Note on operadic non-associative deformations

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Abstract

Deformation equation of a non-associative deformation in operad is proposed. Its integrability condition (the Bianchi identity) is considered. Algebraic meaning of the latter is explained.

Key words: Operad, deformation, Sabinin principle, Bianchi identity.
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1 Introduction and outline of the paper

Non-associativity is sometimes said to be an algebraic equivalent of the differential geometric concept of curvature [3]. To see the equivalence, one must represent an associator in curvature terms. In particular, this can be observed for the geodesic loops of a manifold with an affine connection [1, 2].

In this paper, the equivalence is clarified from an operad theoretical point of view. By using the Gerstenhaber brackets and a coboundary operator in an operad algebra, the (formal) associator can be represented as a curvature form in differential geometry. This equation is called a deformation equation. Its integrability condition is the Bianchi identity.

2 Operad

Let $K$ be a unital associative commutative ring, char $K \neq 2,3$, and let $C^n (n \in \mathbb{N})$ be unital $K$-modules. For homogeneous $f \in C^n$, we refer to $n$ as the degree of $f$ and write (when it does not cause confusion) $f$ instead of $\text{deg } f$. For example, $(-1)^f = (-1)^n$, $C^f \cong C^n$ and $o_f = o_n$. Also, it is convenient to use the reduced degree $|f| = n - 1$. Throughout the paper we assume that $\otimes = \otimes_K$.

Definition 1. A linear operad with coefficients in $K$ is a sequence $C \cong \{C^n\}_{n \in \mathbb{N}}$ of unital $K$-modules (an $\mathbb{N}$-graded $K$-module), such that the following conditions hold.
(1) For $0 \leq i \leq m - 1$ there exist partial compositions
\[ \circ_i \in \text{Hom}(C^m \otimes C^n, C^{m+n-1}), \quad |\circ_i| = 0 \]

(2) For all $h \otimes f \otimes g \in C^h \otimes C^f \otimes C^g$, the composition relations hold,
\[
(h \circ_i f) \circ_j g = \begin{cases} 
(1)^{|f||g|}(h \circ_j g) \circ_{i+|g|} f & \text{if } 0 \leq j \leq i - 1, \\
 h \circ_i (f \circ_{j-i} g) & \text{if } i \leq j \leq i + |f|, \\
(1)^{|f||g|}(h \circ_{j-|f|} g) \circ_i f & \text{if } i + f \leq j \leq |h| + |f|.
\end{cases}
\]

(3) There exists a unit $I \in C^1$ such that
\[ I \circ_0 f = f = f \circ_i I, \quad 0 \leq i \leq |f| \]

In the 2nd item, the first and third parts of the defining relations turn out to be equivalent.

**Example 2 (endomorphism operad [4]).** Let $L$ be a unital $K$-module and $\mathcal{E}_L^n \cong \text{End}_L^n \cong \text{Hom}(L^\otimes n, L)$. Define the partial compositions for $f \otimes g \in \mathcal{E}_L^f \otimes \mathcal{E}_L^g$ as
\[ f \circ_i g = (1)^{|i||g|} f \circ (\text{id}_L^i \otimes g \otimes \text{id}_L^{|i||f|-i}), \quad 0 \leq i \leq |f| \]
Then $\mathcal{E}_L \doteq \{\mathcal{E}_L^n\}_{n \in \mathbb{N}}$ is an operad (with the unit $\text{id}_L \in \mathcal{E}_L^1$) called the endomorphism operad of $L$.

Thus algebraic operations turn out to be elements of an endomorphism operad. It is convenient to call homogeneous elements of an abstract operad the operations as well.

### 3 Gerstenhaber brackets and associator

**Definition 3 (total composition).** The total composition $\bullet : C^f \otimes C^g \to C^{f+|g|}$ is defined by
\[ f \bullet g = \sum_{i=0}^{|f|} f \circ_i g \in C^{f+|g|}, \quad |\bullet| = 0 \]

The pair $\text{Com} C \doteq \{C, \bullet\}$ is called a composition algebra of $C$.

**Lemma 4 (Gerstenhaber identity).** The composition algebra multiplication $\bullet$ is non-associative and satisfies the Gerstenhaber identity
\[
(h, f, g) = (h \bullet f) \bullet g - h \bullet (f \bullet g) = (1)^{|f||g|}(h, g, f)
\]

**Definition 5 (Gerstenhaber brackets).** The Gerstenhaber brackets $[\cdot, \cdot]$ are defined in $\text{Com} C$ by
\[ [f, g] \doteq f \bullet g - (1)^{|f||g|} g \bullet f = -(1)^{|f||g|}[g, f], \quad |[\cdot, \cdot]| = 0 \]

The commutator algebra of $\text{Com} C$ is denoted as $\text{Com}^- C \doteq \{C, [\cdot, \cdot]\}$. 
Theorem 6. Com$^{-C}$ is a graded Lie algebra.

Proof. The anti-symmetry of the Gerstenhaber brackets is evident. To prove the (graded) Jacobi identity

\[(−1)^{|f||h|}[[[f, g], h], f] + (−1)^{|g||f|}[[g, h], f] + (−1)^{|h||f|}[[h, f], g] = 0\]

use the Gerstenhaber identity.

Let \(\{L, \mu\}\) be a non-associative algebra with a multiplication \(\mu : L \otimes L \to L\). The multiplication \(\mu\) can be seen as an element of the component \(E^2_L\) of an endomorphism operad \(E_L\). One can easily check that the associator of \(\mu\) reads

\[A = \mu \circ (\mu \otimes \text{id}_L - \text{id}_L \otimes \mu) = \mu \cdot \mu = \frac{1}{2} [\mu, \mu] = \mu^2, \quad \mu \in E^2_L\]

So the total composition and Gerstenhaber brackets can be used for representing the associator in operadic terms. This was first noticed by Gerstenhaber [4].

Proposition 7. If \(K\) is a field of characteristic 0, then every binary operation \(\mu \in C^2\) generates a power-associative subalgebra in Com$^{-C}$.

Proof. Use the Albert criterion [5] that a power-associative algebra over a field \(K\) of characteristic 0 can be given by the identities

\[\mu^2 \cdot \mu = \mu \cdot \mu^2, \quad (\mu^2 \cdot \mu) \cdot \mu = \mu^2 \cdot \mu^2\]

Both identities easily follow from the corresponding Gerstenhaber identities

\[(\mu, \mu, \mu) = 0, \quad (\mu^2, \mu, \mu) = 0\]

4 Coboundary operator

Let \(h \in C\) be an operation from an operad \(C\). By using the Gerstenhaber brackets, define an adjoint representation \(h \mapsto \partial_h\) of Com$^{-C}$ by

\[\partial_h f = \text{ad}^{\text{right}}_h f = [f, h], \quad |\partial_h| = |h|\]

It follows from the Jacobi identity in Com$^{-C}$ that \(\partial_h\) is a (right) derivation of Com$^{-C}$,

\[\partial_h [f, g] = [f, \partial_h g] + (−1)^{|g||h|}[\partial_h f, g]\]

and the following commutation relation holds:

\[[\partial_f, \partial_g] = \partial_f \partial_g - (−1)^{|f||g|}\partial_g \partial_f = \partial_{[f, g]}\]

Let \(h = \mu \in C^2\) be a binary operation. Then, since \(|\mu| = 1\) is odd, one has

\[\partial_\mu = \frac{1}{2} [\partial_\mu, \mu] = \frac{1}{2} \partial_{[\mu, \mu]} = \partial_{\mu \cdot \mu} = \partial_{\mu^2} = \partial_{A}\]

So associativity \(\mu^2 = 0\) implies \(\partial_\mu^2 = 0\). In this case, \(\partial_\mu\) is called a coboundary operator. In particular, for \(C = E_L\) one obtains the Hochschild coboundary operator [6]

\[-\partial_{\mu} f = \mu \circ (\text{id}_L \otimes f) - \sum_{i=0}^{L} (−1)^i f \circ (\text{id}_L^\otimes i \otimes \mu \otimes \text{id}_L^\otimes (|f|−i)) + (−1)^{|f|} \mu \circ (f \otimes \text{id}_L)\]
5 Deformation equation

Definition 8 (deformation). For an operad $C$, let $\mu, \mu_0 \in C^2$ be two binary operations. The difference $\omega \doteq \mu - \mu_0$ is called a deformation.

Let $\partial \doteq \partial_{\mu_0}$ and denote the (formal) associators of $\mu$ and $\mu_0$ as follows:

$A \doteq \mu \bullet \mu = \frac{1}{2}[\mu,\mu], \quad A_0 \doteq \mu_0 \bullet \mu_0 = \frac{1}{2}[\mu_0,\mu_0]$

Definition 9 (associative deformation). The deformation is called associative if $A = 0 = A_0$.

Theorem 10 (deformation equation). One has

\[
\begin{align*}
A - A_0 &= \partial \omega + \frac{1}{2}[\omega,\omega] \\
\text{deformation} &= \partial \omega + \frac{1}{2}[\omega,\omega] \\
\text{operadic curvature}
\end{align*}
\]

Proof. Calculate

\[
\begin{align*}
A &= \frac{1}{2}[\mu,\mu] \\
&= \frac{1}{2}[\mu_0 + \omega, \mu_0 + \omega] \\
&= \frac{1}{2}[\mu_0, \mu_0] + \frac{1}{2}[\mu_0, \omega] + \frac{1}{2}[\omega, \mu_0] + \frac{1}{2}[\omega, \omega] \\
&= A_0 - \frac{1}{2}(-1)^{|\mu_0|}[\omega, \mu_0] + \frac{1}{2}[\omega, \mu_0] + \frac{1}{2}[\omega, \omega] \\
&= A_0 + [\omega, \mu_0] + \frac{1}{2}[\omega, \omega] \\
&= A_0 + \partial \omega + \frac{1}{2}[\omega, \omega]
\end{align*}
\]

6 Sabinin’s principle

The deformation equation can be seen as a differential equation for $\omega$ with given associators $A_0, A$. Note that if the associator is fixed, i.e. $A = A_0$, we obtain the Maurer-Cartan equation, well-known from the theory of associative deformations:

$A = A_0 \iff \partial \omega + \frac{1}{2}[\omega, \omega] = 0$

Thus the deformation equation may be called the generalized Maurer-Cartan equation as well. The Maurer-Cartan expression

$\partial \omega + \frac{1}{2}[\omega, \omega]$

is a well-known defining form for curvature in modern differential geometry. One can see that the associator (deformation) is a formal (operadic) curvature while the deformation is working as a connection. By reformulating the Sabinin principle, one can say that associator is an operadic equivalent of the curvature.
7 Bianchi identity

By following a differential geometric analogy, one can state the

**Theorem 11 (Bianchi identity).** The associator of the deformed algebra satisfies the Bianchi identity

$$\partial A + [A, \omega] = 0$$

**Proof.** First differentiate the deformation equation,

$$\partial(A - A_0) = \partial^2 \omega + \frac{1}{2} \partial[\omega, \omega]$$

$$\quad = \partial^2 \omega + \frac{1}{2} (-1)^{|\partial||\omega|} \partial[\omega, \omega] + \frac{1}{2} [\omega, \partial \omega]$$

$$\quad = \partial^2 \omega - \frac{1}{2} [\partial \omega, \omega] + \frac{1}{2} [\omega, \partial \omega]$$

$$\quad = \partial^2 \omega - \frac{1}{2} [\partial \omega, \omega] - \frac{1}{2} (-1)^{|\partial||\omega|} [\partial \omega, \omega]$$

$$\quad = \partial^2 \omega - [\partial \omega, \omega]$$

Again using the deformation equation, we obtain

$$\partial(A - A_0) = \partial^2 \omega - [\partial \omega, \omega]$$

$$\quad = \partial^2 \omega - [A - A_0 - \frac{1}{2} [\omega, \omega], \omega]$$

$$\quad = \partial^2 \omega - [A - A_0, \omega] + \frac{1}{2} [\omega, \omega], \omega]$$

It follows from the Jacobi identity that

$$\partial A_0 = [A_0, \mu_0] = \frac{1}{2}[[\mu_0, \mu_0], \mu_0] = 0, \quad [\omega, \omega], \omega] = 0$$

By using these relations we obtain

$$\partial A = \partial^2 \omega - [A - A_0, \omega]$$

Recall that $\partial^2 = \partial A_0$ and calculate

$$\partial A + [A, \omega] = \partial A_0 \omega + [A_0, \omega] = [\omega, A_0] + [A_0, \omega]$$

$$\quad = -(-1)^{|\omega||A_0|} [A_0, \omega] + [A_0, \omega]$$

$$\quad = 0$$

**Remark 12.** To clarify algebraic meaning of the Bianchi identity, let us give another proof of the Bianchi identity:

$$\partial A + [A, \omega] = [A, \mu_0] + [A, \mu - \mu_0] = [A, \mu] = \frac{1}{2}[[\mu, \mu], \mu] = 0$$

where the latter equality is evident from the Jacobi identity. But $A = \mu \bullet \mu$ and so the Bianchi identity strikingly reads

$$(\mu \bullet \mu) \bullet \mu = \mu \bullet (\mu \bullet \mu)$$

The latter identity can be easily seen from the Gerstenhaber identity.
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References


