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An asymptotic expansion of the q-gamma function $\Gamma_q(x)$

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Abstract

In this paper, we get an asymptotic expansion of the q-gamma function $\Gamma_q(x)$. Also, we deduced q-analogues of Gauss' multiplication formula and Legendre's relation which give the known results when q tends to 1.

1 Introduction

Analogous to Gauss' infinite product representation for the gamma function [6]

$$\Gamma(x) = x^{-1} \prod_{n=1}^{\infty} [(1 + 1/n)^x (1 + x/n)^{-1}] \tag{1.1}$$

the $q-$gamma function $\Gamma_q(x)$ is defined by [4]

$$\Gamma_q(x) = \frac{(q,q)_\infty}{(q^x,q)_\infty}(1-q)^{1-x}, \qquad 0 < q < 1, \tag{1.2}$$

where the $q-$shifted factorials are defined by [5]
$(a,q)_0 = 1,$
$(a_1, ..., a_r; q)_k = \prod_{i=1}^{r} \prod_{j=o}^{k-1} (1 - a_i q^j), \qquad k = 0, 1, 2, \ldots ,$
$(a;q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i).$
This function is a $q-$analogue of the gamma function since we have

$$\lim_{q\to 1} \Gamma_q(x) = \Gamma(x) \tag{1.3}$$

The $q-$gamma function satisfies the functional equation

$$\Gamma_q(x+1) = (1-q^x)/(1-q)\Gamma_q(x), \quad \Gamma_q(1) = 1, \tag{1.4}$$

which is a $q-$extension of the well-known functional equation

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1) = 1. \tag{1.5}$$

Boher, H. and Mollerup , J. (1922) proved the following theorem for $\Gamma(x)$ function

Theorem 1. *If a function $f(x)$ satisfies the following three conditions, then it is identical in its domain of definition with the gamma function:*

$$\begin{array}{ll} (1) & f(x+1) = xf(x), \\ (2) & f(1) = 1, \\ (3) & The\ domain\ of\ definition\ of\ f(x)\ contains\ all\ x > 0, \\ & and\ is\ \log\ convex\ for\ these\ x. \end{array}$$

For the proof of this theorem see [2]. In 1978, R. Askey [3] gave the $q-$analogy of this theorem

Theorem 2. *If a function $f(x)$ satisfies the following three conditions*

$$\begin{array}{ll} (1) & f(x+1) = [x]_q f(x) \quad for\ some\ q,\ 0 < q < 1, \\ (2) & f(1) = 1, \\ (3) & \log f(x)\ is\ convex\ for\ \ x > 0, \end{array}$$

then $f(x) = \Gamma_q(x)$, where $[x]_q = \frac{1-q^x}{1-q}$.

2 The behavior of the function $\Gamma_q(x)$ for large x

In order to study the behavior of the function $\Gamma_q(x)$ for large x, we consider a function of the form

$$f(x) = (1-q)^{1/2-x} e^{\mu(x)}. \tag{2.1}$$

Our goal is to make $f(x)$ satisfy the basic conditions for the gamma function by choosing $\mu(x)$ in an appropriate way.

$$\frac{f(x+1)}{f(x)} = \frac{e^{\mu(x+1)-\mu(x)}}{1-q} \tag{2.2}$$

Then $f(x)$ satisfy condition (2) in Theorem (2) iff

$$\mu(x) - \mu(x+1) = -\log(1-q^x), \tag{2.3}$$

holds for $\mu(x)$.
Let $g(x)$ is the write side of the equation (2.3). If we set

$$\mu(x) = \sum_{n=0}^{\infty} g(x+n) \tag{2.4}$$

then equation (2.3) holds, provided that the series in equation (2.4) converges. In order to study the convergence, we will combine this with an approximation of the function $\mu(x)$. Let us begin by considering the expansion

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}, \qquad |z| < 1 \tag{2.5}$$

If we put $z = q^x$ the expansion is valid whenever $x > 0$ and $0 < q < 1$.

$$\begin{aligned} g(x) &= -\log(1-q^x) \\ &= \sum_{n=1}^{\infty} \frac{q^{nx}}{n} \end{aligned}$$

Now we can approximate $g(x)$. If the integers $1, 2, 3, ...$ are all replaced by 1, then the result is an infinite geometric series having $\frac{q^x}{1-q^x}$. But $g(x)$ is positive, hence

$$0 < g(x) < \frac{q^x}{1-q^x} \tag{2.6}$$

Since every term of the series in equation (2.4) is positive, it suffices to show the convergence of

$$\sum_{n=0}^{\infty} \frac{q^{x+n}}{1-q^{x+n}} \tag{2.7}$$

which converges from the ratio test. This gives the approximation

$$0 < \mu(x) < \frac{q^x}{(1-q)-q^x} \tag{2.8}$$

i.e.

$$\mu(x) = \frac{\theta q^x}{(1-q)-q^x}, \tag{2.9}$$

where θ is a number independent of x between 0 and 1.

Now let us consider the condition (3) of theorem (2). The factor $(1-q)^{1/2-x}$ in equation (2.1) is log convex because the second derivative of its logarithm equal to zero for all x. If the factor $e^{\mu(x)}$ is log convex, in other words $\mu(x)$ is convex, then $f(x)$ also satisfies condition (2.4). The function $\mu(x)$ is convex if the general term of the series $g(x+n)$ is convex. To show this, it suffices to prove the convexity of $g(x)$ itself. But we have

$$g''(x) = \frac{q^x \log^2(q)}{(1-q^x)^2} > 0 \tag{2.10}$$

By a suitable choice of the constant a, we get

$$\Gamma_q(x) = a(1-q)^{1/2-x} e^{\frac{\theta q^x}{(1-q)-q^x}} \tag{2.11}$$

If we let x be an integer n, we get the approximation

$$[n]_q! = \Gamma_q(n+1) = a(1-q)^{-1/2-n} e^{\frac{\theta q^{n+1}}{(1-q)-q^{n+1}}} \tag{2.12}$$

Now we will determine the exact value of the constant a. Let p be a positive integer. We consider the function

$$f(x) = [p]_q^x \Gamma_{q^p}(x/p)\Gamma_{q^p}((x+1)/p)...\Gamma_{q^p}((x+p-1)/p), \qquad x > 0 \tag{2.13}$$

The second derivative of $\log[p]_q^x$ is zero, and $\Gamma_{q^p}((x+k)/p)$ is log convex $\forall k = 1, 2, 3, ...$ then $f(x)$ is log convex. Also,

$$\begin{aligned} f(x+1) &= [p]_q \frac{\Gamma_{q^p}((x+p)/p)}{\Gamma_{q^p}(x/p)} f(x) \\ &= [p]_q [x/p]_{q^p} f(x) \\ &= [x]_q f(x) \end{aligned}$$

Then the function $f(x)$ satisfies the conditions (1) and (3) in theorem (2). Then

$$[p]_q^x \Gamma_{q^p}(x/p)\Gamma_{q^p}((x+1)/p)...\Gamma_{q^p}((x+p-1)/p) = a_p \Gamma_q(x), \tag{2.14}$$

where a_p is a constant depending on p and by putting $x = 1$, we get

$$a_p = [p]_q \Gamma_{q^p}(1/p)\Gamma_{q^p}(2/p)...\Gamma_{q^p}(p/p). \tag{2.15}$$

But the $q-$gamma function $\Gamma_q(x)$ function is defined by

$$\Gamma_q(x) = \lim_{n\to\infty} \frac{(q,q)_n}{(q^x,q)_n}(1-q)^{1-x}, \tag{2.16}$$

then

$$\Gamma_{q^p}(k/p) = \lim_{n\to\infty} \frac{(q^p,q^p)_n}{(q^k,q^p)_n}(1-q^p)^{1-k/p} \tag{2.17}$$

and

$$a_p = [p]_q(1-q^p)^{(p-1)/p} \lim_{n\to\infty} \frac{((q^p,q^p)_n)^p}{\prod_{k=1}^p (q^k,q^p)_n} \tag{2.18}$$

By using equation (2.12) and the relation

$$[n]_q! = \frac{(q,q)_n}{(1-q)^n}, \tag{2.19}$$

then

$$((q^p,q^p)_n)^p = a^p(1-q^p)^{-p/2} e^{\frac{\theta_1 q^{p(n+1)}}{(1-q)-q^{p(n+1)}}}. \tag{2.20}$$

Also,

$$\begin{aligned}\prod_{k=1}^p (q^k,q^p)_n &= (q,q)_{np} \\ &= a(1-q)^{-1/2} e^{\frac{\theta_2 q^{pn+1}}{(1-q)-q^{pn+1}}}.\end{aligned}$$

Then

$$a_p = [p]_q^{1/2} a^{p-1} \tag{2.21}$$

From the equations (2.15) and (2.21), we have

$$a_2 = [2]_q^{1/2} a, \qquad a_2 = [2]_q \Gamma_{q^2}(1/2), \tag{2.22}$$

then

$$a = [2]_q^{1/2} \Gamma_{q^2}(1/2) \tag{2.23}$$

and

$$a_p = [p]_q^{1/2} ([2]_q \Gamma_{q^2}^2(1/2))^{(p-1)/2} \tag{2.24}$$

Now in this paper we get the following expressions

$$\Gamma_q(x) = [2]_q^{1/2} \Gamma_{q^2}(1/2)(1-q)^{1/2-x} e^{\frac{\theta q^x}{(1-q)-q^x}}, \quad 0 < \theta < 1, \tag{2.25}$$

$$[n]_q! = [2]_q^{1/2} \Gamma_{q^2}(1/2)(1-q)^{-1/2-n} e^{\frac{\theta q^{n+1}}{(1-q)-q^{n+1}}}. \tag{2.26}$$

$$\Gamma_{q^p}(x/p)\Gamma_{q^p}((x+1)/p)...\Gamma_{q^p}((x+p-1)/p) = [p]_q^{1/2-x}([2]_q\Gamma^2_{q^2}(1/2))^{(p-1)/2}\Gamma_q(x). \tag{2.27}$$

In particular, for $p = 2$

$$\Gamma_{q^2}(x/2)\Gamma_{q^2}((x+1)/2) = [2]_q^{1-x}\Gamma_{q^2}(1/2)\Gamma_q(x). \tag{2.28}$$

The formulas in equations (2.25) and (2.26) are similar to Stirling's formulas in the usual case. The functional equation (2.27) is called $q-$Gauss' multiplication formula. Also, equation (2.28) is called $q-$Legendre's relation.
If we take the limit as $q \to 1$, then we get

$$\Gamma(x/p)\Gamma((x+1)/p)...\Gamma((x+p-1)/p) = p^{1/2-x}(2\Gamma^2(1/2))^{(p-1)/2}\Gamma(x), \tag{2.29}$$

and

$$\Gamma(x/2)\Gamma((x+1)/2) = 2^{1-x}\Gamma(1/2)\Gamma(x). \tag{2.30}$$

These relations are called Gauss' multiplication formula and Legendre's relation (resp.) [1].

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