A factorization for $\mathbb{Z} \times \mathbb{Z}$-matrices yielding solutions of Toda-type hierarchies

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Abstract

In this paper one considers the problem of finding solutions to a number of Toda-type hierarchies. All of them are associated with a commutative subalgebra of the $k \times k$-matrices. The first one is formulated in terms of upper triangular $\mathbb{Z} \times \mathbb{Z}$-matrices, the second one in terms of lower triangular ones and the third is a combination of the two foregoing types. It is shown that in an appropriate group setting solutions of the linearization of these Lax equations can be constructed by using a Birkhoff-type decomposition in the relevant group.

1 Introduction

A well-known example of a system of differential difference equations is the equations of motion of an infinite number of particles on a straight line, the so-called infinite Toda-chain. Recall from [11] that these equations in dimensionless form have the form

$$\frac{dq_n}{dt} = p_n \quad \text{and} \quad \frac{dp_n}{dt} = e^{-(q_n-q_{n-1})} - e^{-(q_{n+1}-q_n)}, \quad n \in \mathbb{Z}. \quad (1.1)$$

Here $q_n$ is the displacement of the $n$-th particle. One can reformulate these equations as an equality between infinite matrices by defining

$$a_n := \frac{1}{2} e^{-(q_n-q_{n-1})} \quad \text{and} \quad b_n := \frac{1}{2} p_n.$$

The equations (1.1) get then the form

$$\frac{da_n}{dt} = a_n(b_n - b_{n-1}) \quad \text{and} \quad \frac{db_n}{dt} = 2(a_{n-1}^2 - a_n^2), \quad n \in \mathbb{Z}. \quad (1.2)$$

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If one introduces the $\mathbb{Z} \times \mathbb{Z}$-matrices $L$ and $B$ by

\[
L = \begin{pmatrix}
\ddots & \ddots & \ddots & \cdots & 0 \\
\ddots & b_{n-1} & a_n & 0 & \ddots \\
\ddots & a_n & b_n & a_{n+1} & \ddots \\
0 & a_{n+1} & b_{n+1} & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
\ddots & \ddots & \ddots & \cdots & 0 \\
\ddots & 0 & -a_n & 0 & \ddots \\
\ddots & a_n & 0 & -a_{n+1} & \ddots \\
0 & a_{n+1} & 0 & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},
\]

then a direct computation shows that the equations (1.2) amount to the matrix equation

\[
\frac{dL}{dt} = BL - LB = [B, L].
\]

This is an example of a so-called Lax equation, because it suggests that the matrix $L$ is obtained by conjugating a matrix that does not depend on $t$ with a $t$-dependent one, a phenomenon first observed in the KdV-setting by P. Lax, see [8]. Several variations and extensions on the above situation have been considered, see e.g. [2], [12] and [6], the last one describing the algebraic structure behind various formulations. These systems of equations play a role in various parts of mathematics, like random matrices and orthogonal polynomials, see [1] and [5], but also in a diversity of subjects from theoretical physics, such as matrix models, quantum gravity and string theory. To get an impression of these connections we refer to [10], [7], [3] and [9].

In this paper one considers the problem of finding solutions to a number of Toda-type hierarchies. All of them are associated with a commutative subalgebra of the $k \times k$-matrices. The first one is formulated in terms of upper triangular $\mathbb{Z} \times \mathbb{Z}$-matrices, the second one in terms of lower triangular ones and the third is a combination of the two foregoing types. It is shown that in an appropriate group setting solutions of the linearization of these Lax equations can be constructed by using the Birkhoff decomposition in the relevant group. A description of the various sections is as follows: the first introduces the relevant notations and general properties of $\mathbb{Z} \times \mathbb{Z}$-matrices. The next section discusses the various hierarchies with their corresponding linearizations and oscillating matrices of a certain type. It also gives a useful sufficiency criterion for oscillating matrices to yield solutions of the hierarchies. In the subsequent section, a Banach setting is presented in which the formal products from the linearization become genuine products. Also the decomposition of the group of commuting flows is discussed there. In the final section one concludes with the construction of the solutions.

2 The space $M_{\mathbb{Z}}(R)$

The hierarchies of Toda-type that form the subject of this paper consist of nonlinear equations for a number of $\mathbb{Z} \times \mathbb{Z}$-matrices whose coefficients are depending of the flow parameters. Therefore the basic prerequisites of this space will be recalled.

Let $R$ be a commutative ring. The ring of $k \times k$-matrices with coefficients from $R$ is denoted by $M_k(R)$. Likewise one writes $M_{\mathbb{Z}}(R)$ for the $R$-module of $\mathbb{Z} \times \mathbb{Z}$-matrices with coefficients from $R$. The ordering of the columns and rows in $M_{\mathbb{Z}}(R)$ that will be used is
the one that is compatible with the finite dimensional case, i.e. any matrix $A = (\alpha_{ij})$ in $M_{\mathbb{Z}}(R)$ is denoted by

$$A = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots \\ \alpha_{n-1,n-1} & \alpha_{n-1,n} & \alpha_{n-1,n+1} & \ldots \\ \alpha_{n-1,n} & \alpha_{n,n} & \alpha_{n,n+1} & \ldots \\ \alpha_{n-1,n+1} & \alpha_{n,n+1} & \alpha_{n,n+2} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$  

There are a number of special elements in $M_{\mathbb{Z}}(R)$ that will be used frequently. First of all, there is the basic matrix $E_{(i,j)}$, $i$ and $j \in \mathbb{Z}$, given by

$$(E_{(i,j)})_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}.$$  

Thus one can describe every $A = (A_{ij}) \in M_{\mathbb{Z}}(R)$ as a formal linear combination of the basic matrices

$$A = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} A_{ij} E_{(i,j)}.$$  

A dominant role in this paper is played by the shift matrix $\Lambda$ given by

$$\Lambda = \sum_{i \in \mathbb{Z}} E_{(i-1,i)}.$$  

With every collection $\{d(ks)|s \in \mathbb{Z}\}$ of matrices in $M_k(R)$ one associates the diagonal of $k$-blocks $\text{diag}(d(ks))$ in $M_{\mathbb{Z}}(R)$ given by

$$\text{diag}(d(ks)) := \sum_{s \in \mathbb{Z}} \sum_{\alpha=1}^k \sum_{\beta=1}^k d(ks)_{\alpha\beta} E_{(s+\alpha-1,s+\beta-1)}.$$  

Its matrix looks as follows

$$\begin{pmatrix} \ldots & \ldots & \ldots & \ldots \\ d(kn-k) & 0 & 0 & \ldots \\ 0 & d(kn) & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$  

For each $k \geq 1$, denote the ring of $k$-block diagonal matrices in $M_{\mathbb{Z}}(R)$ by

$$\mathcal{D}_k(R) = \{d = \text{diag}(d(ks))|d(ks) \in M_k(R) \text{ for all } s \in \mathbb{Z}\}.$$  

One has a ringhomomorphism $i_k$ from $M_k(R)$ into $\mathcal{D}_k(R)$ by taking for an $A \in M_k(R)$ all diagonal blocks of $i_k(A)$ equal to $A$. In particular every $\mathcal{D}_k(R)$ becomes a $M_k(R)$-algebra in this way. The elements $\Lambda^{km}$, $m \in \mathbb{Z}$, act on $\mathcal{D}_k(R)$ according to

$$\Lambda^{km} \text{diag}(d(ks)) \Lambda^{-km} = \text{diag}(d(ks + km)). \quad (2.1)$$
A specific choice of the element \( w \) relation implies further that for each \( m \neq 0 \) any \( d \Lambda^{km} \) with \( d = \text{diag}(d(ks)) \in \mathcal{D}_k(R) \) invertible can be written as

\[
\text{diag}(d(ks)) \Lambda^{km} = w_0 \Lambda^{km} w^{-1}_0, \quad \text{with } w_0 \in \mathcal{D}_k(R).
\]

A specific choice of the element \( w_0 = \text{diag}(w_0(ks)) \) is as follows: first of all one takes for all \( s_0 \) with \( 0 \leq s_0 < |m| \), the matrix \( w_0(s_0k) \) equal to the identity. Next one defines for all these \( s_0 \) and all \( t \geq 1 \)

\[
\begin{align*}
w_0((s_0 + tm)k) &:= d((s_0 + (t-1)m)k)^{-1} \cdots d(s_0)^{-1}, \\
w_0((s_0 - tm)k) &:= d((s_0 - tm)k) \cdots d((s_0 - m)k)
\end{align*}
\]

(2.2)

Each matrix in \( M_\mathbb{Z}(R) \) can be divided into so-called \( k \)-block diagonals. For, one defines namely

**Definition 1.** For each \( j \in \mathbb{Z} \), the \( j \)-th \( k \)-block diagonal of any matrix \( A = (A_{ij}) \in M_\mathbb{Z}(R) \), is the \( \mathbb{Z} \times \mathbb{Z} \)-matrix

\[
\sum_{i \in \mathbb{Z}} \sum_{\alpha = 1}^{k} \sum_{\beta = 1}^{k} A_{(ki-kj+\alpha-1, ki+\beta-1)} E_{(ki-kj+\alpha-1, ki+\beta-1)}.
\]

From equation (2.1) it is clear that the \( j \)-th \( k \)-block diagonal of a \( \mathbb{Z} \times \mathbb{Z} \)-matrix \( A \) can uniquely be written in the form \( \text{diag}(d(ks)) \Lambda^{kj} \) or \( \Lambda^{kj} \text{diag}(c(ks)) \) with \( \text{diag}(d(ks)) \) and \( \text{diag}(c(ks)) \in \mathcal{D}_k(R) \). Thus each \( A = (A_{(i,j)}) \in M_\mathbb{Z}(R) \) can uniquely be written as

\[
A = \sum_{j \in \mathbb{Z}} d_j \Lambda^{kj} \quad \text{or} \quad A = \sum_{j \in \mathbb{Z}} \Lambda^{kj} c_j,
\]

(2.3)

with \( d_j \) and \( c_j \) in \( \mathcal{D}_k(R) \). In particular any matrix that commutes with \( \Lambda^k \) has the form (2.3) with \( d_j \) and \( c_j \) in the image of \( i_k \).

To the first decomposition in (2.3) one links two notations: if \( A = \sum_{j \in \mathbb{Z}} d_j \Lambda^j \) as in (2.3) then one writes

\[
A_+(k) = \sum_{j \geq 0} d_j \Lambda^{kj} \quad \text{and} \quad A_-(k) = \sum_{j < 0} d_j \Lambda^{kj}.
\]

Inside \( M_\mathbb{Z}(R) \) two subspaces are considered that are rings w.r.t. the usual product

**Definition 2.** An element \( A \) in \( M_\mathbb{Z}(R) \) is called upper \( k \)-block triangular of level \( m \), if it can be written as

\[
A = \sum_{j \geq m} d_j \Lambda^{kj}, \quad \text{with } d_j \in \mathcal{D}_k(R).
\]

One calls \( m \) the order of \( A \) in \( \Lambda^k \), if \( d_m \) is nonzero. The collection of all these elements is denoted by \( UT_m, UT_m(R) \) or \( UT_m^{(k)}(R) \), depending, if one has to stress where the coefficients come from, what the size of the blocks along the diagonal is or both. Likewise one uses the notations

\[
UT(R) := \bigcup_{k \in \mathbb{Z}} UT_k =: UT
\]

for the set of all uppertriangular matrices.
One verifies directly that $UT$ with the product forms an $R$-algebra. All commutative $R$-subalgebras of $UT$ that contain the element $\Lambda^k$ have the following form: choose any commutative $R$-subalgebra $C$ of $M_k(R)$, then the required algebra consists of the

$$U(C) := \{ \sum_{i \geq N} i_k(c_i)\Lambda^{ki} \mid \text{with } c_i \in C \text{ for all } i \}. $$

Likewise one introduces the opposite class of matrices

**Definition 3.** An element $A$ in $M_Z(R)$ is called lower $k$-block triangular of level $m$, if it can be written as

$$A = \sum_{j \leq m} d_j \Lambda^{kj}, \text{ with } d_j \in D(R). $$

Like for $UT$ we call $m$ the order of $A$ in $\Lambda^k$, if $d_m$ is nonzero. The collection of all these elements is denoted by $LT_m, LT_m(R)$ or $LT_m^k(R)$, depending of the dependence that has to be stressed. Similarly, the notations

$$LT(R) := \bigcup_{k \in Z} LT_k =: LT$$

are used for the set of all lower triangular matrices.

Again one verifies easily that $LT$ with the usual product forms an algebra over $R$. The commutative $R$-subalgebras of $UT$ containing the element $\Lambda^k$ can be described as follows: let $C$ as above be any commutative $R$-subalgebra of $M_k(R)$, then

$$L(C) := \{ \sum_{i \leq N} i_k(c_i)\Lambda^{ki} \mid \text{with } c_i \in C \text{ for all } i \}$$

is the required algebra. If $C$ is maximal commutative inside $M_k(R)$, then the same holds for $L(C)$.

Note that, if $U \in UT$ and $V \in LT$ have the form respectively

$$U = \sum_{i \geq 0} u_i \Lambda^{ik} \text{ and } V = \sum_{i \leq 0} v_i \Lambda^{ik}, $$

with $u_0$ and $v_0$ invertible in $D_k(R)$, then the elements $U$ and $V$ are invertible and the diagonal $k$-block components of their inverses can be computed recursively.

## 3 The hierarchies of Toda-type

From now on one assumes that $R$ is an algebra over $\mathbb{C}$ such that $\mathbb{C}$ is isomorphic to the subalgebra $\{ \alpha.1 \mid \alpha \in \mathbb{C} \}$ of $R$. Then $M_k(\mathbb{C})$ is naturally a subring of $M_k(R)$.

The first hierarchy is similar to the multicomponent KP-hierarchy and its variations. Only now the hierarchy is formulated in terms of $Z \times Z$-matrices, not of pseudodifferential operators. One starts with a commutative subalgebra $h_L$ of $M_k(\mathbb{C})$. Let $\{ E_\alpha \mid 1 \leq \alpha \leq m_L \}$ be a basis of $h_L$. Inside $LT$ one is interested in matrices of the form

$$\mathcal{L} := \sum_{i \leq 1} l_i \Lambda^{ki} \text{ and } U_\alpha = \sum_{i \leq 0} u_{i,\alpha} \Lambda^{ki}, \text{ } 1 \leq \alpha \leq m_L, \quad (3.1)$$

where $l_i$, $u_{i,\alpha}$ are elements of $R$.
with \( l_k = i_k \text{(Id)} \) and \( u_{0,\alpha} = i_k(E_\alpha) \). They should be seen as perturbations inside \( LT \) of the special cases \( \mathcal{L} = \Lambda^k \) and \( U_\alpha = i_k(E_\alpha) \). As such the perturbation should first of all respect the commutation relations that existed between the basic generators

\[
[\mathcal{L}, U_\alpha] = 0 \quad \text{and} \quad [U_\alpha, U_\beta] = 0 \quad \text{for all } \alpha \text{ and } \beta,
\]

and secondly the algebraic relations inside \( h_L \)

\[
U_\alpha U_\beta = \sum_\gamma C_{\alpha\beta}^\gamma U_\gamma, \quad \text{where } E_\alpha E_\beta = \sum_\gamma C_{\alpha\beta}^\gamma E_\gamma.
\]

More importantly, the matrices \( \mathcal{L} \) and the \( U_\alpha \) should satisfy a number of nonlinear differential equations. For all the indices \( i \geq 0 \) and all \( \alpha, 1 \leq \alpha \leq m_L \), one writes \( \mathcal{P}_{\alpha} := \mathcal{L}^i U_\alpha \). Then the algebra \( R \) should be equipped with a collection of commuting \( \mathbb{C} \)-linear derivations \( \partial_{\mathcal{P}_{\alpha}} : R \to R \). In principle one thinks of \( \partial_{\mathcal{P}_{\alpha}} \) as representing differentiation in the direction \( \Lambda^k i_k(E_\alpha) \). The so-called Lax equations of the lower triangular \( h_L \)-hierarchy are then by definition

\[
\partial_{\mathcal{P}_{\alpha}}(\mathcal{L}) = [(\mathcal{P}_{\alpha})_+, \mathcal{L}] \quad \text{and} \quad \partial_{\mathcal{P}_{\alpha}}(U_\beta) = [(\mathcal{P}_{\alpha})_+, U_\beta]. \tag{3.2}
\]

These equations are equivalent to zero curvature relations for all the finite-band matrices \( \{ \mathcal{B}_{\alpha} := (\mathcal{P}_{\alpha})_+ \} \). The Lax equations also result from the so-called linearization. Thereto one considers the function

\[
\psi_0 := \exp(\sum_{i=0}^{\infty} t_{\mathcal{P}_{\alpha}} i_k(E_\alpha) \Lambda^k).
\]

It belongs to \( UT(\mathbb{C}[t_{\mathcal{P}_{\alpha}}]) \). It is the generator of the \( LT(R) \)-module \( M^{(\infty)} \) of oscillating matrices at infinity. This module consists of formal products

\[
\{ \sum_{j=-\infty}^{N} d_j \Lambda^k \} \exp(\sum_{i=0}^{\infty} t_{\mathcal{P}_{\alpha}} i_k(E_\alpha) \Lambda^k), \quad \text{where } d_j \in D_k(R).
\]

The action of \( LT(R) \) on \( M^{(\infty)} \) is defined as follows: for all \( p_1 \) and \( p_2 \in LT(R) \) one puts

\[
p_1 p_2 \exp(\sum_{i=0}^{\infty} t_{\mathcal{P}_{\alpha}} i_k(E_\alpha) \Lambda^k) = \{ p_1 p_2 \} \exp(\sum_{i=0}^{\infty} t_{\mathcal{P}_{\alpha}} i_k(E_\alpha) \Lambda^k).
\]

Clearly, it is a free \( LT(R) \)-module with \( \psi_0 \) as generator. Since the matrix \( \psi_0 \) commutes with \( \Lambda^k \) and \( i_k(E_\alpha) \), multiplication from the right with these elements is well-defined on \( M^{(\infty)} \). Also the action of the derivations \( \{ \partial_{\mathcal{P}_{\alpha}} \} \) can be extended to \( M^{(\infty)} \) as if the product was real

\[
\partial_{\mathcal{P}_{\alpha}} \{ \sum_{j=-\infty}^{N} d_j \Lambda^k \} \psi_0 = \{ \sum_{j=-\infty}^{N} \partial_{\mathcal{P}_{\alpha}}(d_j) \Lambda^k \} + \sum_{j=-\infty}^{N} d_j \Lambda^k(i_{j+1}k) \psi_0.
\]

For matrices \( \mathcal{L} \) and the \( U_\alpha \) of the form (3.1) one can consider inside \( M^{(\infty)} \) the equations

\[
\mathcal{L} \psi = \psi \Lambda^k, \quad U_\beta \psi = \psi i_k(E_\beta) \quad \text{and} \quad \partial_{\mathcal{P}_{\alpha}}(\psi) = \mathcal{B}_{\alpha} \psi, \tag{3.3}
\]
for all \( i \geq 0, 1 \leq \alpha \leq m_L \). This is the linearization of the lower triangular \( h_L \)-hierarchy. Under a mild condition on \( \psi \in M^{(\infty)} \) they imply that \( \mathcal{L} \) and the \( U_\alpha \) satisfy the Lax equations (3.2). Assume namely that \( \psi = \hat{\psi}\delta_L \psi_0 \), where \( \delta_L \) is an invertible element of \( LT(C) \) that commutes with \( \Lambda^k \) and all the \( i_k(E_\alpha) \) and where \( \hat{\psi} \in LT(R) \) is such that its leading coefficient is invertible. In particular \( \psi \) is invertible in \( LT(R) \) and this \( \psi \) is also a free generator of \( M^{(\infty)} \). An oscillating matrix at infinity of this form is called of type \( \delta_L \).

The first two equations of the linearization imply then that \( \mathcal{L} \) and the \( U_\beta \) are totally determined by \( \hat{\psi} \). Namely, they are equivalent to

\[
\mathcal{L} := \hat{\psi}\delta_L \Lambda^k \delta_L^{-1} \hat{\psi}^{-1} \quad \text{and} \quad U_\beta = \hat{\psi}\delta_L i_k(E_\beta) \delta_L^{-1} \hat{\psi}^{-1}
\]

To get the Lax equations for \( \mathcal{L} \) one applies \( \partial_{P_\alpha} \) to the first equation in (3.3) and substitute the last one. This yields

\[
\partial_{P_\alpha}(\mathcal{L}\psi - \psi\Lambda^k) = \partial_{P_\alpha}(\mathcal{L})\psi + \mathcal{L}(\partial_{P_\alpha}(\psi)) - (\partial_{P_\alpha}(\psi)\Lambda^k) = \frac{\partial_{P_\alpha}(\mathcal{L})}{\partial_{P_\alpha}(\psi)} - (\alpha, \mathcal{L})_\mathcal{L} \psi = 0.
\]

As \( \hat{\psi} \) is a free generator of the \( LT(R) \)-module \( M^{(\infty)} \),

\[
\partial_{P_\alpha}(\mathcal{L}) - [B_\alpha, \mathcal{L}] = 0.
\]

Similarly, applying \{\( \partial_{P_\alpha} \)\} to the second equation of the linearization and substituting the last one, renders

\[
\partial_{P_\alpha}(U_\beta \psi - \psi i_k(E_\alpha)) = \partial_{P_\alpha}(U_\beta) \psi + U_\beta (\partial_{P_\alpha}(\psi)) - (\partial_{P_\alpha}(\psi)E_\beta) = \frac{\partial_{P_\alpha}(U_\beta)\psi + U_\beta B_\alpha \psi - B_\alpha \psi E_\beta = \{\partial_{P_\alpha}(U_\beta) - [B_\alpha, U_\beta]\}\psi = 0}.
\]

This gives then the Lax equations for the \( U_\beta \). An oscillating matrix at infinity of type \( \delta_L \), \( \psi = \hat{\psi}\delta_L \psi_0 \), is also called a wavematrix at infinity of type \( \delta_L \) for the operators \( \mathcal{L} := \hat{\psi}\Lambda^k \hat{\psi}^{-1} \) and \( U_\beta = \hat{\psi} i_k(E_\beta) \hat{\psi}^{-1} \), if it satisfies the equations (3.3). To prove the equations (3.3) for an oscillating matrix at infinity \( \psi \) of the right form, one uses the fact that it suffices to prove a weaker result, namely

**Proposition 1.** Let \( \psi = \hat{\psi}\delta_L \psi_0 \), with \( \hat{\psi} - \Lambda^{kj} \in LT_{j-1} \), be an oscillating matrix at infinity of type \( \delta_L \). If it satisfies for all \( i \geq 0 \) and all \( 1 \leq \alpha \leq m_L \),

\[
\partial_{P_\alpha}(\psi) = F_{\alpha, \psi}, \quad \text{with} \quad F_{\alpha, \psi} \in LT(R) \cap UT_0(R),
\]

then \( F_{\alpha, \psi} = (\mathcal{L}^T U_\alpha)_+ \), where \( \mathcal{L} := \hat{\psi}\Lambda^k \hat{\psi}^{-1} \) and \( U_\beta = \hat{\psi} i_k(E_\beta) \hat{\psi}^{-1} \). In particular the \( \mathcal{L} \) and \( U_\beta \) form a solution to the lower triangular \( h_L \)-hierarchy.

**Proof.** From the definition of the action of \{\( \partial_{P_\alpha} \)\} on \( M^{(\infty)} \) and the fact that \( M^{(\infty)} \) is a free \( LT(R) \)-module with generator \( \psi_0 \), one obtains inside \( LT(R) \) the matrix equality

\[
\partial_{P_\alpha}(\hat{\psi}) + \hat{\psi}(\Lambda^k)^{i_k(E_\alpha)} = F_{\alpha, \hat{\psi}}.
\]

Since \( \partial_{P_\alpha}(\hat{\psi}) \hat{\psi}^{-1} \in LT_{-1} \), multiplying this equation from the right with \( \hat{\psi}^{-1} \) and taking the upper triangular part gives the desired result.\[\square\]
The second hierarchy is concerned with perturbations inside $UT(R)$ of the commutative algebra generated by $\Lambda^{-k}$ and $i_k(h_U)$ with $h_U$ a commutative algebra of $M_k(\mathbb{C})$. Let $\{F_\alpha | 1 \leq \alpha \leq m_U\}$ be a basis of $h_U$. Concretely, one searches then for matrices $\mathcal{M}$ and $V_\alpha$ of the form

$$\mathcal{M} := \sum_{i \geq -1} m_i \Lambda^{ki} \text{ and } V_\alpha := w_0 i_k(F_\alpha) w_0^{-1} + \sum_{i > 0} v_{i,\alpha} \Lambda^{ki}, \quad (3.7)$$

where the element $m_{-1}$ is invertible and $w_0 = \text{diag}(w_0(ks))$ is the gauge corresponding to

$$m_{-1} = w_0 \Lambda^{-k} w_0^{-1}$$

as given in equation (2.2). If the multiplication inside $h_U$ is given by

$$F_\alpha F_\beta = \sum_\gamma D^{\gamma}_{\alpha\beta} F_\gamma,$$

then the matrices $\mathcal{M}$ and $V_\alpha$ should first of all satisfy the algebraic relations of their unperturbed counterparts

$$[\mathcal{M}, V_\alpha] = 0 \text{ and } V_\alpha V_\beta = \sum_\gamma D^\gamma_{\alpha\beta} V_\gamma. \quad (3.8)$$

Note that these equations are automatically satisfied if one takes $\mathcal{M}$ and $V_\alpha$ of the form

$$\mathcal{M} = W \Lambda^{-k} W^{-1} \text{ and } V_\alpha = W i_k(F_\alpha) W^{-1},$$

with $W = \sum_{j \geq m} w_j \Lambda^{kj}, w_j \in D_j(R)$ and $w_m$ invertible. For all $j \geq 1$ and all $\beta, 1 \leq \beta \leq m_U$, one writes $Q_{\beta} := \mathcal{M}^j V_\beta$ and $C_{j\alpha} = (Q_{j\alpha})_{-}$. The search is for a $\mathbb{C}$-algebra $R$ equipped with a collection of $\mathbb{C}$-linear commuting derivations $\{\partial_{Q_{\beta}}, j \geq 1, 1 \leq \beta \leq m_U\}$. The nonlinear differential equations one wants $\mathcal{M}$ and the $V_\beta$ to satisfy are

$$\partial_{Q_{j\alpha}}(\mathcal{M}) = [C_{j\alpha}, \mathcal{M}] \text{ and } \partial_{Q_{j\beta}}(V_\beta) = [C_{j\alpha}, V_\beta] \quad (3.9)$$

and are called the **Lax equations of the upper triangular $h_U$-hierarchy**. They follow from a linear system that requires the introduction of a suitable left $UT(R)$-module. Again the actual form of the elements in the module is guided by the trivial solution $\mathcal{M} = \Lambda^{-k}$ and $V_\alpha = i_k(F_\alpha)$ of the hierarchy. Thinking of $\partial_{Q_{k\beta}}$ as taking the partial derivative $\partial_{s_{k\beta}}$ w.r.t. the parameter $s_{k\beta}$ along the direction $i_k(F_\alpha)\Lambda^{-jk}$, consider the $\mathbb{Z} \times \mathbb{Z}$-matrix

$$\phi_0 := \exp(\sum_{j=1}^\infty \sum_{\beta=1}^{m_U} s_{j\beta} i_k(F_\alpha) \Lambda^{-jk}).$$

This matrix belongs to $LT(\mathbb{C}[s_{k\beta}])$. The module for the linearization will consist of perturbations in $UT(R)$ of this matrix $\phi_0$. Consider namely the collection $M^{(0)}$ consisting of formal products

$$\left\{ \sum_{j=1}^{\infty} d_j \Lambda^{kj} \right\} \exp(\sum_{j=1}^\infty \sum_{\beta=1}^{m_U} s_{j\beta} i_k(F_\beta) \Lambda^{-kj}), \text{ where } d_j \in D_k(R).$$
The elements of $M^{(0)}$ are called oscillating matrices at zero. In general these formal products do not give a well-defined $\mathbb{Z} \times \mathbb{Z}$-matrix. Nevertheless there is a well-defined left action of $UT(R)$ on it. For all $u_1$ and $u_2 \in UT(R)$ one puts namely

$$u_1\{u_2\}\phi_0 = \{u_1u_2\}\phi_0.$$  

Note that $M^{(0)}$ is a free $UT(R)$-module with generator $\phi_0$. Other generators are elements of the form $\phi = \hat{\phi}\delta_U\phi_0$ in $M^{(0)}$, where $\delta_U$ is an invertible element of $UT(\mathbb{C})$ that commutes with $\Lambda^{-k}$ and all the $i_k(F_\beta)$ and $\hat{\phi} \in UT(R)$ is such that

$$\hat{\phi} = \sum_{i=m}^{\infty} d_i\Lambda^{ki}, \text{ with } d_m \text{ invertible.} \quad (3.10)$$

An element of $M^{(0)}$ of this form is called an oscillating matrix at zero of type $\delta_U$. Note that the right multiplication with $\Lambda^{-k}$ and $i_k(F_\beta)$ is well-defined on elements of $M^{(0)}$, since both matrices commute with the generator $\phi_0$. An action of the derivations $\partial_{Q_{j\beta}}$ on $M^{(0)}$ can be defined as follows

$$\partial_{Q_{j\beta}}\{\sum_{j=N}^{\infty} d_j\Lambda^j\}\phi_0 = \{\sum_{j=N}^{\infty} \partial_{Q_{j\beta}}(d_j)\Lambda^j + \sum_{j=N}^{\infty} d_j\Lambda^j(\Lambda^{-k}i_k(F_\beta))\}\phi_0.$$  

For matrices $M$ and $V_\alpha$ of the required form and satisfying the conditions (3.8) the linearization of the $h_U$-hierarchy consists of the following equations inside $M^{(0)}$

$$M\phi = \phi\Lambda^{-k}, V_\alpha\phi = \phi i_k(F_\alpha) \text{ and } \partial_{Q_{j\beta}}\phi = C_{j\beta}\phi. \quad (3.11)$$

Note that if $\phi = \hat{\phi}\delta_U\phi_0$ in these equations is of the form (3.10), then the first two equations imply that the matrices $M$ and $V_\alpha$ are given by

$$M = \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1} \text{ and } V_\alpha = \hat{\phi}i_k(F_\alpha)\hat{\phi}^{-1}. \quad (3.12)$$

This is silently assumed from now on. To get the Lax equations for $M$ one applies the derivation $\partial_{Q_{j\beta}}$ to the first equation in (3.11) and substitutes the last one. This leads to the following manipulations

$$\partial_{Q_{j\beta}}(M\phi - \phi\Lambda^{-k}) = \partial_{Q_{j\beta}}(M)\phi + M(\partial_{Q_{j\beta}}(\phi)) - (\partial_{Q_{j\beta}}(\phi))\Lambda^{-k} =$$

$$\partial_{Q_{j\beta}}(M)\phi + MC_{j\beta}\phi - C_{j\beta}\phi\Lambda^{-k} = \{\partial_{Q_{j\beta}}(M) - [C_{j\beta}, M]\}\phi = 0.$$  

Since $\phi$ may be scratched from this equation, one obtains in this way the Lax equations for $M$. For the operator $V_\alpha$ one applies $\partial_{Q_{j\beta}}$ to the second equation in (3.11) and substitutes the last one. Thus one gets

$$\partial_{Q_{j\beta}}(V_\alpha\phi - \psi F_\alpha) = \partial_{Q_{j\beta}}(V_\alpha)\psi + V_\alpha(\partial_{Q_{j\beta}}(\phi)) - (\partial_{Q_{j\beta}}(\phi))F_\alpha =$$

$$\partial_{Q_{j\beta}}(V_\alpha)\phi + V_\alpha C_{j\beta}\phi - C_{j\beta}\phi F_\alpha = \{\partial_{Q_{j\beta}}(V_\alpha) - [C_{j\beta}, V_\alpha]\}\phi = 0.$$  

and, since one can leave out $\phi$ again, this yields the Lax equations for the $V_\alpha$.

An oscillating matrix at zero of type $\delta_U$, $\phi = \hat{\phi}\delta_U\phi_0$, is called a wavematrix at zero of type $\delta_U$ for the matrices $M = \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1}$ and $V_\alpha = \hat{\phi}i_k(F_\alpha)\hat{\phi}^{-1}$, if it satisfies the equations (3.11). Since the manipulations to get the Lax equations are well-defined on such a $\phi$, the set of matrices $(M, V_\alpha)$ forms a solution of the upper triangular $h_U$-hierarchy.

If one wants to prove the equations (3.11) for an oscillating matrix at zero $\phi$ of the right form, it suffices to prove a weaker result, for there holds
Proposition 2. Let $\phi = \hat{\phi} U_\delta \phi_0$ be an oscillating matrix at zero of type $\delta_U$. If it satisfies for all $j \geq 1$ and all $\alpha, \beta \leq \beta \leq m_U$

$$\partial_{Q_{j\beta}}(\phi) = G_{j\beta} \psi, \text{ with } G_{j\beta} \in UT(R) \cap LT_0(R),$$

then $G_{j\beta} = (M^j V_\beta)_-$, where $M := \hat{\phi} \Lambda^{-k} \hat{\phi}^{-1}$ and $V_\beta = \hat{\phi} i_k(F_\beta) \hat{\phi}^{-1}$. In particular $M$ and the $V_\beta$ form a solution to the upper triangular $h_U$-hierarchy.

**Proof.** From the definition of the action of $\partial_{Q_{j\beta}}$ on $M^{(0)}$ and the fact that $M^{(0)}$ is a free $UT(R)$-module with generator $\phi_0$, we get the operator equation

$$\partial_{k\beta}(\hat{\phi}) + \hat{\phi}(\Lambda)^{-kj} i_k(F_\beta) = G_{j\beta} \hat{\phi}. $$

Multiplying this equation from the right with $\hat{\phi}^{-1}$ and taking the lowertriangular part gives the desired result. \hfill \blacksquare

The third hierarchy is a combination of the two foregoing ones and is called the $(h_L, h_U)$-hierarchy. First of all one has the corresponding potential solutions, namely the matrices $L$ and $U_\sigma$ in $LT(R)$ of the form (3.1) and the matrices $M$ and $V_\sigma$ in $UT(R)$ of the form (3.7). Further one assumes the $C$-algebra $R$ to be equipped with two collections of $C$-linear commuting derivations namely the $\{\partial_{P_{ia}}, i \geq 0, 1 \leq \alpha \leq m_L\}$ and the $\{\partial_{Q_{j\beta}}, j \geq 1, 1 \leq \beta \leq m_U\}$. The Lax equations of this hierarchy consist not only of those in (3.2) and (3.9), but also include the following evolution of $L$ and $U_\alpha$ w.r.t. $\partial_{Q_{j\beta}}$

$$\partial_{Q_{j\beta}}(L) = [C_{j\beta}, L] \text{ and } \partial_{Q_{j\beta}}(U_\alpha) = [C_{j\beta}, U_\alpha]$$

and that of $M$ and $V_\sigma$ w.r.t. $\partial_{P_{ia}}$

$$\partial_{P_{ia}}(M) = [B_{ia}, M] \text{ and } \partial_{P_{ia}}(V_\sigma) = [B_{ia}, V_\sigma].$$

From these last two sets of equations one sees that the unperturbed choice

$$L = \Lambda^k, U_\alpha = i_k(E_\alpha), M = \Lambda^{-k} \text{ and } V_\sigma = i_k(F_\sigma),$$

is a solution of these Lax equations if and only if the algebras $h_L$ and $h_U$ commute. This will be assumed from now on, without further mentioning.

Again there exists a linearization of the $(h_L, h_U)$-hierarchy from which these Lax equations can be deduced. It consists of the equations

$$L \psi = \psi \Lambda^k, U_\gamma \psi = \psi i_k(E_\gamma), \quad (3.13)$$

$$\partial_{Q_{j\beta}}(\psi) = C_{j\beta}(\psi), \text{ and } \partial_{P_{ia}}(\psi) = B_{ia} \psi, \quad (3.14)$$

$$M \phi = \phi \Lambda^{-k}, V_\sigma \phi = \phi i_k(F_\sigma), \quad (3.15)$$

$$\partial_{P_{ia}}(\phi) = B_{ia} \phi, \text{ and } \partial_{Q_{j\beta}}(\phi) = C_{j\beta} \phi. \quad (3.16)$$

Here the action of the $\{\partial_{Q_{j\beta}}, j \geq 1, 1 \leq \beta \leq m_U\}$ on the elements of $M^{(\infty)}$ is defined by

$$\partial_{Q_{j\beta}}(\sum_{r=-\infty}^{N} d_r \Lambda^k) \psi_0 = \sum_{r=-\infty}^{N} \partial_{Q_{j\beta}}(d_r \Lambda^k) \psi_0$$

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and that of the \( \{ \partial P_{\alpha}, i \geq 0, 1 \leq \alpha \leq m_L \} \) on \( M^{(0)} \)

\[
\partial P_{\alpha} \left( \{ \sum_{s=N}^{\infty} d_s \Lambda^{ks} \} \phi_0 \right) = \{ \sum_{s=N}^{\infty} \partial P_{\alpha} (d_s) \Lambda^{ks} \} \phi_0.
\]

Let as before \( \delta_L \) be an invertible element of \( UT(\mathbb{C}) \cap LT(\mathbb{C}) \) that commutes with \( \Lambda^k \) and all the \( i_k(E_\alpha) \) and let \( \delta_U \) be an invertible element of \( UT(\mathbb{C}) \cap LT(\mathbb{C}) \) that commutes with \( \Lambda^{-k} \) and all the \( i_k(F_\beta) \). Assume that \( \delta_L \) and \( \delta_U \) commute. For the unperturbed solution the oscillating functions

\[
\psi = \left\{ \exp \left( \sum_{j=1}^{m_U} \sum_{\beta=1}^{0} s_{j\beta} i_j (E_\alpha) \Lambda^{-jk} \right) \delta_U \right\} \delta_L \psi_0
\]

and

\[
\phi = \left\{ \exp \left( \sum_{i=0}^{\infty} t_{i\alpha} i_k (E_\alpha) \Lambda^{ki} \right) \delta_L \right\} \delta_U \phi_0
\]

satisfy the linearization for the derivations \( \partial P_{\alpha} = \frac{\partial}{\partial t_{i\alpha}} \) and \( \partial Q_{j\beta} = \frac{\partial}{\partial s_{j\beta}} \). Assume \( \psi = \hat{\psi} \delta_L \psi_0 \in M^{(\infty)} \) in the equations (3.13) and (3.14) is an oscillating matrix at infinity of type \( \delta_L \) and let \( \phi = \hat{\phi} \delta_U \phi_0 \in M^{(0)} \) in the equations (3.15) and (3.16) be an oscillating matrix at zero of type \( \delta_U \). It follows from (3.13) that the matrices \( L \) and \( U_\alpha \) are given by (3.4) and the matrices \( M \) and \( V_\alpha \) by (3.12). By applying again both sets of derivations to the equations (3.13) resp. (3.15) and by substituting those of (3.14) resp. (3.16) and scratching the function \( \psi \) resp. \( \phi \) one obtains the Lax equations for \( L \) and the \( U_\alpha \) and those for \( M \) and the \( V_\beta \). A pair \( (\psi, \phi) = (\hat{\psi} \delta_L \psi_0, \hat{\phi} \delta_U \phi_0) \) in \( M^{(\infty)} \times M^{(0)} \) of an oscillating matrix \( \psi \) at infinity of type \( \delta_L \) and an oscillating matrix \( \phi \) at zero of type \( \delta_U \) is called a pair of wavematrices of the \((h_L, h_U)\)-hierarchy of type \((\delta_L, \delta_U)\), if they satisfy the equations in (3.13),(3.14),(3.15) and (3.16) for the \( \mathbb{Z} \times \mathbb{Z} \)-matrices \( L := \hat{\psi} \Lambda^k \hat{\psi}^{-1} \), \( U_\alpha := \hat{\psi} E_\alpha \hat{\psi}^{-1} \), \( M := \hat{\phi} \Lambda^{-k} \hat{\phi}^{-1} \) and \( V_\alpha := \hat{\phi} E_\alpha \hat{\phi}^{-1} \). As one has seen this collection of matrices forms then a solution of the Lax equations of the \((h_L, h_U)\)-hierarchy.

Also in the coupled case, it suffices that an apparently weaker version of the equations (3.14) resp. (3.16) holds for a candidate pair \( (\psi, \phi) \). By combining the propositions (1) and (2) one gets namely

**Proposition 3.** Consider a pair \( (\psi, \phi) = (\hat{\psi} \delta_L \psi_0, \hat{\phi} \delta_U \phi_0) \) in the space \( M^{(\infty)} \times M^{(0)} \) consisting of an oscillating matrix \( \psi \) at infinity of type \( \delta_L \) and an oscillating matrix \( \phi \) at zero of type \( \delta_U \). If they satisfy the equations

\[
\partial P_{i\alpha} (\psi) = F_{i\alpha} \psi \quad \text{and} \quad \partial P_{i\alpha} (\phi) = F_{i\alpha} \phi, \quad \text{with} \quad F_{i\alpha} \in \text{LT}(R) \cap UT_0(R),
\]

\[
\partial Q_{j\beta} (\psi) = G_{j\beta} \psi \quad \text{and} \quad \partial Q_{j\beta} (\phi) = G_{j\beta} \phi, \quad \text{with} \quad G_{j\beta} \in \text{UT}(R) \cap LT_0(R),
\]

then \( F_{i\alpha} = (\mathcal{L}^i U_\alpha)_{+1} \), where \( \mathcal{L} := \hat{\psi} \Lambda^k \hat{\psi}^{-1} \) and \( U_\alpha := \hat{\psi} i_j (E_\alpha) \hat{\psi}^{-1} \), and \( G_{j\beta} = (M^j V_\beta)_{-1} \), with \( M := \hat{\phi} \Lambda^{-k} \hat{\phi}^{-1} \) and \( V_\beta := \hat{\phi} i_k (F_\beta) \hat{\phi}^{-1} \). In particular the set \( (\mathcal{L}, U_\alpha, M, V_\beta) \) is a solution of the \((h_L, h_U)\)-hierarchy.

**Remark 1.** Note that if one chooses the algebra \( h_U = \{0\} \), then the \((h_L, \{0\})\)-hierarchy is the upper triangular \( h_L \)-hierarchy and the choice \( h_L = \{0\} \) yields the upper triangular \( h_U \)-hierarchy.
Under suitable convergence conditions the formal products occurring in as well the space $M^{(\infty)}$ as $M^{(0)}$ turn into real products. Such an analytic setting is described in the following section. It will allow you to generate families of solutions of the $(h_L, h_U)$-hierarchy.

4 An analytic setting

One starts with a complex Banach space $(H, ||\cdot||_H)$ equipped with a topological basis $\{e_i \mid i \in \mathbb{Z}\}$. That is to say every $h \in H$ decomposes uniquely as

$$h = \sum_{i \in \mathbb{Z}} \alpha_i e_i \text{ and } h = \lim_{N \to \infty} \sum_{i=-N}^{N} \alpha_i e_i.$$ 

To each bounded linear operator $A : H \mapsto H$ one can associate the $\mathbb{Z} \times \mathbb{Z}$-matrix $[A] = (\alpha_{ji})$ w.r.t. this basis defined by

$$A(e_i) = \sum_{j \in \mathbb{Z}} \alpha_{ji} e_j.$$ 

In view of the character of the flows of the hierarchies it is convenient to realize $H$ as a space of vector-valued series. More concretely, let $\{f_i \mid 0 \leq i \leq k - 1\}$ be the standard basis of $C^k$, where $f_i$ has a one as the $i + 1$-th entry and zeros elsewhere. Then we make for all $j \in \mathbb{Z}$ and all $s, 0 \leq s \leq k - 1$, the identification

$$e_{s+kj} := f_s z^j.$$ 

and thus get that any element $h \in H$ can be uniquely written as

$$h = \sum_{j \in \mathbb{Z}} h(j) z^j, \text{ with } h(j) \in C^k.$$ 

In order that we can carry out the construction of the solutions of the $(h_L, h_U)$-hierarchy the space $H$ has to satisfy a number of assumptions. First of all multiplying with $z$

$$\sum_{j \in \mathbb{Z}} h(j) z^j \mapsto \sum_{j \in \mathbb{Z}} h(j) z^{j+1}$$

should be a bounded invertible operator $M_z : H \mapsto H$ whose operator norm is equal to $||M_z||$. Its matrix $[M_z]$ is the matrix $\Lambda^k$. Also for each complex $k \times k$-matrix $A$ multiplication with $A$,

$$\sum_{j \in \mathbb{Z}} h(j) z^j \mapsto \sum_{j \in \mathbb{Z}} A(h(j)) z^j$$

should be a bounded operator $M_A : H \mapsto H$. Its matrix $[M_A]$ is clearly $i_k(A)$. Examples of spaces satisfying these conditions are the $L^p(S^1, C^k)$.

For each $i \in \mathbb{Z}$, let $H^{(i)}$ be the complex subspace of $H$ spanned by the

$$\{f_s z^i \mid 0 \leq s \leq k - 1\}.$$
The projection $H \mapsto H^{(i)}$ given by $\sum_{j \in \mathbb{Z}} h(j)z^j \mapsto h(i)z^i$ is denoted by $p^{(i)}$. The space $H$ decomposes as the direct sum

$$H = \bigoplus_{i \in \mathbb{Z}} H^{(i)}$$

and this determines for each bounded linear operator $B \in B(H)$ the corresponding block decomposition $(p^{(i)} \circ B \mid H^{(j)})$. Inside $GL(H)$ we introduce two fundamental groups that occur in the basic decomposition. First there is the parabolic group

$$P = \left\{ g \mid g \in GL(H), \quad p^{(i)} \circ g \mid H^{(j)} = p^{(i)} \circ g^{-1} \mid H^{(j)} = 0 \right \}$$

with its Lie algebra

$$L(P) = \left\{ g \mid g \in B(H), p^{(i)} \circ g \mid H^{(j)} = 0 \text{ for all } i, j \in \mathbb{Z}, i < j \right \}.$$  

Further there is the unipotent part of the opposite parabolic

$$U_- = \left\{ g \mid g \in GL(H), \quad p^{(i)} \circ g \mid H^{(i)} = \text{Id} \text{ for all } i \in \mathbb{Z}, \quad p^{(i)} \circ g \mid H^{(j)} = 0 \text{ for all } i, j \in \mathbb{Z}, i > j \right \}$$

with its Lie algebra

$$L(U_-) = \left\{ g \in B(H), p^{(i)} \circ g \mid H^{(j)} = 0 \text{ for all } i, j \in \mathbb{Z}, i \geq j \right \}.$$  

Clearly $B(H) = L(P) \oplus L(U_-)$ and the map $\chi : L(P) \oplus L(U_-) \mapsto GL(H)$ defined by $\chi(u, p) = \exp(u)\exp(p)$ is a local diffeomorphism at $(0, 0)$. Hence the set $\Omega := U_-P$ is open in $GL(H)$. The Birkhoff-type decomposition of the elements of $\Omega$ enables one to construct solutions of the hierarchies.

Next the commuting flows relevant for the $(h_L, h_U)$-hierarchy will be discussed. Let $U$ be an open connected neighbourhood in the complex plane of the circle

$$S(||M_z||) = \{z \mid z \in \mathbb{C}, |z| = ||M_z||\}.$$  

For any commutative subalgebra $h$ of $gl_k(\mathbb{C})$ let $\Gamma(U, h)$ be the set of holomorphic maps $\gamma : U \mapsto h$ such that $det(\gamma(u)) \neq 0$ for all $u \in U$. It is a group for the pointwise multiplication in $GL_k(\mathbb{C})$. If two such neighbourhoods $U_1$ and $U_2$ satisfy $U_2 \subset U_1$ then one has a natural embedding of $\Gamma(U_1, h)$ into $\Gamma(U_2, h)$ and the inductive limit is denoted by $\Gamma(h)$. Each $\gamma \in \Gamma(h)$ has a Fourier series

$$\sum_{i \in \mathbb{Z}} \gamma_i z^i, \text{ with } \gamma_i \in h.$$  

and the multiplication with this series defines a bounded operator $M_z : H \mapsto H$. This determines an embedding of $\Gamma(h)$ into $GL(H)$. Let $h_{ss}$ denote the subset of semi simple elements in $h$ and let $h_n$ be the collection of nilpotent elements in $h$. From the fact that $h$ is the direct sum of these subspaces one deduces that the group $\Gamma(h)$ is the direct product of the groups

$$\Gamma(h)_{ss} := \{ \gamma \mid \gamma(u) \in h_{ss} \text{ for all } u \}$$

and

$$\Gamma(h)_{u} := \{ \gamma \mid \gamma(u) \text{ is unipotent for all } u \}.$$
Now it is easy to see that any \( \gamma \in \Gamma(h)_u \) can be written as

\[
\gamma = \exp\left( \sum_{s \in \mathbb{Z}} k_s z^s \right), \quad \text{where } k_s \in h_n \text{ for all } s \in \mathbb{Z},
\]

\[
= \exp\left( \sum_{s \geq 0} k_s z^s \right) \exp\left( \sum_{s < 0} k_s z^s \right)
\]

This shows that the elements of \( \Gamma(h)_u \) split up perfectly in those that have an analytic continuation to the interior of \( S(||M_z||) \) and those that extend holomorphically around “infinity”.

As for the group \( \Gamma(h)_{ss} \), recall, see e.g. [4], that, if \( U \) is an open connected neighbourhood of \( S(||M_z||) \), any holomorphic \( f : U \rightarrow \mathbb{C}^r \) decomposes as

\[
f(z) = \{1 + \sum_{i < 0} b_i z^i\} \sum_{j \geq 0} c_j z^j, \quad \text{with } c_0 \neq 0 \text{ and } m \in \mathbb{Z}.
\]

By applying this to the group \( \Gamma(h)_{ss} \), one arrives at the following decomposition of \( \Gamma(h) \)

**Proposition 4.** There is a subgroup \( \Delta(h) \) of \( \Gamma(h)_{ss} \) isomorphic to \( \mathbb{Z}^r \), where \( r \) is the dimension of \( h_{ss} \) such that

\[
\Gamma(h)_+ = \{ \gamma \mid \gamma = \exp\left( \sum_{s \geq 0} \gamma_s z^s \right), \text{ with } \gamma_s \in h \text{ for all } s \geq 0 \}
\]

and

\[
\Gamma(h)_- = \{ \gamma \mid \gamma = \exp\left( \sum_{s < 0} \gamma_s z^s \right), \text{ with } \gamma_s \in h \text{ for all } s < 0 \}.
\]

In the case that \( h \) equals the diagonal matrices, one can take

\[
\Delta(h) = \left\{ \begin{pmatrix}
 z^{m_1} & 0 & \cdots & 0 \\
 0 & \ddots & 0 & 0 \\
 \vdots & 0 & \ddots & 0 \\
 0 & \cdots & 0 & z^{m_k}
\end{pmatrix} \mid m_i \in \mathbb{Z} \right\}
\]

If \( \{H_\sigma \mid 1 \leq \sigma \leq m\} \) is a basis of \( h \), then there is for each element \( \gamma_+ \) of \( \Gamma(h)_+ \) an \( N > ||M_z|| \) such that

\[
\gamma_+ = \exp\left( \sum_{i=0}^{\infty} \sum_{\sigma=1}^{m} h_{i\sigma} H_\sigma z^i \right), h_{i\sigma} \in \mathbb{C}, \sum_{i,\sigma} |h_{i\sigma}| N^i < \infty
\]

and if \( \gamma_- \in \Gamma(h)_- \), then there is a \( M < ||M_z|| \) such that

\[
\gamma_- = \exp\left( \sum_{j=1}^{\infty} \sum_{\sigma=1}^{m} h_{j\sigma} H_\sigma z^{-j} \right), h_{j\sigma} \in \mathbb{C}, \sum_{j,\sigma} |s_{j\sigma}| M^j < \infty.
\]

In other words, the \( t = \{t_{i\alpha}\} \) are the coordinates on \( \Gamma(h_U)_+ \) w.r.t. the basis \( \{E_\alpha\} \) and the \( s = \{s_{j\beta}\} \) are the coordinates on \( \Gamma(h_L)_- \) w.r.t. the basis \( \{F_\beta\} \). The next step will be the construction of pairs of wavematrices in this analytic context.
5 The construction of the solutions

Let $g$ belong to $\text{GL}(H)$. Consider the map $G_g$ from the product of the flows $\Gamma(h_L)_+ \times \Delta(h_L) \times \Gamma(h_U)_- \times \Delta(h_U)$ to group $\text{GL}(H)$ defined by

$$ G_g(\gamma_+(\mathbf{1}), \Delta_L, \gamma_-(\mathbf{2}), \Delta_U) = \gamma_+(\mathbf{1})\Delta_Lg\gamma_-(\mathbf{2})^{-1}\Delta_U^{-1}. $$

Let $\Omega(h_L, h_U)$ be the inverse image under $G_g$ of the open set $\Omega$ in $\text{GL}(H)$. For the ring of functions $R$ one chooses now those that are holomorphic on $\Omega(h_L, h_U)$ and for the two sets of commuting derivations of $R$ one takes the

$$ \{\partial_{P_{\alpha}} = \partial_{t_{\alpha}} := \frac{\partial}{\partial t_{\alpha}}, i \geq 0, 1 \leq \alpha \leq m_L\} $$

and the

$$ \{\partial_{Q_{\beta}} = \partial_{s_{\beta}} := \frac{\partial}{\partial s_{\beta}}, j \geq 1, 1 \leq \beta \leq m_U\}. $$

By definition, the operator $G_g(\gamma_+(\mathbf{1}), \Delta_L, \gamma_-(\mathbf{2}), \Delta_U)$ splits for each point in $\Omega(h_L, h_U)$ as

$$ G_g(\gamma_+(\mathbf{1}), \Delta_L, \gamma_-(\mathbf{2}), \Delta_U) = \hat{\Phi}(\infty)(\mathbf{1}, \Delta_L, \mathbf{2}, \Delta_U)^{-1}\hat{\Phi}(0)(\mathbf{1}, \Delta_L, \mathbf{2}, \Delta_U), $$

where one operator $\hat{\Phi}(\infty) := \hat{\Phi}(\infty)(\mathbf{1}, \Delta_L, \mathbf{2}, \Delta_U)$ belongs to $U_-$ and the other one $\hat{\Phi}(0) := \hat{\Phi}(0)(\mathbf{1}, \Delta_L, \mathbf{2}, \Delta_U)$ to $P$. Note that if one introduces the operator

$$ \hat{\Phi}(\infty)_L = \hat{\Phi}(\infty)_L\Delta_L^+=\hat{\Phi}(\infty)_L, $$

then its $\mathbb{Z} \times \mathbb{Z}$-matrix $[\Phi(\infty)_L]$ is an oscillating matrix at infinity of type $\delta_L := [\Delta_L]$ and likewise if one considers the operator

$$ \Phi(0)_L := \hat{\Phi}(0)_L\Delta_L^+=\hat{\Phi}(0)_L, $$

then its matrix $[\Phi(0)_L]$ is an oscillating matrix at zero of type $\delta_U := [\Delta_U]$. Moreover one has $\Phi(\infty)_L g = \Phi(0)_L$, and the same identity holds for the corresponding matrices. The present construction works for all $g$ in the open set $\Gamma(h_L)\Omega(\Gamma(h_U))$ because

$$ \Gamma(h_L)\Omega(\Gamma(h_U)) = \Gamma(h_L)_+\Delta(h_L)\Gamma(h_L)_-\Delta(h_L)+\Gamma(h_U)_+\Delta(h_U)\Gamma(h_U)_-\Delta(h_U). $$

This set however does not have to equal $\text{GL}(H)$. The final result is now

**Theorem 1.** 1. Let the element $g$ belong to the open set $\Gamma(h_L)\Omega(\Gamma(h_U))$. Then the pair $([\Phi(\infty)_L], [\Phi(0)_L])$, as constructed above, is a pair of wavematrices of the $(h_U, h_L)$-hierarchy of type $(\delta_L, \delta_U)$. In particular, the matrices

$$ L = [\hat{\Phi}(\infty)_L]\Lambda^k[\hat{\Phi}(\infty)_L]^{-1}, U_{\alpha} = [\hat{\Phi}(\infty)_L]E_{\alpha}[\hat{\Phi}(\infty)_L]^{-1} $$

$$ M = [\hat{\Phi}(0)_L]\Lambda^{-k}[\hat{\Phi}(0)_L]^{-1} \text{ and } V_{\beta} = [\hat{\Phi}(0)_L]F_{\beta}[\hat{\Phi}(0)_L]^{-1} $$

are a solution to the $(h_U, h_L)$-hierarchy.
2. For each \( \gamma(L) \in \Gamma(h_L)_- \) and each \( \gamma(U) \in \Gamma(h_U)_+ \), the solutions of the \((h_U, h_L)\)-hierarchy corresponding to \( g \) and \( \gamma(L)g\gamma(U) \) are the same.

**Proof.** To prove the first part of the theorem, we make use of proposition 3. We begin with the derivative of \([\Phi^{(\infty)}_{\delta_L}]\) w.r.t. the parameter \( t_{i\alpha} \)

\[
\partial_{t_{i\alpha}}([\Phi^{(\infty)}_{\delta_L}]) = \{\partial_{t_{i\alpha}}([\Phi^{(0)}]) + [\Phi^{(\infty)}]_{0}^{\infty} E_{\alpha}\} \delta_{L}[\gamma_+(L)]
\]

\[
= \{\partial_{t_{i\alpha}}([\Phi^{(0)}])\} \delta_{U}[\gamma_-(\xi)] + L^t U_{\alpha}\} [\Phi^{(\infty)}_{\Delta L}]
\]

\[
= F_{t_{i\alpha}}[\Phi^{(\infty)}_{\Delta L}]
\]

with \( F_{t_{i\alpha}} \) lower k-block triangular of level \( \leq i \). On the other hand one knows that

\[
[\Phi^{(\infty)}_{\Delta L}] = [\Phi^{(0)}_{\delta_U}] [g]^{-1}
\]  

(5.1)

and substituting this relation gives

\[
\partial_{t_{i\alpha}}([\Phi^{(0)}_{\delta_U}]) = \partial_{t_{i\alpha}}([\Phi^{(0)}_{\delta_L}]) [g]^{-1}
\]

\[
= \partial_{t_{i\alpha}}([\Phi^{(0)}]) [\delta_U][\gamma_-(\xi)] [g]^{-1}
\]

\[
= \{\partial_{t_{i\alpha}}([\Phi^{(0)}])\} [\Phi^{(0)}_{\delta_L}]^{-1} [\Phi^{(\infty)}_{\delta_L}].
\]

Since \( \partial_{t_{i\alpha}}([\Phi^{(0)}]) [\Phi^{(0)}_{\delta_L}]^{-1} \) is upper k-block triangular of level \( \geq 0 \) and \( [\Phi^{(\infty)}_{\delta_L}] \) is a generator of \( M^{(\infty)} \), this shows that \( F_{t_{i\alpha}} \) has a k-block decomposition in a finite sum of positive powers of \( \Lambda^k \). From the relation (5.1) follows

\[
\partial_{t_{i\alpha}}([\Phi^{(0)}_{\delta_U}]) = \partial_{t_{i\alpha}}([\Phi^{(0)}_{\delta_L}]) [g] = F_{t_{i\alpha}}[\Phi^{(0)}_{\delta_U}].
\]

This concludes the proof that the first set of equations from proposition 3 is satisfied. Consider now the derivative of \([\Phi^{(0)}_{\delta_U}]\) w.r.t. the parameter \( s_{j\beta} \)

\[
\partial_{s_{j\beta}}([\Phi^{(0)}_{\delta_U}]) = \{\partial_{s_{j\beta}}([\Phi^{(0)}]) + [\Phi^{(\infty)}_{\delta_U}] E_{\beta}\} \delta_U[\gamma_-(\xi)]
\]

\[
= \{\partial_{s_{j\beta}}([\Phi^{(0)}])\} [\Phi^{(0)}_{\delta_U}]^{-1} + M^t V_{\beta}\} [\Phi^{(0)}_{\delta_U}]
\]

\[
= G_{j\beta}[\Phi^{(0)}_{\delta_U}],
\]

with \( G_{j\beta} \) upper k-block triangular of level \( \geq -j \). By using the relation (5.1) one sees on the other hand

\[
\partial_{s_{j\beta}}([\Phi^{(0)}_{\delta_U}]) = \partial_{s_{j\beta}}([\Phi^{(\infty)}_{\delta_L}]) [g]
\]

\[
= \partial_{s_{j\beta}}([\Phi^{(0)}_{\delta_U}]) [\delta_{L}[\gamma_+(\xi)]
\]

\[
= \{\partial_{s_{j\beta}}([\Phi^{(\infty)}_{\delta_L}]) [\Phi^{(0)}_{\delta_L}]^{-1}\} [\Phi^{(0)}_{\delta_U}].
\]

The matrix \( \partial_{s_{j\beta}}([\Phi^{(\infty)}_{\delta_L}]) [\Phi^{(\infty)}_{\delta_L}]^{-1} \) is however lower k-block triangular of level \( < 0 \). Since \([\Phi^{(0)}_{\delta_U}]\) is a generator of the module \( M^{(0)} \), one may conclude now that the matrix \( G_{j\beta} \) has a k-block decomposition in a finite number of negative powers of \( \Lambda^k \) just as required in proposition 3. Thus we have proved part (1) of the theorem.
As for the second part, one has by definition the identity
\[
G_{\gamma(L)g\gamma(U)}(\gamma_+, \Delta_L, \gamma_-, \Delta_U) = \gamma_+ \Delta_L \gamma(L) g \gamma(U) \gamma_-^{-1} \Delta_U^{-1} \\
= \gamma(L) \hat{\Phi}^{(\infty)} \hat{\Phi}^{(0)} \gamma(U)
\]
and this gives directly the decomposition of \( G_{\gamma(L)g\gamma(U)}(\gamma_+, \Delta_L, \gamma_-, \Delta_U) \). Hence the solutions corresponding to \( \gamma(L)g\gamma(U) \) are
\[
L = [\hat{\Phi}^{(\infty)}] \gamma(L) \Lambda^{-k} \gamma(L) [\hat{\Phi}^{(\infty)}]^{-1} = [\hat{\Phi}^{(\infty)}] \Lambda^k [\hat{\Phi}^{(\infty)}]^{-1},
\]
\[
U_\alpha = [\hat{\Phi}^{(\infty)}] \gamma(L)^{-1} E_\alpha \gamma(L) [\hat{\Phi}^{(\infty)}]^{-1} = [\hat{\Phi}^{(\infty)}] E_\alpha [\hat{\Phi}^{(\infty)}]^{-1},
\]
\[
M = [\hat{\Phi}^{(0)}] \gamma(U) \Lambda^{-k} \gamma(U)^{-1} \hat{\Phi}^{(0)} = [\hat{\Phi}^{(0)}] \Lambda^k [\hat{\Phi}^{(0)}],
\]
\[
V_\beta = [\hat{\Phi}^{(0)}] \gamma(U) F_\beta \gamma(U)^{-1} \hat{\Phi}^{(0)} = [\hat{\Phi}^{(0)}] F_\beta [\hat{\Phi}^{(0)}].
\]
This proves the claims in part (2) of the theorem. ■

References