varepsilon(\lambda) = 1 + \frac{\varepsilon_1}{\lambda} + \frac{\varepsilon_2}{\lambda^2} + \frac{\varepsilon_3}{\lambda^3} + \cdots
\]
such that the product $\lambda^N \varepsilon(\lambda)G(\lambda)$ is a polynomial in $\lambda$:
\[
\hat{G}(\lambda) = \varepsilon(\lambda)G(\lambda) = 1 + \frac{\hat{g}_1}{\lambda} + \frac{\hat{g}_2}{\lambda^2} + \cdots + \frac{\hat{g}_n}{\lambda^n}.
\]

Proof. We have
\[
D_0(g_{n+1}) = D_n(g_1) = -[\varepsilon_1 D_{n-1} + \cdots + \varepsilon_n D_0](g_1) = -D_0[\varepsilon_1 g_n + \cdots + \varepsilon_n g_1] 
\]
since $D_n g_1 = D_0 g_{n+1}$ (see Theorem 1). We use now that
\[
D_0(g) = D_0(h) \iff g = h + \text{const}
\]
which means that there exist a constant $\varepsilon_{n+1}$ such that
\[
g_{n+1} + \varepsilon_1 g_n + \cdots + \varepsilon_n g_1 + \varepsilon_{n+1} = 0.
\]

It remains to notice that, for any $m$,
\[
-g_{m+1} = \varepsilon_1 g_m + \cdots + \varepsilon_m g_1 + \varepsilon_{m+1} = -g_{m+2} = \varepsilon_1 g_{m+1} + \cdots + \varepsilon_m g_2 + \varepsilon_{m+1} g_1 + \varepsilon_{m+2}
\]
since
\[
D_0(g_{m+2}) = D_1(g_{m+1}) - <g_1, g_{m+1}>.
\]

Indeed, insert $g_{m+1} = -\varepsilon_1 g_m - \cdots - \varepsilon_m g_1$; we obtain
\[
-<g_1, g_{m+1}> = \varepsilon_1 <g_1, g_m> + \cdots + \varepsilon_m <g_1, g_2> - \varepsilon_{m+1} D_0 g_1,
\]
\[
-D_1(g_{m+1}) = D_1[\varepsilon_1 g_m + \cdots + \varepsilon_m g_1] =
\]
\[
D_0[\varepsilon_1 g_{m+1} + \cdots + \varepsilon_m g_2] + \varepsilon_1 <g_1, g_m> + \cdots + \varepsilon_m <g_1, g_2>.
\]

Corollary. The roots $\gamma_1, \ldots, \gamma_N$ of the polynomial $\lambda^N \varepsilon(\lambda)G(\lambda)$ satisfy the hyperbolic quasilinear system as follows
\[
(D_1 + \varepsilon_1 D_0)\gamma_j = (\sum_{k \neq j} \gamma_k) D_0 \gamma_j, \quad j = 1, 2, \ldots, N.
\]

Indeed: the modified generating function $\hat{G} = \varepsilon(\lambda)G(\lambda)$ satisfies the same equations
\[
D_n \hat{G} = <A_n, \hat{G}>.
as the original generating function and
\[
\hat{G}(\lambda) = \prod_{1}^{N} \left(1 - \frac{\gamma_{j}}{X}\right) \Rightarrow D_{n}\hat{G}|_{\lambda = \gamma_{j}} = -\frac{D_{n}\gamma_{j}}{\gamma_{j}} \prod_{k \neq j}^{N} \left(1 - \frac{\gamma_{k}}{\gamma_{j}}\right). \tag{1.10}
\]

If \( n = 1 \), this yields (1.9) since
\[
A_{1} = \lambda + g_{1} = \lambda - \sum_{k}^{N} \gamma_{k} - \varepsilon_{1} \Rightarrow A_{1}|_{\lambda = \gamma_{j}} = -\sum_{k \neq j}^{N} \gamma_{k} - \varepsilon_{1}.
\]

The hyperbolic system (1.9) has many applications including the gas dynamics [4] as well as finite-gap potentials in spectral theory [6].

Using the roots variables \( \vec{\gamma} = (\gamma_{1}, \ldots, \gamma_{N}) \) defined in (1.10) as dynamical ones we can write down the whole set of equations \( D_{n}\hat{G} = < A_{n}, \hat{G} > \), where \( n = 1, 2, \ldots, N - 1 \), in the form analogous to (1.9):
\[
D_{n}(\gamma_{j}) = a_{jn}(\vec{\gamma})D_{0}(\gamma_{j}), \quad n = 1, \ldots, N - 1.
\]

Introduce now an auxiliary vector field
\[
D_{0}(\vec{\gamma}) = X(\vec{\gamma}), \quad X = (X^{1}, \ldots, X^{N}).
\]

Then the above \( N - 1 \) equations for the roots variables can be rewritten as \( N - 1 \) dynamical systems
\[
D_{n}(\vec{\gamma}) = \hat{A}_{n}(\vec{\gamma}), \quad \text{where} \quad n = 1, 2, \ldots, N - 1. \tag{1.11}
\]

These dynamical systems can be integrated in quadratures with the help of

**Ferapontov’s Lemma** \(([5])\) Let
\[
D_{0}(\gamma_{j}) = z_{j}(\gamma_{j}) \left(\prod_{k \neq j}^{N} \gamma_{jk}\right)^{-1}, \quad \gamma_{jk} \overset{\text{def}}{=} \gamma_{j} - \gamma_{k} \tag{1.12}
\]
where the \( N \) functions in one variable \( z_{j}(\gamma_{j}) \) are arbitrary.

Then \( N \) dynamical systems (1.12) together with (1.11) for root variables \( \vec{\gamma} = (\gamma_{1}, \ldots, \gamma_{N}) \) define \( N \) commuting vector fields.

**Remark.** Together with (1.12) the dynamical systems (1.11) can be rewritten as
\[
d\vec{\gamma} = d\vec{t}A(\vec{\gamma}) \quad \text{or, equivalently, as} \quad d\vec{t} = d\vec{\gamma} \hat{A}(\vec{\gamma}) \tag{1.13}
\]
where
\[
d\vec{\gamma} \overset{\text{def}}{=} (z_{1}(\gamma_{1})d\gamma_{1}, \ldots, z_{N}(\gamma_{N})d\gamma_{N})
\]
and the differential form in the right hand side (1.13) is a closed one by commutativity of the corresponding vector fields.
2 Generalized spectral problems

Applications of the $G$-model to the second order spectral problems

$$\psi_{xx} = U(x, \lambda)\psi$$  \hspace{1cm} (2.1)

are based on a mapping $G \mapsto U$ defined by the exact formula

$$U(x, \lambda) = \{D_x, \frac{1}{G(x, \lambda)}\} + \frac{z(\lambda)}{G(x, \lambda)^2}, \quad D_x \equiv D_0$$  \hspace{1cm} (2.2)

where $z(\lambda)$ is an $x$-independent normalizing coefficient and

$$\{D_x, F(x)\} \text{ def} = \frac{3}{4} \frac{F_x^2}{F^2} - \frac{1}{2} \frac{F_{xx}}{F} = \varphi_{xx} + \varphi_x^2, \quad \text{if} \quad F = e^{-2\varphi}.$$  \hspace{1cm} (2.3)

Rewriting (2.2) in the bilinear form

$$4UG_x^2 + G_x^2 - 2GG_{xx} = z(\lambda)$$  \hspace{1cm} (2.4)

and differentiating it with respect to $x$ we get the third order linear differential equation

$$G_{xxx} = 4UG_x + 2U_xG.$$  \hspace{1cm} (2.5)

On the other hand, the differentiating (2.2) with respect the “time” $\tau$ and using equations (1.6), (2.5) one gets

$$D_\tau(G) = \langle A, G \rangle \Rightarrow 2D_\tau(U) = 4UA_x + 2U_xA - A_{xxx}, \quad x \equiv t_0.$$  \hspace{1cm} (2.6)

**Remark.** Recall that if $\psi_1, \psi_2$ constitute a basis of the linear space of solutions (2.1) then $G = \{\psi_1^2, \psi_2^2, \psi_1\psi_2\}$ is a fundamental system of solutions of (2.5). Vice versa, in order to reconstruct $\psi_1, \psi_2$ from a given $G$ and the wronskian $w = \langle \psi_1, \psi_2 \rangle = \psi_1\psi_{2,x} - \psi_2\psi_{1,x}$ we have

$$G = \psi_1\psi_2 \Rightarrow \frac{G_x}{G} = \frac{\psi_{1,x}}{\psi_1} + \frac{\psi_{2,x}}{\psi_2}, \quad \frac{\langle \psi_1, \psi_2 \rangle}{G} = \frac{\psi_{2,x}}{\psi_2} - \frac{\psi_{1,x}}{\psi_1}.$$  

Thus

$$\frac{\psi_{1,x}}{\psi_1} = \frac{1}{2} \left( \frac{G_x}{G} - \frac{w}{G} \right), \quad \frac{\psi_{2,x}}{\psi_2} = \frac{1}{2} \left( \frac{G_x}{G} + \frac{w}{G} \right).$$

Recall also that the second equation in (2.6) which describes the evolution of the potential $U$ is the consistency condition for the equations

$$\psi_{xx} = U\psi, \quad \text{and} \quad \psi_\tau = A\psi_x - \frac{1}{2} A_x\psi.$$  

Lastly, one can notice that latter equation for $\psi$ together with $G = \psi_1\psi_2$ yield the first of equations (2.6) $D_\tau(G) = \langle A, G \rangle$.

In the general case the mapping $G \mapsto U$ defined by (2.2), where

$$z(\lambda) = \lambda^m + \alpha_1\lambda^{m-1} + \cdots + \alpha_0 + \sum_{j>0} \beta_j\lambda^{-j},$$
and $G$ represented by the asymptotic expansion, yields the formal Laurent series

$$U = U(x, \lambda) = \lambda^m + u_1(x)\lambda^{m-1} + \cdots + u_m(x) + \sum_{k=1}^{\infty} \lambda^{-k}v_k(x), \quad \text{where} \quad x = t_0. \quad (2.7)$$

Equation (2.1) with the potential $U$ represented as $\lambda \to \infty$ by its asymptotic expansion (2.7) is said to be the generalized spectral problem.

**Theorem 3.** Let the potential $U(x, \lambda)$ of the generalized spectral problem (2.1) be represented by the formal Laurent series (2.7) with $m > 0$. Then the third order equation (2.5) associated with the spectral problem possesses a unique normalized solution $G = Y = \sum y_k\lambda^{-k}$ such that

$$Y(x, \lambda) = 1 + \sum_{k=1}^{\infty} \lambda^{-k}y_k(x), \quad -2Y_{xx}Y + Y_x^2 + 4UY^2 = 4\lambda^m. \quad (2.8)$$

*Proof.* We need to prove that equation (2.8) uniquely defines the coefficients $y_k$, where $k = 1, 2, \ldots$ in terms of the coefficients of the formal powers series (2.7).

It suffices to verify that consequently equating in (2.8) the coefficients of equal powers of $\lambda$

$$\lambda^m, \lambda^{m-1}, \ldots, \lambda^0, \lambda^{-1}, \ldots$$

we get a triangular type system. Thus we find

$$y_1 = -\frac{1}{2}\tilde{u}_1, \ldots, y_n = -\frac{1}{2}\tilde{u}_n + \Phi_n(y_1, \ldots, y_{n-1}; \tilde{u}_1, \ldots, \tilde{u}_{n-1}), \ldots,$$

where

$$\tilde{u}_j = \begin{cases} u_j & \text{for } j \leq m \\ v_j - m & \text{for } j > m \end{cases}$$

and functions $\Phi_n(\cdot; \cdot)$ are differential polynomials in their arguments. The first $m$ coefficients are obtained in a purely algebraic way since the equation (2.8) rewritten in the form (2.2) implies that the coefficients $u_1, \ldots, u_m$ of the potential coincide with the corresponding coefficients of the series $\lambda^mY^{-2}$.

Now we can incorporate $G$-model into the theory of spectral problems. Namely, choose $Y(x, \lambda)$ defined in Theorem 3 as the initial data at $t_n = 0$ for the dynamical equations (1.6). Then, thanks to eq. (2.6), the mapping (2.2) transforms solutions of this initial data problem into solutions of the hierarchy of evolutionary equations

$$2D_{t_n}(U) = 4UA_n,x + 2U_xA_n - A_{n,xxx}, \quad A_n = (\lambda^nY)_+, \quad n = 1, 2, 3, \ldots \quad (2.9)$$

for the potential $U = U(t, \lambda)$, where $t_0 = x$, of the generalized spectral problem. In particular, for the generalized Schrödinger spectral problem with

$$U = \lambda + u_1 + \frac{v_1}{\lambda} + \frac{v_2}{\lambda^2} + \ldots$$
we find, by setting \( A_1 = \lambda + y_1 \) and \( D_t = 2D_t \), the following generalization of Korteweg-de Vries equation:

\[
\begin{align*}
  u_t - \frac{1}{2}u_{xxx} + 3u_{ux} &= 2v_{1,x}, \\
  v_{1,t} + 2v_1 u_x + uv_{1,x} &= 2v_{2,x}, \ldots 
\end{align*}
\] (2.10)

which is equivalent to eq. (1.7).

Proof of Theorem 3 describes the inversion of the mapping introduced by the basic eq. (2.2) with \( z(\lambda) = \lambda^m \). This mapping \( Y = G \mapsto \tilde{Y} \) defines a differential substitution \( \tilde{u}_j = \tilde{\Psi}_j(\tilde{y}) \) transforming equations (1.6) into (2.9). For example, expressing \( U \) in terms of \( Y \) for \( m = 1 \) and \( m = 2 \) with

\[
U = \lambda + u_1 + \sum_{k \geq 1} \lambda^{-k}v_k, \quad \text{and} \quad U = \lambda^2 + \lambda u_1 + u_2 + \sum_{k \geq 1} \lambda^{-k}v_k,
\]

respectively, we find, in addition to eq. \( u_1 + 2y_1 = 0 \) above, that

\[
\begin{align*}
  v_1 &= -2y_2 + 3y_1^2 + \frac{1}{2}y_{1,xx}, \\
  v_2 &= -4y_3 + 12y_1y_2 - 8y_1^3 + y_{2,xx} - y_1y_{1,x} - \frac{1}{2}y_{1,x}^2 \quad (2.11)
\end{align*}
\]

if \( m = 1 \) and

\[
\begin{align*}
  u_2 &= -2y_2 + 3y_1^2, \\
  v_1 &= -4y_3 + 12y_1y_2 - 8y_1^3 + y_{1,xx} \quad (2.12)
\end{align*}
\]

if \( m = 2 \).

**The polynomial case**

Under the conditions of Theorem 2 we have, with a slightly misused notations,

\[
G(x, \lambda) = \lambda^N \varepsilon(\lambda)Y(\lambda) = \prod_{k \neq j} (\lambda - \gamma_k(x)).
\] (2.13)

The differential equations for the roots \( \gamma_1, \ldots, \gamma_N \) are now obtained (Cf Ferapontov’s Lemma) by substitution \( \lambda = \gamma_j \), where \( j = 1, \ldots, N \) in (2.4). This yields

\[
\frac{d}{dx} \gamma_j = \left( \prod_{k \neq j} \gamma_{jk} \right)^{-1} \tilde{z}(\gamma_j), \quad \tilde{z}(\lambda) = \frac{1}{2} \sqrt{z(\lambda)}
\] (2.14)

since the potential \( U(x, \lambda) \) is assumed to be regular without movable singularities in the \( \lambda \)-plane.

Dubrovin’s equations (2.14) are completely defined by the function \( z(\lambda) \). In its turn, \( z(\lambda) \) is defined by the right hand side of (2.4) and has to be an \((m + 2N)\)th degree polynomial in \( \lambda \) for spectral problems with \( m \)th degree polynomial potentials. The other way round, the formula (2.2) defines the potential as a degree \( m \) polynomial in \( \lambda \) for any polynomial \( z(\lambda) \) of degree \( m + 2N \) thanks to the following

**Dubrovin’s Lemma**  Let \( z(\lambda) \) be meromorphic and the roots \( \gamma_j \) of an \( N \)-th degree unitary polynomial \( G(x, \lambda) \) satisfy equations (2.14). Then

\[
\lambda = \gamma_j \Rightarrow G(x, \lambda) = 0 \Rightarrow 2G_{xx}(x, \lambda) + \frac{z(\lambda) - G^2_x(x, \lambda)}{G(x, \lambda)} = 0.
\]
Corollary Let us, under conditions of Dubrovin’s Lemma, denote by $\hat{z}(\lambda; \vec{\gamma})$ an $N$–th degree polynomial in $\lambda$ such that

$$\hat{z}(\lambda; \vec{\gamma})|_{\lambda=\gamma_j}=z(\lambda)|_{\lambda=\gamma_j}, \quad \hat{z}'(\lambda; \vec{\gamma})|_{\lambda=\gamma_j}=z'(\lambda)|_{\lambda=\gamma_j}.$$ 

Then eq. (2.2) for potential $U$ of the generalized spectral problem can be expressed as follows

$$4U \prod (\lambda - \gamma_j)^2 = z(\lambda) - \hat{z}(\lambda; \vec{\gamma}).$$

3 The problem of constraints

The change of variables in the $G$-model defined by (2.2) allows us to reformulate the problem of reductions (of an infinite set of dynamical variables to a finite one) in terms of equations (2.9) for coefficients of the formal series (2.7). Thus the truncation $v_j = 0$ for $j \geq 1$ in (2.10) yields the classical KdV equation while the next possibility $v_j = 0$ for $j \geq 2$ leads to the system of equations for $u = u_1$ and $v = 4v_1$:

$$2u_t = u_{xxx} - 6uu_x + v_x, \quad v_t + 2vu_x + uv_x = 0.$$

The polynomial constraints have been considered in [3]. They correspond to the case where $U(t, \lambda)$ and $z(\lambda)$ in (2.2) are polynomials of the same degree in $\lambda$ and give rise to some interesting modifications of the classical solitonic hierarchies (see next subsection). These polynomial constraints are in good agreement with Dubrovin’s Lemma but one has to keep in mind that this lemma and its corrolary do not forbid pole singularities in $\lambda$.

In order to highlight the possibility of more direct approach to the problem of constraints let us consider a simple example related with the Bürgers hierarchy. The infinite system of equations

$$D_t G = \langle \lambda + g_1, G \rangle$$

for the coefficients $g_j$ can be reduced, apparently, to a single evolutionary equation for the function $u = g_1$ in a very elementary way:

$$g_2 = f(g_1, g_{1,x}) \Rightarrow u_t = Df(u, u_x), \quad \text{where} \quad u \equiv g_1, \quad D_t \equiv D_1.$$ (3.2)

Lemma Let an evolutionary equation of the second order $u_t = D_0 f(u, u_x)$ possess the conservation law with density $g = g(u, u_x)$. Then $g = a(u)u_x + b(u)$.

Proof Denoting by $\sim$ equivalence modulo $\text{Im} D_0$ and setting $g_0 = \partial_u g(u, u_1)$ and $g_1 = \partial_{u_1} g(u, u_1)$ we obtain by integration by parts

$$D_t(g) = g_0 D_0 f + g_1 D_0^2 f \sim D_0 f(g_0 - D_0 g_1) = f_1 g_{11} u_{xx}^2 + \ldots$$

By assumption $D_t g \in \text{Im} D_0$, so the right hand side should be represented by $D_0 h(u, u_x)$ and thus $f_1 g_{11} = 0$. 

The constraint (3.2) should be compatible with the original system if equations (3.1) allow one to express all coefficients $g_j$ as differential functions of $u$ (i.e., functions of $u$ and derivatives of $u$ with respect to $x$). Using Theorem 1 and the above Lemma one can readily prove the following
Proposition  The constraint (3.2) is compatible with the system of equations (3.1) if and only if
\[ g_2 = g_1^2 + \varepsilon_1 g_{1,x} + \varepsilon_2 g_1 + \varepsilon_3, \]
where \( \varepsilon_j \) are arbitrary constants.

Camassa-Holm equation

Straightforward application of Theorem 3 to the polynomial potentials (2.7) gives rise to the constraints \( v_1 = \Psi(\vec{y}) = 0 \). Using formulae (2.11), (2.12) it yields, for example,
\[ m = 1 \Rightarrow 2y_2 = 3y_1^2 + \frac{1}{2} y_{1,xx}, \quad m = 2 \Rightarrow 4y_3 = 12y_1y_2 - 8y_1^3 + D_0^2 y_1. \]
These constraints correspond to KdV and NLS hierarchies, respectively.

Invoking Theorem 1 and hodograph type transformations one can substantially enlarge the set of differential constraints defined by (2.2) in the polynomial case (see [3]). For simplicity we confine ourselves to the case \( m = 1 \) and, accordingly, set \( z(\lambda) = \lambda + \varepsilon \) in (2.2). In terms of \( H = G^{-1} \) this yields (Cf (2.3)):
\[ U(t, \lambda) = \{ D_0, H \} + (\lambda + \varepsilon) H^2. \quad (3.3) \]
The main point is that (3.3) allows us now to relate the Taylor series expansion of \( H \) at \( \lambda = 0 \) (Cf (1.5)) with the asymptotic expansion \( H = 1 + h_1 \lambda^{-1} + \ldots \) at the \( \lambda \)-infinity used in Theorem 3 and which implies that \( U = \lambda + \varepsilon + 2h_1 \) in (3.3). Thus, at \( \lambda = 0 \) eq. (3.3) implies a new form of the constraint:
\[ H = a_0 + \lambda a_1 + \ldots \Rightarrow 2h_1 = \{ D_0, a_0 \} + \varepsilon a_0^2 - \varepsilon \]
and a modified KdV equation (Cf [2]) as follows
\[ (4D_1 - 2\varepsilon D_0) \varphi = D_0^3 \varphi - \frac{1}{2} (D_0 \varphi)^3 - 6\varepsilon \varphi D_0 \varphi, \quad a_0 = e^\varphi. \]

Considering equations with negative numbers in the hierarchy (1.6) it is natural to introduce renormalization as follows
\[ \hat{H} = b_0 H, \quad b_0 a_0 = 1 \Rightarrow H \longrightarrow 1 (\lambda \longrightarrow \infty), \quad \hat{H} \longrightarrow 1 (\lambda \longrightarrow 0). \quad (3.4) \]
Combined with the “hodograph” (0.1) change of variables \( t \mapsto x \) and \( \hat{D}_0 = b_0 D_0 \) it transforms (3.3) into
\[ \hat{U}(x, \lambda) = \lambda b_0^2 + \varepsilon = \{ \hat{D}_0, \hat{H} \} + (\lambda + \varepsilon) \hat{H}^2, \quad a_0(t)b_0(x) = 1 \quad (3.5) \]
since
\[ \{ \hat{D}_0, \hat{H} \} = b_0^2 (\{ D_0, H \} - \{ D_0, a_0 \}), \quad \hat{D}_0 = b_0 D_0, \quad \hat{H} = b_0 H. \]
In the notations
\[ D_x = \hat{D}_0, \quad D_\tau = \hat{D}_{-1}, \quad \hat{H} = 1 + \lambda u + \ldots, \quad \lambda \longrightarrow 0 \]
we obtain, with Theorem 1,
\[ \lambda D_\tau(\hat{H}) = D_x[(1 - \lambda u)\hat{H}] \Rightarrow b_0 \tau + (u b_0)_x = 0. \quad (3.6) \]
Eq. (3.5) yields the constraint
\[ b_0^2 = 1 + 2\varepsilon u - \frac{1}{2} u_{xx} \]
which together with the last equation in (3.6) is equivalent to the Camassa-Holm equation [1].
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