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On the matrix 3×3 exact solvable models of the type G_2

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Abstract

We study the exact solvable 3×3 matrix model of the type G_2. We apply the construction similar to that one, which give the 2×2 matrix model. But in the studied case the construction does not give symmetric matrix potential. We conceive that the exact solvable 3×3 matrix potential model of the type G_2 does not exist.

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1 Introduction

In this note we continue in the study of the matrix exact solvable models [1]. We discuss the 3×3 matrix model of the type G_2 of the Calogero model [2]. For a comprehensive review of these systems connected with different root systems see [3].

We applied the method developed in [4] to the 2×2 matrix models of the type A_2 [4], BC_2 [5] and G_2 type in [6] and to the matrix 3×3 models of the type A_2 [4] and BC_2 [6]. Some general results for $N \times N$ matrix models of the type A_2 was obtained in [7]. It is shown that the method and especially the simplification used for 2×2 matrix models or 3×3 matrix models of the A_2 and BC_2 do not gives the symmetric 3×3 model of the G_2 type.

This is a reason for our conjecture that the exact solvable 3×3 matrix model of the type G_2 does not exist.

2 General construction

Let us consider the differential operator

$$\mathbf{H} = \eta^{ik}\partial_{ik} - \mathbf{U}\,, \tag{2.1}$$

where η^{ik} is symmetric constant matrix, $\partial_k = \frac{\partial}{\partial x_k}$ and $\mathbf{U}$ is matrix function of the type $N \times N$. The aim of our general construction is to find operator (2.1), which leads after transformation

$$\widehat{\mathbf{H}} = \mathbf{G}^{-1}\mathbf{H}\mathbf{G}\,,$$

where $\mathbf{G}$ is a regular matrix function, and change of variables $y_r = y_r(x_k)$ to the differential operator

$$\widehat{\mathbf{H}} = g^{rs}(y)\partial_{rs} + 2\mathbf{b}^r(y)\partial_y + \mathbf{V}(y)\,, \tag{2.2}$$

for which we know finite dimensional invariant spaces. In the paper [4] are shown the conditions, which the matrix functions $\mathbf{b}^r$ and $\mathbf{V}$ have to fulfill, and construction of the operator (2.1) by means of this functions. We briefly remind these conditions.

If we write

$$\partial_{x_k}\mathbf{G} = \mathbf{G}\mathbf{X}_k(x) \quad \text{or} \quad \partial_{y_r}\mathbf{G} = \mathbf{G}\mathbf{Y}_r(y)\,, \tag{2.3}$$

the matrix functions $\mathbf{X}_k(x)$ or $\mathbf{Y}_r(y)$ must fulfil the compatibility conditions

$$\partial_k\mathbf{X}_i - \partial_i\mathbf{X}_k = [\mathbf{X}_i\mathbf{X}_k] \quad \text{or} \quad \partial_s\mathbf{Y}_r - \partial_r\mathbf{Y}_s = [\mathbf{Y}_r, \mathbf{Y}_s]\,. \tag{2.4}$$

The matrix functions $\mathbf{b}^r$ and $\mathbf{Y}_r$ are connected by the relation

$$\mathbf{b}^r = g^{rs}\mathbf{Y}_s - \tfrac{1}{2}\Gamma^r\,,$$

where we denote

$$\Gamma^r = g^{st}\Gamma^r_{st}\,, \quad \Gamma^r_{st} = g^{rk}\Gamma_{st,k}\,, \quad \text{and} \quad \Gamma_{st,k} = \tfrac{1}{2}\left(-\partial_k g_{st} + \partial_s g_{tk} + \partial_t g_{sk}\right),$$

and g_{rs} is inverse of the g^{rs}.

If we denote $\mathbf{b}_r = g_{rs}\mathbf{b}^s$ and introduce

$$T_r = \tfrac{1}{N}\,\mathrm{Tr}\,\mathbf{b}_r\,, \quad \widehat{\mathbf{b}}_r = \mathbf{b}_r - T_r\,,$$

we can rewrite the compatibility conditions (2.4) in the form

$$\partial_s\left(T_r + \tfrac{1}{2}\Gamma_r\right) - \partial_r\left(T_s + \tfrac{1}{2}\Gamma_s\right) = 0 \tag{2.5}$$

$$\partial_s\widehat{\mathbf{b}}_r - \partial_r\widehat{\mathbf{b}}_s = [\widehat{\mathbf{b}}_r, \widehat{\mathbf{b}}_s]\,. \tag{2.6}$$

From the equation (2.5) follows that exist function $F(y)$ such that

$$\partial_r F = T_r + \tfrac{1}{2}\Gamma_r\,. \tag{2.7}$$

Denoting $\mathbf{G} = \mathrm{e}^F\,\widehat{\mathbf{G}}$ we can write the equation (2.3) in the form

$$\partial_r\widehat{\mathbf{G}} = \widehat{\mathbf{G}}\widehat{\mathbf{b}}_r\,. \tag{2.8}$$

If the $\widehat{\mathbf{G}}_0$ is solution of the equation (2.8) and $\mathbf{G}_0 = \mathrm{e}^F \widehat{\mathbf{G}}_0$, the matrix potential $\mathbf{U}_0(x)$ corresponding the matrix functions $\mathbf{G}_0(x)$ and $\mathbf{V}$ can be find from relation

$$\mathbf{U}_0 = \Big(\eta^{ik}\partial_{ik}\mathbf{G}_0(x) - \mathbf{G}_0(x)\mathbf{V}\Big)\mathbf{G}_0^{-1}(x)\,. \qquad (2.9)$$

As the equation (2.8) is linear their general solution can be written in the form $\widehat{\mathbf{G}} = \mathbf{C}\widehat{\mathbf{G}}_0$, where $\mathbf{C}$ is constant matrix. The potential $\mathbf{U}(x)$ corresponding to such solution of (2.8) is

$$\mathbf{U}(x) = \mathbf{C}\mathbf{U}_0(x)\mathbf{C}^{-1}\,. \qquad (2.10)$$

the main problem of our construction is to find for given transformations $y_r = y_r(x_k)$ matrix functions $\mathbf{b}^r(y)$, $\mathbf{V}(y)$ and constant regular matrix $\mathbf{C}$ to the matrix potential (2.10) be symmetric.

3 Models of the G_2 type

We will consider matrix models with[1]

$$\eta^{11} = \eta^{22} = \tfrac{2}{3}\,, \quad \eta^{12} = \eta^{21} = -\tfrac{1}{3}$$

and transformation

$$y_1 = -x_1^2 - x_1x_2 - x_2^2\,, \quad y_2 = -x_1x_2(x_1 + x_2)\,.$$

In this case we obtain

$$g^{11} = -2y_1\,, \quad g^{12} = g^{21} = -3y_2\,, \quad g^{22} = \tfrac{2}{3}\,y_1^2\,. \qquad (3.1)$$

It is easy to see that the differential operator $g^{rs}\partial_{rs}$ has invariant subspaces of two type: $V_N^{(1)}$ spaces of polynomials generated by $y_1^{n_1}y_2^{n_2}$, where $n_1 + n_2 \le N$ and $V_N^{(2)}$ spaces of polynomials generated by $y_1^{n_1}y_2^{2n_2}$, where $n_1 + 2n_2 \le N$. In the scalar case the choose of the invariant spaces $V_N^{(1)}$ leads to the models of the A_2 type and the choose invariant spaces $V_N^{(2)}$ to the models of the G_2 type. Matrix model of the A_2 type we study in [4]. In this paper we will study the matrix models of type G_2, i.e. we will consider invariant subspaces $V_N^{(2)}$.

Therefore we choose matrix functions $\mathbf{b}^r(y)$ in the form

$$\mathbf{b}^1 = \mathbf{C}_0^1 + \mathbf{C}_1^1\,, \quad \mathbf{b}^2 = \mathbf{C}_3^2 y_2 + y_2^{-1}\left(\mathbf{C}_0^2 + \mathbf{C}_1^2 y_1 + \mathbf{C}_2^2 y_1^2\right)$$

and $\mathbf{V}$ as a constant matrix.

In this case the compatibility conditions (2.4) are

$$\begin{aligned}
&[\mathbf{C}_1^1, \mathbf{C}_3^2] = 0\,, && [\mathbf{C}_1^1, \mathbf{C}_2^2] = 0\,,\\
&[\mathbf{C}_0^1, \mathbf{C}_2^2] + [\mathbf{C}_1^1, \mathbf{C}_1^2] = 0\,, && [\mathbf{C}_0^1, \mathbf{C}_3^2] = -3\mathbf{C}_1^1 + 2\mathbf{C}_3^2\,,\\
&[\mathbf{C}_0^1, \mathbf{C}_1^2] + [\mathbf{C}_1^1, \mathbf{C}_0^2] = -2\mathbf{C}_1^2\,, && [\mathbf{C}_0^1, \mathbf{C}_0^2] = -4\mathbf{C}_0^2\,.
\end{aligned} \qquad (3.2)$$

[1]This η^{ik} is connected with Laplace operator in three dimension in center of mass coordinates.

In the case of 2×2 matrix model [6] we was successful with solution of the system (3.2), when we put $\mathbf{C}_1^1=0$. Therefore we choose $\mathbf{C}_1^1=0$ in case 3×3 matrix, too. With this choose the conditions (3.2) gives

$$[\mathbf{C}_0^1,\mathbf{C}_2^2]=0\,,\quad [\mathbf{C}_0^1,\mathbf{C}_3^2]=2\mathbf{C}_3^2\,,\quad [\mathbf{C}_0^1,\mathbf{C}_1^2]=-2\mathbf{C}_1^2\,,\quad [\mathbf{C}_0^1,\mathbf{C}_0^2]=-4\mathbf{C}_0^2\,. \tag{3.3}$$

It seems to sensible to chose the traceless matrix $\widehat{\mathbf{C}}_0^1$ in the (3.3) as diagonal. We will study two cases:

a) $\widehat{\mathbf{C}}_0^1=A(\mathbf{e}_{11}-\mathbf{e}_{33})$ and

b) $\widehat{\mathbf{C}}_0^1=A(\mathbf{e}_{11}-2\mathbf{e}_{22}+\mathbf{e}_{33})$,

where A is a constant and $\mathbf{e}_{rs}$ are 3×3 matrices $(\mathbf{e}_{rs})_{ik}=\delta_{ri}\delta_{sk}$.

3.1 Solution in the case a)

In the case a) the general solution of (3.3) is

$$\begin{array}{ll}
\mathbf{C}_0^1=-3\mu-3\nu-1+2(\mathbf{e}_{11}-\mathbf{e}_{33})\,, & \mathbf{C}_1^1=-2\omega\,,\\
\mathbf{C}_0^2=A_0^2\mathbf{e}_{31}\,, & \mathbf{C}_1^2=A_1^2\mathbf{e}_{21}+B_1^2\mathbf{e}_{32}\,,\\
\mathbf{C}_2^2=\frac{2}{3}\,\nu+A_2^2(\mathbf{e}_{11}-\mathbf{e}_{22})+B_2^2(\mathbf{e}_{22}-\mathbf{e}_{33})\,, & \mathbf{C}_3^2=-3\omega+A_3^2\mathbf{e}_{12}+B_3^2\mathbf{e}_{23}
\end{array}$$

or for traceless matrices $\widehat{\mathbf{C}}_s^r$

$$\begin{array}{ll}
\widehat{\mathbf{C}}_0^1=2(\mathbf{e}_{11}-\mathbf{e}_{33})\,, & \widehat{\mathbf{C}}_1^1=0\,,\\
\widehat{\mathbf{C}}_0^2=A_0^2\mathbf{e}_{31}\,, & \widehat{\mathbf{C}}_1^2=A_1^2\mathbf{e}_{21}+B_1^2\mathbf{e}_{32}\,,\\
\widehat{\mathbf{C}}_2^2=A_2^2(\mathbf{e}_{11}-\mathbf{e}_{22})+B_2^2(\mathbf{e}_{22}-\mathbf{e}_{33})\,, & \widehat{\mathbf{C}}_3^2=A_3^2\mathbf{e}_{12}+B\mathbf{e}_{23}
\end{array} \tag{3.4}$$

The system of equations (2.8) is in this case equivalent to three systems of equations

$$\begin{aligned}
(4y_1^3+27y_2^2)\partial_1X &= -(4+9A_2^2)y_1^2X-9A_1^2y_1Y-9A_0^2Z\\
(4y_1^3+27y_2^2)\partial_1Y &= -9A_3^2y_2^2X+9(A_2^2-B_2^2)y_1^2Y-9B_1^2y_1Z\\
(4y_1^3+27y_2^2)\partial_1Z &= -9B_3^2y_2^2Y+(4+9B_2^2)y_1^2Z\\
y_2(4y_1^3+27y_2^2)\partial_2X &= -6(3y_2^2-A_2^2y_1^3)X+6A_1^2y_1^2Y+6A_0^2y_1Z\\
y_2(4y_1^3+27y_2^2)\partial_2Y &= 6A_3^2y_1y_2^2X-6(A_2^2-B_2^2)y_1^3Y+6B_1^2y_1^2Z\\
y_2(4y_1^3+27y_2^2)\partial_2Z &= 6B_3^2y_1y_2^2Y+6(3y_2^2-B_2^2y_1^3)Z
\end{aligned} \tag{3.5}$$

where $X=\widehat{G}_{k1}$, $Y=\widehat{G}_{k2}$ and $Z=\widehat{G}_{k3}$, $k=1,\,2,\,3$.

It is easy to see that from the system (3.5) follow

$$2y_1\partial_1X+3y_2\partial_2X=-2X\,,\quad 2y_1\partial_1Y+3y_2\partial_2Y=0\,,\quad 2y_1\partial_1Z+3y_2\partial_2Z=2Z\,,$$

which gives

$$X=y_1^{-1}F(t)\,,\quad Y=G(t)\,,\quad Z=y_1H(t)\,,\qquad t=\frac{y_2^2}{y_1^3}\,.$$

The functions F, G and H then fulfill the system of equations

$$\begin{aligned} t(4+27t)F' &= 3(A_2^2-3t)F + 3A_1^2G + 3A_0^2H \\ t(4+27t)G' &= 3A_3^2tF - 3(A_2^2-B_2^2)G + 3B_1^2H \\ t(4+27t)H' &= 3B_3^2tG - 3(B_2^2-3t)H\,. \end{aligned} \tag{3.6}$$

To find three independent solution of system (3.6) we choose with analogy of 2 × 2 matrix model special value of constants A_s^r and B_s^r, which essentially simplify the system (3.6)[2].

If we chose

$$\begin{aligned} &A_0^2 = \tfrac{4}{9}\,, \quad A_1^2 = -\tfrac{16}{9}\,, \quad A_2^2 = -\tfrac{4}{3}\,, \quad A_3^2 = -3\,, \\ &B_1^2 = \tfrac{4}{9}\,, \quad B_2^2 = -\tfrac{8}{9}\,, \quad B_3^2 = -6\,, \end{aligned} \tag{3.7}$$

three independent solution of the system (3.6) are

$$\begin{aligned} F_1 &= G_1 = H_1 = \frac{t^{2/3}}{4+27t} \\ F_2 &= G_2 = \frac{4t^{1/3}}{4+27t}\,, \quad H_2 = -\frac{27t^{4/3}}{4+27t}\,, \\ F_3 &= \frac{4(81t+4)}{t(4+27t)}\,, \quad G_3 = \frac{27(4-27t)}{4+27t}\,, \quad H_3 = -\frac{1458t}{4+27t} \end{aligned}$$

In our case the function e^F is

$$\mathrm{e}^F = \Big((x_1-x_2)(2x_1+x_2)(x_1+2x_2)\Big)^{\mu}\Big(x_1x_2(x_1+x_2)\Big)^{\nu}\mathrm{e}^{-\omega(x_1^2+x_1x_2+x_2^2)}\,, \tag{3.8}$$

which gives matrix $\mathbf{G}_0(x)$. By direct calculation it is possible to show that corresponding matrix potential $\mathbf{U}_0$ can not be symmetrize by any choose of constant matrices $\mathbf{V}$ and $\mathbf{C}$.

3.2 Solution in the case b)

In this case the general solution of (3.3) is

$$\begin{aligned} \mathbf{C}_0^1 &= -3\mu - 3\nu - 1 + \tfrac{2}{3}\,(\mathbf{e}_{11} - 2\mathbf{e}_{22} + \mathbf{e}_{33})\,, \\ \mathbf{C}_1^1 &= -2\omega\,, \\ \mathbf{C}_0^2 &= 0\,, \\ \mathbf{C}_1^2 &= A_1^2\mathbf{e}_{21} + B_1^2\mathbf{e}_{23}\,, \\ \mathbf{C}_2^2 &= \tfrac{2}{3}\,\nu + \alpha(\mathbf{e}_{11} - \mathbf{e}_{22}) + \beta(\mathbf{e}_{22} - \mathbf{e}_{33}) + A_2^2\mathbf{e}_{13} + B_2^2\mathbf{e}_{31}\,, \\ \mathbf{C}_3^2 &= -3\omega + A_3^2\mathbf{e}_{12} + B_3^2\mathbf{e}_{32} \end{aligned}$$

or for traceless matrices $\widehat{\mathbf{C}}_s^r$

$$\begin{aligned} \widehat{\mathbf{C}}_0^1 &= \tfrac{2}{3}\,(\mathbf{e}_{11} - 2\mathbf{e}_{22} + \mathbf{e}_{33})\,, \\ \widehat{\mathbf{C}}_1^1 &= 0\,, \\ \widehat{\mathbf{C}}_0^2 &= 0\,, \\ \widehat{\mathbf{C}}_1^2 &= A_1^2\mathbf{e}_{21} + B_1^2\mathbf{e}_{23}\,, \\ \widehat{\mathbf{C}}_2^2 &= \alpha(\mathbf{e}_{11} - \mathbf{e}_{22}) + \beta(\mathbf{e}_{22} - \mathbf{e}_{33}) + A_2^2\mathbf{e}_{13} + B_2^2\mathbf{e}_{31}\,, \\ \widehat{\mathbf{C}}_3^2 &= A_3^2\mathbf{e}_{12} + B_3^2\mathbf{e}_{32} \end{aligned} \tag{3.9}$$

[2] In the other case in the solution of (3.6) appear hypergeometric functions.

To solve the system (2.8) we have to find three independent solutions of the system

$$\begin{aligned}(4y_1^3+27y_2^2)\partial_1 X &= -\tfrac{1}{3}\,(4+27\alpha)y_1^2X - 9A_1^2y_1Y - 9B_2^2y_1^2Z\\(4y_1^3+27y_2^2)\partial_1 Y &= -9A_3^2y_2^2X + \tfrac{1}{3}\,(8+27\alpha-27\beta)y_1^2Y - 9B_3^2y_2^2Z\\(4y_1^3+27y_2^2)\partial_1 Z &= -9A_2^2y_1^2X - 9B_1^2y_1Y - \tfrac{1}{3}\,(4-27\beta)y_1^2Z\\y_2(4y_1^3+27y_2^2)\partial_2 X &= -6(y_2^2-\alpha y_1^3)X + 6A_1^2y_1^2Y + 6B_2^2y_1^3Z\\y_2(4y_1^3+27y_2^2)\partial_2 Y &= 6A_3^2y_1y_2^2X + 6(2y_2^2-(\alpha-\beta)y_1^3)Y + 6B_3^2y_1y_2^2Z\\y_2(4y_1^3+27y_2^2)\partial_1 Z &= 6A_2^2y_1^3X + 6B_1^2y_1^2Y - 6(y_2^2+\beta y_1^3)Z\end{aligned}\tag{3.10}$$

From the system (3.10) we obtain relations

$$6y_1\partial_1X + 9y_2\partial_2X = -2X\,,\quad 6y_1\partial_1Y + 9y_2\partial_2Y = 4Y\,,\quad 6y_1\partial_1Z + 9y_2\partial_2Z = -2Z\,,$$

from which follow

$$X = y_1^{-1/3}F(t)\,,\quad Y = y_1^{2/3}G(t)\,,\quad Z = y_1^{-1/3}H(t)\,,\qquad t = \frac{y_2^2}{y_1^3}\,.$$

Functions $F(t)$, $G(t)$ and $H(t)$ fulfill system differential equations

$$\begin{aligned}t(4+27t)F' &= 3(\alpha-t)F + 3A_1^2G + 3B_2^2H\\t(4+27t)G' &= 3A_3^2tF - 3(\alpha-\beta-2t)G + 3B_3^2tH\\t(4+27t)H' &= 3A_2^2F + 3B_1^2G - 3(\beta+t)H\end{aligned}\tag{3.11}$$

To solve (3.11) we again choose convenient constants.

First possibility is to choose

$$\begin{aligned}&A_1^2 = \tfrac{8}{9}\,, && A_2^2 = \tfrac{10}{9}\,(1+p)\,, && A_3^2 = -3\,,\\&B_1^2 = \tfrac{10}{9}\,(1-p)\,, && B_2^2 = 0\,, && B_3^2 = 0\,,\\&\alpha = \tfrac{4}{27}\,, && \beta = \tfrac{32}{27}\,.\end{aligned}\tag{3.12}$$

In this case we have

$$\begin{aligned}&F_1 = G_1 = H_1 = \frac{t^{7/9}}{(4+27t)^{8/9}}\\&F_2 = G_2 = 0\,,\ \ H_2 = \frac{(4+27t)^{7/9}}{t^{8/9}}\\&F_3 = \frac{4t^{1/9}}{(4+27t)^{8/9}}\,,\ \ G_3 = -\frac{27t^{10/9}}{(4+27t)^{8/9}}\,,\ \ H_3 = \frac{4-20p+135(1-p)t}{27t^{8/9}(4+27t)^{8/9}}\,.\end{aligned}\tag{3.13}$$

The second possibility is to choose

$$\begin{aligned}&A_1^2 = -\tfrac{8}{3}\,, && A_2^2 = -\tfrac{2}{3}\,(1+p)\,, && A_3^2 = -3\,,\\&B_1^2 = -\tfrac{2}{3}\,(1-p)\,, && B_2^2 = 0\,, && B_3^2 = 0\,,\\&\alpha = \tfrac{4}{3}\,, && \beta = 0\end{aligned}\tag{3.14}$$

and three solutions of (3.11) are

$$\begin{aligned} F_1 &= G_1 = H_1 = \frac{(4+27t)^{8/9}}{t}\,, \\ F_2 &= G_2 = 0\,,\quad H_2 = \frac{1}{(4+27t)^{1/9}}\,, \\ F_3 &= \frac{4(4+45t)}{t(4+27t)^{7/9}}\,,\quad G_3 = \frac{16+180t+405t^2}{t(4+27t)^{7/9}}\,,\quad H_3 = \frac{16+45(1-p)t}{t(4+27t)^{7/9}}\,. \end{aligned} \tag{3.15}$$

In the third case we choose

$$\begin{aligned} &A_1^2 = -\tfrac{4}{3}\,, && A_2^2 = -\tfrac{10}{9}\,(1+p)\,, && A_3^2 = -3\,, \\ &B_1^2 = -\tfrac{10}{9}\,(1-p)\,, && B_2^2 = 0\,, && B_3^2 = 0\,, \\ &\alpha = \tfrac{4}{27}\,, && \beta = -\tfrac{28}{27} && \end{aligned} \tag{3.16}$$

and independent solutions of (3.11) are

$$\begin{aligned} F_1 &= G_1 = H_1 = \frac{(4+27t)^{7/9}}{t^{8/9}}\,, \\ F_2 &= G_2 = 0\,,\quad H_2 = \frac{t^{7/9}}{(4+27t)^{8/9}}\,, \\ F_3 &= \frac{2(4+27t)^{1/9}}{t^{8/9}}\,,\quad G_2 = \frac{(4+27t)^{1/9}(9t+2)}{t^{8/9}}\,,\quad H_3 = \frac{8+45(1-p)t}{t^{8/9}(4+27t)^{8/9}}\,. \end{aligned} \tag{3.17}$$

In the last interesting case the constants are

$$\begin{aligned} &A_1^2 = \tfrac{20}{9}\,, && A_2^2 = \tfrac{2}{3}\,(1+p)\,, && A_3^2 = -3\,, \\ &B_1^2 = \tfrac{2}{3}\,(1-p)\,, && B_2^2 = 0\,, && B_3^2 = 0\,, \\ &\alpha = -\tfrac{28}{27}\,, && \beta = \tfrac{4}{27} && \end{aligned} \tag{3.18}$$

and in this case the three independent solutions are, e.g.

$$\begin{aligned} F_1 &= G1 = H_1 = \frac{t^{8/9}}{4+27t}\,, \\ F_2 &= G_2 = 0\,,\quad H_2 = t^{-1/9}\,, \\ F_3 &= \frac{2(8+135t)}{3t^{7/9}(4+27t)}\,,\quad G_3 = \frac{9t^{2/9}(2-27t)}{4+27t}\,,\quad H_3 = -\frac{4(1+p)-27(1-p)t}{t^{7/9}(4+27t)}\,. \end{aligned} \tag{3.19}$$

The function e^F is in all discussed case given by relation (3.8).

4 Potential

To compute the corresponding potential we first use formulae (2.9).

The most interesting choose of the constant matrix $\mathbf{V}$ is

$$\mathbf{V} = -2\omega(3\mu+3\nu+1) + \tfrac{4}{3}\,\omega(\mathbf{e}_{11} - 2\mathbf{e}_{22} + \mathbf{e}_{33}) + A\mathbf{e}_{12} + B\mathbf{e}_{32}\,, \tag{4.1}$$

where A and B are suitable constants. The other choose of the matrix $\mathbf{V}$ leads to the matrix function in the potential, which must be symmetrize simultaneously with the following matrix potential.

With this choice we obtain by direct computation

$$\mathbf{U}_0 = \Big(\eta^{ik}\partial_{ik}\mathbf{G}_0(x)) - \mathbf{G}_0\mathbf{V}\Big)\mathbf{G}_0^{-1} = U_0^{(s)} + \mathbf{U}_0^{(m)}\,,$$

where

$$\begin{aligned} U_0^{(s)} &= 2\omega^2(x_1^2 + x_1x_2 + x_2^2)+ \\ &+2\left(\mu^2 - \mu + \tfrac{8}{3}\right)\left(\frac{1}{(x_1-x_2)^2} + \frac{1}{(2x_1+x_2)^2} + \frac{1}{(x_1+2x_2)^2}\right) + \\ &+\tfrac{2}{3}\left(\nu^2 - \nu + \tfrac{152}{81}\right)\left(\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{(x_1+x_2)^2}\right) \end{aligned} \tag{4.2}$$

and $\mathbf{U}_0^{(m)}$ is the traceless part of the potential, which is given as follows

$$\begin{aligned} U_{11}^{(m)} &= \frac{-12\mu(x_1^2+x_1x_2+x_2^2)^2}{(x_1-x_2)^4(2x_1+x_2)^4(x_1+2x_2)^4}\times \\ &\times\Big(16x_1^6 + 48x_1^5x_2 + 69x_1^4x_2^2 + 58x_1^3x_2^3 + 69x_1^2x_2^4 + 48x_1x_2^5 + 16x_2^6\Big) + \\ &+\tfrac{16}{27}\left(4\nu - \tfrac{5}{9}\right)\left(\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{(x_1+x_2)^2}\right) + \\ &+\frac{2(6\nu + A + B)(x_1^2+x_1x_2+x_2^2)^2}{(x_1-x_2)^4(2x_1+x_2)^4(x_1+2x_2)^4}\times \\ &\times\Big(8x_1^6 + 24x_1^5x_2 - 87x_1^4x_2^2 - 214x_1^3x_2^3 - 87x_1^2x_2^4 + 24x_1x_2^5 + 8x_2^6\Big) + \\ &+\tfrac{8}{3}\,\frac{(x_1^2+x_1x_2+x_2^2)^2}{(x_1-x_2)^4(2x_1+x_2)^4(x_1+2x_2)^4}\times \\ &\times\Big(94x_1^6 + 282x_1^5x_2 - 111x_1^4x_2^2 - 692x_1^3x_2^3 - 111x_1^2x_2^4 + 282x_1x_2^5 + 94x_2^6\Big) \\ U_{22}^{(m)} &= -\left(\tfrac{8}{27}\,\nu + \tfrac{260}{243}\right)\left(\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{(x_1+x_2)^2}\right) - \\ &-\tfrac{1}{9}(B(3p+7)+48)\left(\frac{1}{(x_1-x_2)^2} + \frac{1}{(2x_1+x_2)^2} + \frac{1}{(x_1+2x_2)^2}\right) \\ U_{33}^{(m)} &= -U_{11}^{(m)} - U_{22}^{(m)} \\ U_{12}^{(m)} &= -\frac{(3p+7)(6\mu+6\nu+A+B+\frac{2}{3})x_1^2x_2^2(x_1+x_2)^2(x_1^2+x_1x_2+x_2^2)^2}{(x_1-x_2)^4(2x_1+x_2)^4(x_1+2x_2)^4} \\ U_{13}^{(m)} &= -\frac{3(6\mu+6\nu+A+B+\frac{2}{3})(x_1x_2(x_1+x_2))^{10/3}}{(x_1-x_2)^4(2x_1+x_2)^4(x_1+2x_2)^4} \\ U_{23}^{(m)} &= -\frac{3B(x_1x_2(x_1+x_2))^{4/3}}{(x_1-x_2)^2(2x_1+x_2)^2(x_1+2x_2)^2} \\ U_{21}^{(m)} &= 4B\left(\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{(x_1+x_2)^2}\right) - \\ &\quad -18B\left(\frac{1}{(x_1-x_2)^2} + \frac{1}{(2x_1+x_2)^2} + \frac{1}{(x_1+2x_2)^2}\right) \end{aligned}$$

$$
\begin{aligned}
U_{31}^{(m)} = & \frac{(x_1^2+x_1x_2+x_2^2)(x_1x_2(x_1+x_2))^{-10/3}}{(x_1-x_2)^4(2x_1+x_2)^4(x_1+2x_2)^4} \times \\
& \times \Big[216\mu x_1^2x_2^2(x_1+x_2)^2 \times \\
& \Big(16x_2^{12} + 96x_1x_2^{11} + 1011x_1^2x_2^{10} + 4175x_1^3x_2^9 + 6570x_1^4x_2^8 + \\
& \quad +2286x_1^5x_2^7 - 2064x_1^6x_2^6 + 2286x_1^7x_2^5 + 6570x_1^8x_2^4 + \\
& \quad +4175x_1^9x_2^3 + 1011x_1^{10}x_2^2 + 96x_1^{11}x_2 + 16x_1^{12} \Big) - \\
& -24\nu \Big(32x_1^{18} + 288x_2x_1^{17} + 2160x_2^2x_1^{16} + 10752x_2^3x_1^{15} + \\
& \quad +24363x_2^4x_1^{14} + 5229x_2^5x_1^{13} - 86307x_2^6x_1^{12} - 181467x_2^7x_1^{11} - \\
& \quad -176436x_2^8x_1^{10} - 142012x_2^9x_1^9 - 176436x_2^{10}x_1^8 - 181467x_2^{11}x_1^7 - \\
& \quad -86307x_2^{12}x_1^6 + 5229x_2{}^13x_1^5 + 24363x_2^{14}x_1^4 + 10752x_2^{15}x_1^3 + \\
& \quad -2160x_2^{16}x_1^2 + 288x_2^{17}x_1 + 32x_2^{18} \Big) + \\
& +\tfrac{64}{3} \Big(38x_1^{18} + 342x_2x_1^{17} + 1134x_2^2x_1^{16} + 1320x_2^3x_1^{15} - \\
& \quad -6282x_2^4x_1^{14} - 39942x_2^5x_1^{13} - 87999x_2^6x_1^{12} - 49464x_2^7x_1^{11} + \\
& \quad +122643x_2^8x_1^{10} + 234518x_2^9x_1^9 + 122643x_2^{10}x_1^8 - 49464x_2^{11}x_1^7 - \\
& \quad -87999x_2^{12}x_1^6 - 39942x_2^{13}x_1^5 - 6282x_2^{14}x_1^4 + 1320x_2^{15}x_1^3 + \\
& \quad +1134x_2^{16}x_1^2 + 342x_2^{17}x_1 + 38x_2^{18} \Big) + \\
& +\tfrac{4}{3}\, A(x_1^2+x_1x_2+x_2^2)^3 \times \\
& \quad \times \Big(8x_1^6 + 24x_1^5x_2 - 87x_1^4x_2^2 - 214x_1^3x_2^3 - 87x_1^2x_2^4 + 24x_1x_2^5 + 8x_2^6 \Big)^2 - \\
& -2B(x_1^2+x_1x_2+x_2^2)^3 \Big(4x_1^6 + 12x_1^5x_2 + 51x_1^4x_2^2 + 82x_1^3x_2^3 + 51x_1^2x_2^4 + \\
& \quad +12x_1x_2^5 + 4x_2^6 + 4px_1^6 + 12px_1^5x_2 - 3px_1^4x_2^2 - 26px_1^3x_2^3 - \\
& \quad -3px_1^2x_2^4 + 12px_1x_2^5 + 4px_2^6 \Big) \times \\
& \times \Big(8x_1^6 + 24x_1^5x_2 - 87x_1^4x_2^2 - 214x_1^3x_2^3 - 87x_1^2x_2^4 + 24x_1x_2^5 + 8x_2^6 \Big) \Big]
\end{aligned}
$$

$$
\begin{aligned}
U_{32}^{(m)} = & \frac{(3p+7)(x_1^2+x_1x_2+x_2^2)(x_1x_2(x_1+x_2))^{-4/3}}{(x_1-x_2)^4(2x_1+x_2)^4(x_1+2x_2)^4} \times \\
& \times \Big[108\mu x_1^2x_2^2(x_1+x_2)^2 \Big(7x_2^6 + 21x_1x_2^5 + 15x_1^2x_2^4 - 5x_1^3x_2^3 + \\
& \quad +15x_1^4x_2^2 + 21x_1^5x_2 + 7x_1^6 \Big) - \\
& -12\nu \Big(-558x_1^8x_2^4 - 594x_1^7x_2^5 - 456x_1^6x_2^6 - 594x_1^5x_2^7 - 558x_1^4x_2^8 - \\
& \quad -185x_1^3x_2^9 + 51x_1^2x_2^{10} + 51x_1^{10}x_2^2 - 185x_1^9x_2^3 + \\
& \quad +48x_1^{11}x_2 + 48x_1x_2^{11} + 8x_1^{12} + 8x_2^{12} \Big) +
\end{aligned}
$$

$$+\frac{8}{3}\Big(38x_1^{12}+228x_1^{11}x_2+411x_1^{10}x_2^2-35x_1^9x_2^3-639x_1^8x_2^4+$$
$$+162x_1^7x_2^5+1128x_1^6x_2^6+162x_1^5x_2^7-639x_1^4x_2^8-$$
$$-35x_1^3x_2^9+411x_1^2x_2^{10}+228x_1x_2^{11}+38x_2^{12}\Big)-$$
$$-\frac{2}{3}\,A(x_1^2+x_1x_2+x_2^2)^3\Big(8x_1^6+24x_1^5x_2-87x_1^4x_2^2-$$
$$-214x_1^3x_2^3-87x_1^2x_2^4+24x_1x_2^5+8x_2^6\Big)+$$
$$+B(x_1^2+x_1x_2+x_2^2)^3\Big(4x_1^6+12x_1^5x_2+51x_1^4x_2^2+$$
$$+82x_1^3x_2^3+51x_1^2x_2^4+12x_1x_2^5+4x_2^6+$$
$$+4px_1^6+12px_1^5x_2-3px_1^4x_2^2-26px_1^3x_2^3-$$
$$-3px_1^2x_2^4+12px_1x_2^5+4px_2^6\Big)\Big]$$

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