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A geometric interpretation of the spectral parameter for surfaces of constant mean curvature

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Abstract

Considering the kinematics of the moving frame associated with a constant mean curvature surface immersed in S^3 we derive a linear problem with the spectral parameter corresponding to elliptic sinh-Gordon equation. The spectral parameter is related to the radius R of the sphere S^3. The application of the Sym formula to this linear problem yields constant mean curvature surfaces in $\mathbb{E}^3$. Independently, we show that the Sym formula itself can be derived by an appropriate limiting process $R \to \infty$.

Integrable nonlinear equations in $1+1$ dimensions are distinguished by the existence of the linear problem or spectral problem, i.e., an associated system of linear equations, containing the so called spectral parameter (see, for instance, [16]). The integrability conditions for the linear problem are equivalent to the considered nonlinear system. Integrable systems played an important role in the classical differential geometry [12], and are more and more important in the modern differential geometry [5, 20, 21]. Some integrable systems are of geometric origin [3, 13, 18, 19].

Given a spectral problem we can construct a local immersion by the so called Sym formula [6, 19]. For instance, starting from the spectral problem for the sine-Gordon equation we get pseudospherical surfaces. The Sym approach gives probably the best correspondence between the geometry and spectral problems [7, 19]. The spectral problem is necessary for the application of various methods of the soliton theory, like the inverse scattering method, the Darboux-Bäcklund transformation or algebro-geometric solutions in terms of Riemann theta functions. The Sym formula allows one to use all these methods in differential geometry.

In the differential geometry of immersed submanifolds we have always a typical pair: the linear system of Gauss-Weingarten equations and their compatibility conditions, the nonlinear system of Gauss-Codazzi-Ricci equations. To obtain a linear problem of the soliton theory we need to insert a spectral parameter into the Gauss-Weingarten equations under consideration (for more details and references see, for instance, [6]).

In this paper we consider surfaces of constant mean curvature $H \neq 0$. Constant mean curvature surfaces immersed in 3-dimensional Euclidean space $\mathbb{E}^3$ appear in the problem of soap bubbles if the (constant) outer pressure on both sides of the bubble surface is different (if the pressure is the same on both sides, we get minimal surfaces, i.e., $H = 0$). The Gauss map for constant mean curvature surfaces in $\mathbb{E}^3$ is harmonic, i.e., the normal vector N satisfies the equation

$$N,_{xx} + N,_{yy} + (N,_x^2 + N,_y^2)N = 0 \ , \qquad N^2 = 1 \ , \tag{1}$$

where x, y are curvature coordinates (or their conformal equivalents). This is the 2-dimensional Euclidean $O(3)$ σ-model which appears in the classical field theory [17] and also describes static solutions of $2+1$-dimensional continuum classical Heisenberg ferromagnet equation

$$S,_t = S \times (S,_{xx} + S,_{yy}) \ , \qquad S^2 = 1 \ . \tag{2}$$

Here we use the approach proposed by Doliwa and Santini [10] which has been successfully applied to the case of submanifolds of negative constant sectional curvature [2, 9]. The Gauss-Weingarten equations for surfaces of constant mean curvature immersed in the sphere $S^3 \subset \mathbb{E}^4$ contain explicitly the radius R of the sphere S^3. The main result of this paper is to show that R plays the role of the spectral parameter (or, more precisely, R is a function of the spectral parameter).

We consider an immersion in the sphere $S^3 \subset E^4$ (of radius R) defined by the position vector $r = r(x, y)$. The unit vector r/R is orthogonal to S^3 and we choose the second normal vector n to be tangent to S^3. The immersion has 2-dimensional normal space spanned by r/R and n. We can always consider conformal coordinates, i.e., such that the first fundamental form is proportional to $dx^2 + dy^2$, while the second fundamental form associated with n is arbitrary:

$$\begin{aligned} I &:= dr \cdot dr = e^{2\vartheta}(dx^2 + dy^2) \ , \\ II &:= -dr \cdot dn = b_{11}dx^2 + 2b_{12}dxdy + b_{22}dy^2 \ . \end{aligned} \tag{3}$$

Moreover, the second fundamental form II' associated with the normal r/R is proportional to the metric. Indeed,

$$II' := -dr \cdot d(r/R) = -\frac{1}{R} dr \cdot dr = -k_0 e^{2\vartheta}(dx^2 + dy^2) \ , \tag{4}$$

where we denoted $k_0 := 1/R$.

We denote unit tangent vectors by $E_1 \equiv e^{-\vartheta} r,_x$, $E_2 \equiv e^{-\vartheta} r,_x$, and the normal vectors by $E_3 \equiv n$, $E_4 \equiv r/R$. The so called mean curvature vector (see, for instance, [1, 4]) is given by

$$\vec{H} = hE_3 - k_0 E_4 \ , \tag{5}$$

where

$$h := \frac{1}{2}(b_{11} + b_{22})\, e^{-2\vartheta} \ . \tag{6}$$

We recall that in d-submanifold case ($d > 2$) the covariant constancy of $\vec{H}$ is a natural generalization of the condition $H = \text{const}$.

Kinematics of the adapted frame (Gauss-Weingarten equations or structural equations) can be expressed in terms of the coefficients of the fundamental forms:

$$
\begin{aligned}
& r_{,xx} = \vartheta_{,x} r_{,x} - \vartheta_{,y} r_{,y} + b_{11} n - R^{-2} e^{2\vartheta} r \ , \\
& r_{,xy} = \vartheta_{,y} r_{,x} + \vartheta_{,x} r_{,y} + b_{12} n \ , \\
& r_{,yy} = \vartheta_{,y} r_{,y} - \vartheta_{,x} r_{,x} + b_{22} n - R^{-2} e^{2\vartheta} r \ , \\
& n_{,x} = -b_{11} e^{-2\vartheta} r_{,x} \ , \quad n_{,y} = -b_{22} e^{-2\vartheta} r_{,y} \ .
\end{aligned}
\tag{7}
$$

Note that $n \equiv E_3$ and $E_4 \equiv k_0 r$ are covariantly constant. Therefore $\vec{H}$ is covariantly constant iff $h = \text{const}$.

We can rewrite the equations (7) in the matrix form:

$$
\begin{aligned}
& \frac{\partial}{\partial x} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} 0 & -\vartheta_{,y} & b_{11} e^{-\vartheta} & -k_0 e^{\vartheta} \\ \vartheta_{,y} & 0 & b_{12} e^{-\vartheta} & 0 \\ -b_{11} e^{-\vartheta} & -b_{12} e^{-\vartheta} & 0 & 0 \\ k_0 e^{\vartheta} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix} , \\
& \frac{\partial}{\partial y} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} 0 & \vartheta_{,x} & b_{12} e^{-\vartheta} & 0 \\ -\vartheta_{,x} & 0 & b_{22} e^{-\vartheta} & -k_0 e^{\vartheta} \\ -b_{12} e^{-\vartheta} & -b_{22} e^{-\vartheta} & 0 & 0 \\ 0 & k_0 e^{\vartheta} & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix} .
\end{aligned}
\tag{8}
$$

Denoting $(E_1, E_2, E_3, E_4)^T \in SO(4)$ by Φ and standard generators of the matrix Lie algebra $so(4)$ by f_{jk}, we have

$$
\begin{aligned}
& \Phi_{,x} = \hat{U} \Phi \equiv \left(-\vartheta_{,y} f_{12} + b_{11} e^{-\vartheta} f_{13} - k_0 e^{\vartheta} f_{14} + b_{12} e^{-\vartheta} f_{23} \right) \Phi \ , \\
& \Phi_{,y} = \hat{V} \Phi \equiv \left(\vartheta_{,x} f_{12} + b_{22} e^{-\vartheta} f_{23} - k_0 e^{\vartheta} f_{24} + b_{12} e^{-\vartheta} f_{13} \right) \Phi \ .
\end{aligned}
\tag{9}
$$

The system of Gauss-Codazzi equations (identical with the compatibility conditions for the above system of matrix linear equations) is given by

$$
\begin{aligned}
& \vartheta_{,xx} + \vartheta_{,yy} + (b_{11} b_{22} - b_{12}^2) e^{-2\vartheta} + k_0^2 e^{2\vartheta} = 0 \ , \\
& b_{12,x} = b_{11,y} - \vartheta_{,y} (b_{11} + b_{22}) \ , \\
& b_{12,y} = b_{22,x} - \vartheta_{,x} (b_{11} + b_{22}) \ .
\end{aligned}
\tag{10}
$$

Introducing complex variables $z = x + iy$, $\bar{z} = x - iy$ and a complex function

$$
Q := \frac{1}{4} (b_{11} - b_{22}) - \frac{1}{2} i b_{12} \ , \tag{11}
$$

known as the Hopf differential (compare [3] where the case of surfaces in $\mathbb{E}^3$ is discussed in detail), we may rewrite the Gauss-Codazzi equations as

$$
\begin{aligned}
& 4\vartheta_{,z\bar{z}} + (h^2 + k_0^2) e^{2\vartheta} - 4 Q \bar{Q} e^{-2\vartheta} = 0 \ , \\
& Q_{,\bar{z}} = \tfrac{1}{2} h_{,z} \, e^{2\vartheta} \ .
\end{aligned}
\tag{12}
$$

If $h=$ const, then $r(x,y)$ describes constant mean curvature surfaces immersed in $\mathbb{E}^4$ and the system (12) reduces to

$$\vartheta,_{z\bar{z}}+\frac{1}{4}H^2e^{2\vartheta}-Q\bar{Q}e^{-2\vartheta}=0\ ,\qquad Q=Q(z)\ , \tag{13}$$

(i.e., $Q(z)$ is an analytic function) where

$$H^2=h^2+k_0^2\ . \tag{14}$$

The system (13) can be transformed into the elliptic sinh-Gordon equation

$$u,_{XX}+u,_{YY}=-\sinh u\cosh u \tag{15}$$

by the change of variables $z\to Z\equiv X+iY$ and $\theta\to u$, where

$$dZ=\sqrt{8HQ(z)}\ dz\ ,\qquad u=\vartheta-\ln(2H^{-1}|Q(z)|)\ . \tag{16}$$

Note that the sign on the left hand side of (15) is negative. Both cases, positive and negative, have some applications in physics, and both are integrable (see, for instance, [14]). The equations (13) can also be interpreted as Gauss-Codazzi equations for surfaces of constant mean curvature H immersed in $\mathbb{E}^3$.

By virtue of (14), for any fixed H linear equations (9) form a one-parameter family of equations (the linear problem with the spectral parameter) parameterized by κ, where

$$h=H\cos\kappa\ ,\qquad k_0=H\sin\kappa\ , \tag{17}$$

or, even better, h and k_0 can be expressed in terms of $\zeta=e^{i\kappa}$, i.e.,

$$h=\frac{H}{2}\left(\zeta+\frac{1}{\zeta}\right)\ ,\qquad \frac{1}{R}\equiv k_0=\frac{H}{2i}\left(\zeta-\frac{1}{\zeta}\right)\ . \tag{18}$$

We point out that the spectral parameter ζ takes values in the unit circle. The coefficients b_{ij} appearing in equations (9), can be expressed in terms of Q and ϑ (compare (6) and (11)):

$$b_{11}=he^{2\vartheta}+2\mathrm{Re}Q\ ,\quad b_{12}=-2\mathrm{Im}Q\ ,\quad b_{22}=he^{2\vartheta}-2\mathrm{Re}Q\ . \tag{19}$$

Finally, we have the following $SO(4)$-valued spectral problem

$$\Phi,_x=\hat{U}\Phi\equiv\sum_{i<j}u_{ij}f_{ij}\Phi\ ,\qquad \Phi,_y=\hat{V}\Phi\equiv\sum_{i<j}v_{ij}f_{ij}\Phi\ , \tag{20}$$

where

$$\begin{aligned}
&\hat{U}=\frac{\zeta H}{2}e^{\vartheta}(f_{13}+if_{14})+\frac{H}{2\zeta}e^{\vartheta}(f_{13}-if_{14})+\hat{U}_0\ ,\\
&\hat{U}_0:=-\vartheta,_y f_{12}+2e^{-\vartheta}f_{13}\mathrm{Re}Q-2e^{-\vartheta}f_{23}\mathrm{Im}Q\ ,\\
&\hat{V}=\frac{\zeta H}{2}e^{\vartheta}(f_{23}+if_{24})+\frac{H}{2\zeta}e^{\vartheta}(f_{23}-if_{24})+\hat{V}_0\ ,\\
&\hat{V}_0:=\vartheta,_x f_{12}-2e^{-\vartheta}f_{23}\mathrm{Re}Q-2e^{-\vartheta}f_{13}\mathrm{Im}Q
\end{aligned} \tag{21}$$

The compatibility conditions for the linear problem (20), (21) are given by (13). For our purposes it is very convenient to use the isomorphism $so(4) \simeq \mathrm{spin}(4)$, i.e.,

$$f_{ij} \qquad \longleftrightarrow \qquad \frac{1}{2}\mathbf{e}_i\mathbf{e}_j \ , \tag{22}$$

where $\mathbf{e}_j$ are so called Clifford numbers (see, for instance,[15]), satisfying

$$\mathbf{e}_j^2 = 1 \quad (j = 1, 2, 3, 4) \ , \qquad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \qquad (i \neq j) \ . \tag{23}$$

The group Spin(4) is the double covering of $SO(4)$. Using the above isomorphism we immediately obtain the following Spin(4)-valued spectral problem

$$\begin{aligned} \Psi,_x = U\Psi = \frac{1}{2}\sum_{i<j} u_{ij}\mathbf{e}_i\mathbf{e}_j\Psi \ , \\ \Psi,_y = V\Psi = \frac{1}{2}\sum_{i<j} v_{ij}\mathbf{e}_i\mathbf{e}_j\Psi \ , \end{aligned} \tag{24}$$

where u_{ij} and v_{ij} are defined by (20), (21), i.e.,

$$\begin{aligned} 2U &= He^{\vartheta}\mathbf{e}_1(\mathbf{e}_3\cos\kappa - \mathbf{e}_4\sin\kappa) - \vartheta,_y\,\mathbf{e}_1\mathbf{e}_2 + 2e^{-\vartheta}(\mathbf{e}_1\mathrm{Re}Q - \mathbf{e}_2\mathrm{Im}Q)\mathbf{e}_3 \ , \\ 2V &= He^{\vartheta}\mathbf{e}_2(\mathbf{e}_3\cos\kappa - \mathbf{e}_4\sin\kappa) + \vartheta,_x\,\mathbf{e}_1\mathbf{e}_2 - 2e^{-\vartheta}(\mathbf{e}_1\mathrm{Im}Q + \mathbf{e}_2\mathrm{Re}Q)\mathbf{e}_3 \ . \end{aligned} \tag{25}$$

One can easily check that the matrices of the spectral problem (25) have the following property:

$$U(-\kappa) = \mathbf{e}_4 U(\kappa)\mathbf{e}_4^{-1} \ , \qquad V(-\kappa) = \mathbf{e}_4 V(\kappa)\mathbf{e}_4^{-1} \ . \tag{26}$$

Therefore we can confine ourselves to solutions Ψ satisfying

$$\Psi(-\kappa) = \mathbf{e}_4\Psi(\kappa)\mathbf{e}_4^{-1} \tag{27}$$

Actually, κ is defined as a positive quantity (compare (17)). Therefore, the equations (26) can be treated as an extension of the obtained spectral problem on negative values of κ (note that $\kappa < 0$ formally means that R is negative as well).

The frame E_k associated with the immersion r can also be expressed in terms of Clifford numbers, namely

$$E_k \quad \longleftrightarrow \quad \mathbf{E}_k := \Psi^{-1}\mathbf{e}_k\Psi \ . \tag{28}$$

Note that $\mathbf{E}_k$ form an orthonormal basis in the 4-dimensional linear space W spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$. We define $\mathbf{\Phi} := (\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_4)^T$ and compute

$$\mathbf{E}_k,_x = (\Psi^{-1}\mathbf{e}_k\Psi),_x = \Psi^{-1}[\mathbf{e}_k, U]\Psi \ , \tag{29}$$

and similar expression for the y-derivative. By virtue of

$$[\mathbf{e}_k, U] = \sum_{j>k} u_{kj}\mathbf{e}_j - \sum_{i<k} u_{ik}\mathbf{e}_i \ , \tag{30}$$

we get again the considered $SO(4)$ spectral problem (defined on the space $W \simeq \mathbb{E}^4$). Thus we proved that $\mathbf{\Phi}$ satisfies (20), (21) provided that Ψ solves (24) (compare an analogical discussion for pseudospherical surfaces [2, 9]).

In particular, using $W \simeq \mathbb{E}^4$, we can express by Ψ the position vector $\mathbf{r} \in \mathbb{E}^4$ of the considered immersion of constant mean curvature ($\mathbf{r} \simeq r$):

$$\mathbf{r} = R\mathbf{E}_4 \equiv R\Psi^{-1}\mathbf{e}_4\Psi \tag{31}$$

Finally, we define (compare [9]):

$$F = \lim_{R\to\infty} (\mathbf{r} - R\mathbf{e}_4) \ . \tag{32}$$

We expect that F should be an immersion into $\mathbb{E}^3$ (for $R \to \infty$ the sphere S^3 locally becomes $\mathbb{E}^3$). The subtraction of $R\mathbf{e}_4$ in the definition (32) means that we choose a more convenient origin of the reference frame (a fixed point of the "blowing" sphere instead of the center of the sphere). This is the North pole or (for $R < 0$) the South pole.

The limit $R \to \infty$ means that $\zeta \to 1$, $\kappa \to 0$ and $k_0 \to 0$. In this limit the matrices U, V given by (25) do not contain $\mathbf{e}_4$. From (27) it follows that

$$\Psi_0\mathbf{e}_4 = \mathbf{e}_4\Psi_0 \ , \qquad \Psi_0'\mathbf{e}_4 = -\mathbf{e}_4\Psi_0' \ , \tag{33}$$

where $\Psi_0 := \Psi(x, y, 0)$ (i.e., Ψ evaluated at $\kappa = 0$) and the prime means differentiation with respect to κ. Thee limit (32) can be computed as follows:

$$F = \lim_{R\to\infty} R(\Psi^{-1}\mathbf{e}_4\Psi - \mathbf{e}_4) = \lim_{k_0\to 0} \frac{\Psi^{-1}\mathbf{e}_4\Psi - \mathbf{e}_4}{k_0} \ . \tag{34}$$

Applying L'Hospital's rule, and (17) and (33), we get

$$F = (-\Psi^{-1}\Psi,_{k_0} \Psi^{-1}\mathbf{e}_4\Psi + \Psi^{-1}\mathbf{e}_4\Psi,_{k_0})|_{k_0=0} = 2H^{-1}\mathbf{e}_4\Psi^{-1}\Psi,_\kappa|_{\kappa=0} \ . \tag{35}$$

Thus we derived the Sym formula. The factor $\mathbf{e}_4$ turns out to be quite convenient because it assures that $F = F(x, y)$ describes an immersion in the space $\mathbb{E}^3$ spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

We expect that the fundamental forms for the immersion F can be obtained from the fundamental forms for r in the limit $k_0 \to 0$. Indeed, we compute

$$\begin{aligned}
F,_x &= 2H^{-1}\mathbf{e}_4\Psi_0^{-1}U,_\kappa(0)\Psi_0 = -\mathbf{e}_4 e^\vartheta\Psi_0^{-1}\mathbf{e}_1\mathbf{e}_4\Psi_0 = e^\vartheta\tilde{\mathbf{e}}_1 \ , \\
F,_y &= 2H^{-1}\mathbf{e}_4\Psi_0^{-1}V,_\kappa(0)\Psi_0 = -\mathbf{e}_4 e^\vartheta\Psi_0^{-1}\mathbf{e}_2\mathbf{e}_4\Psi_0 = e^\vartheta\tilde{\mathbf{e}}_2 \ , \\
N &= \Psi_0^{-1}\mathbf{e}_3\Psi_0 = \tilde{\mathbf{e}}_3 \ , \\
N,_x &= \Psi_0^{-1}[\mathbf{e}_3, U(0)]\Psi_0 = -He^\vartheta\tilde{\mathbf{e}}_1 - 2e^{-\vartheta}(\tilde{\mathbf{e}}_1\mathrm{Re}Q - \tilde{\mathbf{e}}_2\mathrm{Im}Q) \ , \\
N,_y &= \Psi_0^{-1}[\mathbf{e}_3, V(0)]\Psi_0 = -He^\vartheta\tilde{\mathbf{e}}_2 + 2e^{-\vartheta}(\tilde{\mathbf{e}}_1\mathrm{Im}Q + \tilde{\mathbf{e}}_2\mathrm{Re}Q) \ ,
\end{aligned} \tag{36}$$

where $\tilde{\mathbf{e}}_k := \Psi_0^{-1}\mathbf{e}_k\Psi_0$. Taking into account that $\tilde{\mathbf{e}}_1$, $\tilde{\mathbf{e}}_2$, $\tilde{\mathbf{e}}_3$ form an orthonormal frame in $\mathbb{E}^3$, we obtain the fundamental forms (3), (19):

$$\begin{aligned}
I &= e^{2\vartheta}(dx^2 + dy^2) \ , \\
II &= (he^{2\vartheta} + 2\mathrm{Re}Q)dx^2 - 4\mathrm{Im}Q \ dxdy + (he^{2\vartheta} - 2\mathrm{Re}Q)dy^2 \ .
\end{aligned} \tag{37}$$

Computing the mean curvature (according to the formula (6)) we verify that the obtained F is a surface of constant mean curvature in $\mathbb{E}^3$.

It is interesting that the presented geometric interpretation yields constant mean curvature surfaces in $\mathbb{E}^3$ when the Sym formula is applied directly to Spin(4) spectral problem. Usually one expects to get surfaces in $\mathbb{E}^3$ from some Spin(3) (i.e., $SU(2)$) spectral problem. It turns out that such approach is possible in our case as well. Namely, we can use the well known isomorphism $so(4) \simeq so(3) \oplus so(3)$. For instance, we can define

$$\begin{aligned} 2f_1 &= \mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_3\mathbf{e}_4 \ , \quad 2f_2 = \mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_2\mathbf{e}_4 \ , \quad 2f_3 = \mathbf{e}_1\mathbf{e}_4 - \mathbf{e}_2\mathbf{e}_3 \ , \\ 2g_1 &= \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4 \ , \quad 2g_2 = \mathbf{e}_2\mathbf{e}_4 - \mathbf{e}_1\mathbf{e}_3 \ , \quad 2g_3 = \mathbf{e}_1\mathbf{e}_4 + \mathbf{e}_2\mathbf{e}_3 \ . \end{aligned} \tag{38}$$

One can easily check that

$$[f_1, f_2] = f_3 \ , \quad [g_1, g_2] = g_3 \ , \quad [f_j, g_k] = 0 \ , \tag{39}$$

etc., which means that f_1, f_2, f_3 and g_1, g_2, g_3 span two copies of commuting Lie algebras $so(3)$. The projector

$$P := \frac{1}{2}\left(1 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\right) \tag{40}$$

projects $so(4)$ onto $so(3)$ spanned by f_1, f_2, f_3, while $I - P$ projects $so(4)$ on the second copy of $so(3)$. Namely

$$\begin{aligned} &P\mathbf{e}_1\mathbf{e}_2 = f_1 \ , \quad P\mathbf{e}_1\mathbf{e}_3 = f_2 \ , \quad P\mathbf{e}_1\mathbf{e}_4 = f_3 \ , \\ &P\mathbf{e}_3\mathbf{e}_4 = -f_1 \ , \quad P\mathbf{e}_2\mathbf{e}_4 = f_2 \ , \quad P\mathbf{e}_2\mathbf{e}_3 = -f_3 \ . \end{aligned} \tag{41}$$

Note that $Pf_k = f_k$ and $Pg_k = 0$ for $k = 1, 2, 3$.

Performing the projection (40) we transform the spectral problem (24), (25) into

$$\begin{aligned} 2U &= He^{\vartheta}(f_2 \cos\kappa - f_3 \sin\kappa) - \vartheta_{,y} f_1 + 2e^{-\vartheta} f_2 \mathrm{Re}Q + 2e^{-\vartheta} f_3 \mathrm{Im}Q \ , \\ 2V &= He^{\vartheta}(-f_3 \cos\kappa - f_2 \sin\kappa) + \vartheta_{,x} f_1 + 2e^{-\vartheta} f_3 \mathrm{Re}Q - 2e^{-\vartheta} f_2 \mathrm{Im}Q \ . \end{aligned} \tag{42}$$

The projection $I - P$, applied to (24), (25), yields

$$\begin{aligned} 2U &= He^{\vartheta}(-g_2 \cos\kappa - g_3 \sin\kappa) - \vartheta_{,y} g_1 - 2e^{-\vartheta} g_2 \mathrm{Re}Q - 2e^{-\vartheta} g_3 \mathrm{Im}Q \ , \\ 2V &= He^{\vartheta}(g_3 \cos\kappa - g_2 \sin\kappa) + \vartheta_{,x} g_1 - 2e^{-\vartheta} g_3 \mathrm{Re}Q + 2e^{-\vartheta} g_2 \mathrm{Im}Q \ . \end{aligned} \tag{43}$$

The spectral problem (43) can be obtained from (42) by a simple transformation, namely: $g_2 \to -f_2$, $g_3 \to -f_3$, $\kappa \to -\kappa$.

We can apply the Sym formula $F = 2H^{-1}\Psi^{-1}\Psi_{,\kappa}$ to the linear problems (42) and (43). Now we do not confine ourselves to $\kappa = 0$. In both cases we get the same κ-family of surfaces of constant mean curvature H, characterized by the following fundamental forms:

$$\begin{aligned} I &= e^{2\vartheta}(dx^2 + dy^2) \ , \\ II &= (He^{2\vartheta} + 2(\cos\kappa \ \mathrm{Re}Q + \sin\kappa \ \mathrm{Im}Q))dx^2 \\ &+ 4(\sin\kappa \ \mathrm{Re}Q - \cos\kappa \ \mathrm{Im}Q)dxdy \\ &+ (He^{2\vartheta} - 2(\cos\kappa \ \mathrm{Re}Q + \sin\kappa \ \mathrm{Im}Q))dy^2 \end{aligned} \tag{44}$$

The coefficient $2H$ in the Sym formula, the same as in (35), is not very important, but is necessary to get the mean curvature H. Note that for $\kappa = 0$ we get the fundamental forms (37).

Therefore, we derived spectral problems (25) and (42), containing the spectral parameter, associated (by the Sym formula) with surfaces of constant mean curvature in $\mathbb{E}^3$. The 2×2 spectral problems presented in the existing literature can be reduced to the spectral problem (42) by some obvious gauge transformations and changes of variables (like (16)), compare [3, 7, 11].

It turns out that the role of Clifford algebras and Spin groups is quite important in the derivation of the Sym formula (compare the above results with [9]). It would be very interesting to extend our approach on the submanifolds asociated with spectral problems defined in terms of Clifford algebras [8].

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