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Symmetry Classes of Quasilinear Systems in One Space Variable

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Abstract

The family of simple quasilinear systems in one space variable is partitioned into classes of commuting flows, i.e., symmetry classes. The systems in a symmetry class have the same zeroth order conserved densities and the same Hamiltonian structure. The zeroth and first order conservation laws and the Hamiltonian structure of the systems in a complete symmetry class are described. If such a system has a degenerate characteristic speed, then it has conservation laws of arbitrarily high order. Symmetry classes of 2-component hyperbolic systems correspond to coframes on the plane. The invariants of 2-component Hamiltonian hyperbolic symmetry classes are given. An exact symmetry class of 2-component hyperbolic systems is characterized by its canonical representative, and the first order conservation laws of the canonical system correspond to the infinitesimal automorphisms of the coframe. The normal forms of the rank 0 and rank 1 exact classes are listed. A simple symmetry class is tri-Hamiltonian if and only if the metric of its coframe has constant curvature. The normal forms of the tri-Hamiltonian simple classes are listed.

I Introduction

There are many physical systems which are mathematically approximated by first order quasilinear evolution equations. Hyperbolic equations of this type are associated with wave propagation. The symmetry structure, conservation laws, and Hamiltonian structure of first order hyperbolic quasilinear systems in one space variable have been recently studied by Verosky [1–3], Tsarev [4, 5], Nutku [6], Olver and Nutku [7], Serre [8, 9], and Arik, et al.[10]. Their results are unified and extended here.

Section II is an outline of the necessary geometric formalism. It is shown in Section III that the family of simple quasilinear systems is partitioned into symmetry classes.

A symmetry class is a collection of commuting flows. The systems in a class have the same zeroth order conserved densities. Dubrovin and Novikov [11] discovered that each flat metric on the space of field variables intrinsically defines a Poisson bivector. The simplest Hamiltonian systems generated by such a bivector are quasilinear. The systems in a symmetry class have the same Hamiltonian structure.

Complete symmetry classes are discussed in Section IV. These are infinite dimensional spaces of diagonalizable systems. The space of conserved densities for the systems in a complete class is also infinite dimensional. These systems are the simplest higher dimensional generalization of 2-component hyperbolic systems. The determination of the Hamiltonian structure of a complete symmetry class is reduced to the solvability of a system of linear equations. This implies that multiple Hamiltonian formulations are compatible. An example from gas dynamics is used to illustrate the theory. The first order conserved densities of the systems in a complete symmetry class are characterized in Section V. Systems with degenerate characteristic speeds are shown to have conserved densities of arbitrarily high order. The Born-Infeld equation is an example.

Symmetry classes of 2-component hyperbolic systems are studied in Section VI. There is a natural local correspondence between these classes and coframes on the plane. Dynamical properties of the systems in a class are thereby related to geometric features of the corresponding coframe. A surface metric is either definite or indefinite, so there are two local canonical forms for 2-component Hamiltonian quasilinear systems. The invariants of hyperbolic Hamiltonian symmetry classes in canonical form are presented. In particular, the separability condition of Olver and Nutku [7] is shown to be equivalent to the invariant condition of exactness. An exact symmetry class is characterized by its canonical representative, and the first order conservation laws of this system correspond naturally to the infinitesimal automorphisms of the related coframe. The existence of an automorphism permits the derivation of the normal forms of rank 0 and rank 1 exact symmetry classes. Most elementary of all are the simple symmetry classes. Each of these contains an elasticity system and an isentropic gas dynamics system, and the propagation speeds of these systems are related to the invariants of the coframe. For example, polytropy is equivalent to the condition that the coframe have rank 0. The tri-Hamiltonian structure of the polytropic gas dynamics systems was discovered by Nutku [6]. That discovery is generalized here. It is proved that a simple symmetry class has multi-Hamiltonian structure if and only if the metric defined by the coframe of the class has constant curvature, in which case the class is tri-Hamiltonian. The normal forms for the tri-Hamiltonian simple classes are listed. The paper concludes with the remarks of Section VII.

II Geometric Formalism

The geometric context for this work is outlined here. Further details can be found in Olver [12], Doyle [13], and references therein. A coordinate x on $\mathcal{X} \subset \mathbb{R}$ is a *space* variable, and coordinates $u^1, \ldots, u^n$ on $\mathcal{U} \subset \mathbb{R}^n$ are *field* variables. A phase is a map $\mathcal{X} \longmapsto \mathcal{U}$, i.e., an n-tuple $\mathbf{u}(x)$ of functions $u^1(x), \ldots, u^n(x)$. An evolutionary vector field is the infinitesimal generator of a flow on the phase space. The coordinate representation of an evolutionary vector field is

$$\mathbf{V} = Q^\alpha \, \partial_{u^\alpha}, \tag{2.1}$$

where $Q^1[u], \ldots, Q^n[u]$ are differential functions, i.e., smooth functions of x and of a finite number of the derivative coordinates $u_j^\alpha = d^j u^\alpha / dx^j$. A trajectory of the vector field is an n-tuple $\mathbf{u}(x,t)$ which satisfies the differential constraints

$$u_t^\alpha = Q^\alpha[\mathbf{u}].$$

The directional derivative of a differential function $F[u]$ along the vector field (2.1) has the coordinate description

$$\mathbf{V}\, F = \sum (\mathrm{D}_j Q^\alpha) \frac{\partial F}{\partial u_j^\alpha},$$

where

$$\mathrm{D}_j = (\mathrm{D}_x)^j,$$

and

$$\mathrm{D}_x = \sum u_{j+1}^\alpha \, \partial_{u_j^\alpha}$$

is the total derivative with respect to x. Evolutionary derivations commute with total derivatives, i.e.,

$$\mathbf{V}\,(\mathrm{D}_x F) = \mathrm{D}_x(\mathbf{V}\, F)$$

for all differential functions F. The Lie bracket of evolutionary vector fields

$$\mathbf{V} = Q^\alpha \, \partial_{u^\alpha}, \quad \mathbf{W} = R^\alpha \, \partial_{u^\alpha}$$

is the evolutionary vector field

$$[\mathbf{V}, \mathbf{W}] = S^\alpha \, \partial_{u^\alpha}, \qquad S^\alpha = \mathbf{V}\, R^\alpha - \mathbf{W}\, Q^\alpha.$$

The Lie bracket is formally equivalent to differentiating $\mathbf{W}$ along the flow of $\mathbf{V}$, via pull-back. Also,

$$[\mathbf{V},\mathbf{W}]\, F = \mathbf{V}\,(\mathbf{W}\, F) - \mathbf{W}\,(\mathbf{V}\, F).$$

The evolutionary vector fields $\mathbf{V}, \mathbf{W}$ are symmetries of one another if $[\mathbf{V}, \mathbf{W}] = 0$. This infinitesimal criterion is formally equivalent to the requirement that the flow of $\mathbf{V}$ carry trajectories of $\mathbf{W}$ onto new trajectories of $\mathbf{W}$.

A density is a horizontal 1-form $F[u]\, dx$. The total differential of a differential function F is the density $(\mathrm{D}_x F)\, dx$. The action of evolutionary vector fields on densities is

$$\mathbf{V}\,(F\, dx) = \mathbf{V}\, F\, dx.$$

A functional is the formal integral over $\mathcal{X}$ of a density. Algebraically, the linear space of functionals is the space of densities modulo the space of total differentials. The action of evolutionary vector fields on densities factors to an action

$$\mathbf{V} \int F\, dx = \int \; \mathbf{V}\, F \; dx$$

of evolutionary vector fields on functionals. In physical terms, functionals are the dynamical variables of an evolutionary system. A conservation law for the system $\mathbf{V}$ is a functional $\int T\,dx$ such that

$$\mathbf{V}\int T\,dx = 0. \tag{2.2}$$

The condition (2.2) is formally equivalent to the requirement that the functional be constant along the trajectories of $\mathbf{V}$. Note that (2.2) holds if and only if there is a differential function X such that

$$\mathbf{V}\,T + \mathrm{D}_x X = 0. \tag{2.3}$$

This implies

$$\frac{\partial}{\partial t}(T[\mathbf{u}]) + \frac{\partial}{\partial x}(X[\mathbf{u}]) = 0$$

whenever $\mathbf{u}(x,t)$ is a trajectory of $\mathbf{V}$, in which case

$$\frac{d}{dt}\int_a^b T[\mathbf{u}](x,t)\,dx = X[\mathbf{u}](a,t) - X[\mathbf{u}](b,t)$$

for any interval $[a,b]$, so that the time rate of change of the total quantity of $T[\mathbf{u}]$ enclosed in a spatial region is equal to the amount of $X[\mathbf{u}]$ flowing into the region. The densities $T\,dx$ and $X\,dx$ are the conserved density and the flux of the conservation law. The space variable x is fixed for the duration, so it will be convenient to refer simply to the functions T and X as the conserved density and flux.

The tensor calculus of finite dimensional differential geometry generalizes to functional tensor calculus. The objects dual to evolutionary vector fields are functional 1-forms. The coordinate representation of a functional 1-form is

$$\mathbf{\Omega} = \int R_\alpha\,du^\alpha\,dx, \tag{2.4}$$

where $R_1,\ldots,R_n$ are unique differential functions. The contraction of the 1-form (2.4) with the vector field (2.1) is the functional

$$\langle \mathbf{\Omega}, \mathbf{V}\rangle = \int R_\alpha Q^\alpha\,dx.$$

The variational differential of a functional $\int F\,dx$ is the unique functional 1-form $\delta\int F\,dx$ such that

$$\left\langle \delta\int F\,dx, \mathbf{V}\right\rangle = \mathbf{V}\int F\,dx$$

for all evolutionary vector fields $\mathbf{V}$. Formal integration by parts proves

$$\delta\int F\,dx = \int \frac{\delta F}{\delta u^\alpha}\,du^\alpha\,dx,$$

where

$$\frac{\delta}{\delta u^\alpha} = \sum (-1)^j\,\mathrm{D}_j \circ \partial_{u^\alpha_j}$$

is the αth variational derivative. Note that

$$\frac{\delta}{\delta u^\alpha}(\mathrm{D}_x F) = 0$$

for any differential function F. Conversely, a differential function with variational derivatives equal to zero is locally equal to a total derivative.

See Doyle [13] for a thorough discussion of differential geometric bivectors. Only the following abbreviated version is needed here. A nondegenerate 1-weight differential geometric Poisson bivector $\boldsymbol{\Theta}$ is described in coordinates by a Poisson operator

$$\mathcal{D}^{\alpha\beta} = g^{\alpha\beta}(u)\,\mathrm{D}_x + b^{\alpha\beta}_\gamma(u) u^\gamma_x, \tag{2.5}$$

where g is symmetric, $\det g \neq 0$,

$$\frac{\partial g^{\alpha\beta}}{\partial u^\gamma} = b^{\alpha\beta}_\gamma + b^{\beta\alpha}_\gamma, \tag{2.6}$$

and the connection ∇ on $\mathcal{U}$ with coefficients

$$\Gamma^\gamma_{\alpha\beta} = -b^{\lambda\gamma}_\alpha g_{\lambda\beta} \tag{2.7}$$

is symmetric and integrable. The notation $u_x = u_1$ is used here. These conditions on the operator (2.5) ensure that the bracket

$$\left\{\int F\,dx, \int G\,dx\right\} = \left\langle \boldsymbol{\Theta}; \delta\int F\,dx, \delta\int G\,dx\right\rangle = \int \frac{\delta F}{\delta u^\alpha}\mathcal{D}^{\alpha\beta}\frac{\delta G}{\delta u^\beta}\,dx$$

is skew-symmetric and satisfies the Jacobi identity

$$\bigcirc\left\{\left\{\int F\,dx, \int G\,dx\right\}, \int H\,dx\right\} = 0, \tag{2.8}$$

where the notation indicates cyclic summation. Note that (2.6), (2.7) imply that ∇ is the Levi-Civita connection for the flat metric

$$\mathbf{g} = g_{\alpha\beta}\,du^\alpha \otimes du^\beta. \tag{2.9}$$

A functional $\mathcal{H} = \int H\,dx$ generates a Hamiltonian evolutionary vector field $\mathbf{X}_\mathcal{H}$ such that

$$\mathbf{X}_\mathcal{H}\int F\,dx = \left\{\int F\,dx, \int H\,dx\right\} \tag{2.10}$$

for all functionals $\int F\,dx$. In coordinates,

$$\mathbf{X}_\mathcal{H} = Q^\alpha\,\partial_{u^\alpha}, \qquad Q^\alpha = \mathcal{D}^{\alpha\beta}\frac{\delta H}{\delta u^\beta}. \tag{2.11}$$

In physical terms, the generating functional is the energy of the Hamiltonian system. The conservation laws for a Hamiltonian system $\mathbf{X}_\mathcal{H}$ are the functionals $\int F\,dx$ such that

$$\left\{\int F\,dx, \int H\,dx\right\} = 0, \tag{2.12}$$

because of (2.10). For example $\int H\,dx$ is a conservation law for $\mathbf{X}_{\mathcal{H}}$, i.e., the energy of a Hamiltonian system is conserved. The Jacobi identity (2.8) implies that the map

$$\mathcal{H} \longmapsto -\mathbf{X}_{\mathcal{H}}$$

is a Lie algebra homomorphism from the space of functionals to the space of evolutionary vector fields. Hence if $\mathcal{F} = \int F\,dx$ satisfies (2.12) then

$$[\mathbf{X}_{\mathcal{F}}, \mathbf{X}_{\mathcal{H}}] = 0,$$

so that the conservation laws for a Hamiltonian system generate symmetries of the system.

A functional $\mathcal{F}$ is a Casimir of the bivector if $\mathbf{X}_{\mathcal{F}} = 0$, in which case $\mathcal{F}$ is a conservation law for any Hamiltonian system. The densities of the Casimirs of the Poisson bivector $\boldsymbol{\Theta}$ described in coordinates by the operator (2.5) are the exponential coordinates of the flat metric (2.9) (modulo total derivatives). These are called *Darboux* coordinates for $\boldsymbol{\Theta}$, because $\boldsymbol{\Theta}$ is described in these coordinates by the constant coefficient Poisson operator

$$\mathcal{D} = \varepsilon\,\mathrm{D}_x,$$

where ε is a constant, invertible, symmetric matrix. In appropriate Darboux coordinates, the matrix ε is diagonal with diagonal entries equal to ± 1.

III Symmetry Classes of Quasilinear Systems

Consider a system of partial differential equations

$$u^\alpha_t + A^\alpha_\beta(u) u^\beta_x = 0, \qquad \alpha = 1, \ldots, n. \tag{3.1}$$

It is described in field variables $\tilde{u}^1, \ldots, \tilde{u}^n$ by the equations

$$\tilde{u}^\alpha_t + \widetilde{A}^\alpha_\beta(\tilde{u}) \tilde{u}^\beta_x = 0, \qquad \widetilde{A}^\alpha_\beta = \frac{\partial \tilde{u}^\alpha}{\partial u^\gamma}\frac{\partial u^\eta}{\partial \tilde{u}^\beta} A^\gamma_\eta.$$

Hence the (1,1)-tensor

$$\mathbf{A} = A^\alpha_\beta\; \partial_{u^\alpha} \otimes du^\beta$$

on $\mathcal{U}$ is intrinsically associated with the system, which therefore has the geometric description

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0. \tag{3.2}$$

Systems of type (3.2) are *quasilinear.* A quasilinear system is *simple* if the characteristic polynomial of its tensor is simple, and is *semisimple* if the minimal polynomial of its tensor is simple.

For example, the quasilinear system associated with a Hessian tensor

$$\mathbf{A} = -\nabla^2 H,$$

where ∇ is the Levi-Civita connection of a semiRiemannian metric on $\mathcal{U}$ and $H(u)$ is a function on $\mathcal{U}$, is

$$\mathbf{u}_t = \nabla^2 H\, \mathbf{u}_x = \nabla_{\mathbf{u}_x} \nabla H.$$

This equation states that the vector which describes the instantaneous evolution of a phase $\mathbf{u}(x)$ is the covariant derivative of ∇H along $\mathbf{u}(x)$. A Hessian tensor is self-adjoint with respect to the relevant metric. The Hessian systems generated by a definite metric are therefore semisimple.

The evolutionary vector field which represents the quasilinear system (3.2) is

$$\mathbf{V} = Q^\alpha\, \partial_{u^\alpha}, \qquad Q^\alpha[u] = -A^\alpha_\beta(u) u^\beta_x.$$

A functional $\int T\, dx$ is conserved along the flow of $\mathbf{V}$ if and only if

$$\frac{\delta}{\delta u}(\mathbf{V}\, T) = 0, \tag{3.3}$$

by (2.3), because the space of total derivatives is locally equal to the common kernel of the variational derivatives. If $T = T(u)$ then the condition (3.3) is

$$\frac{\partial}{\partial u^\gamma}(T_{,\alpha} A^\alpha_\beta) = \frac{\partial}{\partial u^\beta}(T_{,\alpha} A^\alpha_\gamma),$$

which holds if and only if

$$dT\mathbf{A} = dX$$

for some function $X(u)$, which is then the flux associated with the conserved density T. A quasilinear system is a system of conservation laws if it has functionally independent conserved densities $T^1(u), \dots, T^n(u)$. In the coordinates $u^1 = T^1, \dots, u^n = T^n$, the system then has the form

$$u^\alpha_t + X^\alpha_x = 0,$$

where $X^1(u), \dots, X^n(u)$ are the associated fluxes.

The Hamiltonian system (2.11) generated by the Poisson operator (2.5) and by a functional $\int H(u)\, dx$ has the form (3.1), where

$$A^\alpha_\beta = -g^{\alpha\gamma} H_{,\gamma\beta} - b^{\alpha\gamma}_\beta H_{,\gamma} = -g^{\alpha\gamma}(H_{,\gamma\beta} - \Gamma^\eta_{\gamma\beta} H_{,\eta}),$$

i.e.,

$$\mathbf{A} = -\nabla^2 H,$$

where ∇ is the connection (2.7) of the flat metric (2.9). Such a system is therefore Hessian. For the duration, a quasilinear system is considered to be Hamiltonian if and only if it is Hessian with respect to a flat metric on $\mathcal{U}$. Fix Darboux coordinates in which the bivector is described by the operator $\mathcal{D} = \varepsilon \mathrm{D}_x$, where ε is a diagonal matrix with diagonal entries $\varepsilon_\alpha = \varepsilon^\alpha = \pm 1$. A Hamiltonian system $\mathbf{u}_t = \nabla^2 H \mathbf{u}_x$ is described in these coordinates by

$$u^\alpha_t = (\varepsilon_\alpha H_{,\alpha})_x, \tag{3.4}$$

hence is a system of conservation laws. The conservation laws (3.4) imply the additional conservation laws

$$\left(\frac{1}{2}\varepsilon_\alpha (u^\alpha)^2\right)_t = (u^\alpha H_{,\alpha} - H)_x, \tag{3.5}$$

and

$$H_t = \left(\frac{1}{2}\varepsilon^\alpha (H_{,\alpha})^2\right)_x. \tag{3.6}$$

The Hamiltonian system generated by the density $T(u) = \varepsilon_\alpha (u^\alpha)^2/2$ of the conservation law (3.5) is the translation generator $\mathbf{u}_t = \mathbf{u}_x$, which is a symmetry of $\mathbf{u}_t = \nabla^2 H \mathbf{u}_x$, because the latter is independent of x. The conservation law (3.6) asserts that the integral of the density H is conserved along its own Hamiltonian flow.

The symmetries of $\mathbf{V}$ are the evolutionary vector fields

$$\mathbf{W} = R^\alpha \, \partial_{u^\alpha}$$

such that

$$\mathbf{V}\, R^\alpha = \mathbf{W}\, Q^\alpha. \tag{3.7}$$

For example, if $R^\alpha = R^\alpha(u)$ then $\mathbf{W}$ is simply a vector field on $\mathcal{U}$, and is a symmetry of $\mathbf{V}$ if and only if

$$L_{\mathbf{W}} \mathbf{A} = 0,$$

where $L_{\mathbf{W}}$ is the Lie derivative. Hence there are coordinates in which the matrix representation of $\mathbf{A}$ is constant if and only if there is a basis of commuting vector fields $\mathbf{W}_1, \ldots, \mathbf{W}_n$ such that

$$L_{\mathbf{W}_1} \mathbf{A} = \cdots = L_{\mathbf{W}_n} \mathbf{A} = 0,$$

i.e., (3.2) has a linear coordinate representation if and only if it has an n-dimensional abelian algebra of zeroth order symmetries. If

$$\mathbf{W} = R^\alpha \partial_{u^\alpha}, \quad R^\alpha[u] = -B^\alpha_\beta(u) u^\beta_x$$

is the evolutionary vector field of a second quasilinear system defined by a (1,1)-tensor $\mathbf{B}$ then the symmetry conditions (3.7) are

$$A^\alpha_\gamma B^\gamma_\beta = B^\alpha_\gamma A^\gamma_\beta \tag{3.8}$$

and

$$\begin{aligned}\left(\frac{\partial A^\alpha_\beta}{\partial u^\eta} - \frac{\partial A^\alpha_\eta}{\partial u^\beta}\right) B^\eta_\gamma + \left(\frac{\partial A^\alpha_\gamma}{\partial u^\eta} - \frac{\partial A^\alpha_\eta}{\partial u^\gamma}\right) B^\eta_\beta \quad = \quad & \left(\frac{\partial B^\alpha_\beta}{\partial u^\eta} - \frac{\partial B^\alpha_\eta}{\partial u^\beta}\right) A^\eta_\gamma + \\ & \left(\frac{\partial B^\alpha_\gamma}{\partial u^\eta} - \frac{\partial B^\alpha_\eta}{\partial u^\gamma}\right) A^\eta_\beta.\end{aligned} \tag{3.9}$$

Note that (3.8) holds if and only if the tensors $\mathbf{A}$ and $\mathbf{B}$ commute.

Theorem 3.1 was proved by Verosky [3].

Theorem 3.1 *Two systems of conservation laws*

$$u^\alpha_t + X^\alpha_x = 0, \quad u^\alpha_t + Y^\alpha_x = 0$$

are symmetries if and only if their tensors commute.

P r o o f The tensors of these systems are described in the given coordinates by the matrices

$$A^\alpha_\beta = \frac{\partial X^\alpha}{\partial u^\beta}, \qquad B^\alpha_\beta = \frac{\partial Y^\alpha}{\partial u^\beta},$$

hence (3.9) holds. □

Corollary Two Hamiltonian quasilinear systems generated by the same nondegenerate 1-weight Poisson bivector are symmetries if and only if their tensors commute.

P r o o f These are systems of conservation laws in Darboux coordinates for the bivector. □

A quasilinear symmetry of a simple system is semisimple, and the tensors of the symmetries of a simple system commute with one another. Theorem 3.2 partitions the family of simple systems into symmetry classes, Theorem 3.3 proves that the systems of a symmetry class have the same zeroth order conserved densities, and Theorem 3.4 proves that the systems of a symmetry class have the same Hamiltonian structure.

Theorem 3.2 *Symmetry is an equivalence relation on the family of simple systems.*

P r o o f Fix a simple system $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$. Locally, there are distinct smooth complex functions $\lambda_1, \ldots, \lambda_n$ and nonzero complex vector fields $\mathbf{X}_1, \ldots, \mathbf{X}_n$ such that

$$\mathbf{A}\mathbf{X}_\alpha = \lambda_\alpha \mathbf{X}_\alpha.$$

In coordinates,

$$\mathbf{X}_\alpha = U^\beta_\alpha \partial_{u^\beta}, \tag{3.10}$$

where $U(u)$ is an invertible complex matrix. If A is the matrix representation of $\mathbf{A}$ in these coordinates then

$$A = U\lambda U^{-1}, \tag{3.11}$$

where λ is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. The structure functions of the complex frame $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are defined by

$$[\mathbf{X}_\alpha, \mathbf{X}_\beta] = c^\gamma_{\alpha\beta} \mathbf{X}_\gamma,$$

or

$$c^\gamma_{\alpha\beta} = (\mathbf{X}_\alpha U^\eta_\beta - \mathbf{X}_\beta U^\eta_\alpha) {U^{-1}}^\gamma_\eta. \tag{3.12}$$

If the simple system $\mathbf{u}_t + \mathbf{B}\mathbf{u}_x = 0$ is a symmetry of $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$ then there are distinct functions $\mu_1, \ldots, \mu_n$ such that

$$\mathbf{B}\mathbf{X}_\alpha = \mu_\alpha \mathbf{X}_\alpha,$$

or

$$B = U\mu U^{-1}, \tag{3.13}$$

where $B(u)$ is the matrix representation of $\mathbf{B}$ and μ is the diagonal matrix with diagonal entries $\mu_1, \ldots, \mu_n$. The quantity

$$\left\{ \left(\frac{\partial A^\zeta_\xi}{\partial u^\eta} - \frac{\partial A^\zeta_\eta}{\partial u^\xi} \right) B^\eta_\upsilon + \left(\frac{\partial A^\zeta_\upsilon}{\partial u^\eta} - \frac{\partial A^\zeta_\eta}{\partial u^\upsilon} \right) B^\eta_\xi \right\} U^\xi_\alpha U^\upsilon_\beta U^{-1}{}^\gamma_\zeta$$

is equal to

$$(\mu_\alpha - \mu_\beta)\Big\{ \mathbf{X}_\alpha \lambda^\gamma_\beta - \mathbf{X}_\beta \lambda^\gamma_\alpha + \lambda^\gamma_\eta c^\eta_{\beta\alpha} + (\lambda_\beta \mathbf{X}_\alpha U^\eta_\beta - \lambda_\alpha \mathbf{X}_\beta U^\eta_\alpha) U^{-1}{}^\gamma_\eta \Big\},$$

using (3.10)–(3.13). The conditions (3.9) are therefore equivalent to

$$\begin{aligned} (\mu_\alpha - \mu_\beta)\Big\{ \mathbf{X}_\alpha \lambda^\gamma_\beta - \mathbf{X}_\beta \lambda^\gamma_\alpha \Big\} \quad &- \quad (\lambda_\alpha - \lambda_\beta)\Big\{ \mathbf{X}_\alpha \mu^\gamma_\beta - \mathbf{X}_\beta \mu^\gamma_\alpha \Big\} = \\ \Big\{ (\mu_\alpha - \mu_\beta)(\lambda_\gamma - \lambda_\beta) \quad &- \quad (\lambda_\alpha - \lambda_\beta)(\mu_\gamma - \mu_\beta) \Big\} c^\gamma_{\alpha\beta}, \end{aligned}$$

or

$$\frac{\mathbf{X}_\beta \lambda_\alpha}{\lambda_\alpha - \lambda_\beta} = \frac{\mathbf{X}_\beta \mu_\alpha}{\mu_\alpha - \mu_\beta}, \qquad \alpha \neq \beta \tag{3.14}$$

and

$$\frac{\lambda_\gamma - \lambda_\beta}{\lambda_\alpha - \lambda_\beta}\, c^\gamma_{\alpha\beta} = \frac{\mu_\gamma - \mu_\beta}{\mu_\alpha - \mu_\beta}\, c^\gamma_{\alpha\beta}, \qquad \alpha, \beta, \gamma \text{ distinct.} \tag{3.15}$$

Therefore the conditions (3.8), (3.9) define a transitive relation on the family of simple systems. □

The equivalence classes of simple systems are *symmetry classes.* Note that the distributions $\langle \mathbf{X}_\alpha, \mathbf{X}_\beta \rangle$ are involutive if and only if $c^\gamma_{\alpha\beta} = 0$ for $\gamma \neq \alpha, \beta$, in which case (3.15) is vacuous, e.g. if $n = 2$. Theorem 3.3 was proved by Serre [9].

Theorem 3.3 *The systems in a symmetry class have the same zeroth order conserved densities.*

P r o o f A regular zeroth order conserved density of $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$ can be used as a coordinate. The coordinate function u^1 is a conserved density if and only if

$$\frac{\partial A^1_\alpha}{\partial u^\beta} = \frac{\partial A^1_\beta}{\partial u^\alpha}.$$

If $\mathbf{u}_t + \mathbf{B}\mathbf{u}_x = 0$ is a symmetry then (3.9) implies

$$CA = A^t C, \tag{3.16}$$

where C is the skew-symmetric matrix with entries

$$C_{\alpha\beta} = \frac{\partial B^1_\alpha}{\partial u^\beta} - \frac{\partial B^1_\beta}{\partial u^\alpha}.$$

If $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$ is simple then there is an invertible complex matrix U such that

$$U^{-1} A U = \lambda,$$

where λ is a diagonal matrix with distinct diagonal entries. Condition (3.16) is equivalent to

$$U^tCAU = U^tA^tCU,$$

or

$$U^tCUU^{-1}AU = U^tA^tU^{-t}U^tCU,$$

or

$$U^tCU\lambda = \lambda U^tCU,$$

from which it follows that U^tCU is diagonal. Combining this with

$$(U^tCU)^t = U^tC^tU = -U^tCU$$

proves $C = 0$, i.e.,

$$\frac{\partial B^1_\alpha}{\partial u^\beta} = \frac{\partial B^1_\beta}{\partial u^\alpha}. \quad \square$$

Note that the argument does not require the symmetry $\mathbf{u}_t + \mathbf{B}\mathbf{u}_x = 0$ to be simple. See Verosky [3] for generalizations of Theorems 3.2 and 3.3 in the case $n = 2$.

Theorem 3.4 *The systems in a symmetry class have the same Hamiltonian structure. A symmetry class is Hamiltonian with respect to a 1-weight Poisson bivector* $\mathbf{\Theta}$ *with metric* $\mathbf{g}$ *if and only if the Casimirs of* $\mathbf{\Theta}$ *are conserved densities for the class, and the characteristic spaces of the class are orthogonal with respect to* $\mathbf{g}$.

P r o o f Assume that the Casimirs of $\mathbf{\Theta}$ are conserved densities for a semisimple system $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$, and that the characteristic spaces of $\mathbf{A}$ are orthogonal. If the system is simple then a symmetry $\mathbf{u}_t + \mathbf{B}\mathbf{u}_x = 0$ is semisimple, the Casimirs of $\mathbf{\Theta}$ are conserved densities for $\mathbf{u}_t + \mathbf{B}\mathbf{u}_x = 0$, and the characteristic spaces of $\mathbf{B}$ are orthogonal. The bivector is described in appropriate Darboux coordinates by the operator $\mathcal{D} = \varepsilon \mathrm{D}_x$, where ε is a diagonal matrix with diagonal entries $\varepsilon_\alpha = \varepsilon^\alpha = \pm 1$. The matrix representation of $\mathbf{A}$ in these coordinates is

$$A^\alpha_\beta = X^\alpha_{,\beta}$$

for some functions $X^1(u), \ldots, X^n(u)$, because Darboux coordinates are Casimirs. The assumptions of semisimplicity and orthogonality imply that there is a complex matrix U such that

$$U^t\varepsilon U = I$$

and

$$A = U\lambda U^{-1},$$

where λ is diagonal. Hence $\varepsilon A = \varepsilon U\lambda U^t\varepsilon$ is symmetric, i.e.,

$$(\varepsilon^\alpha X^\alpha)_{,\beta} = (\varepsilon^\beta X^\beta)_{,\alpha},$$

so that

$$\varepsilon^\alpha X^\alpha = -H_{,\alpha}$$

for some function $H(u)$. Therefore

$$A^\alpha_\beta = -\varepsilon_\alpha H_{,\alpha\beta},$$

proving that $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$ is Hamiltonian with respect to the bivector $\mathbf{\Theta}$. If the system is simple then the same argument proves that a symmetry $\mathbf{u}_t + \mathbf{B}\mathbf{u}_x = 0$ is also Hamiltonian. Conversely, the Casimirs of $\mathbf{\Theta}$ are conserved densities for a Hamiltonian system $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$, and $\mathbf{A}$ is Hessian, hence is self-adjoint with respect to $\mathbf{g}$, so the characteristic spaces of $\mathbf{A}$ are orthogonal. □

IV Complete Symmetry Classes

A simple system is *hyperbolic* if its tensor has real characteristic values, which are then smooth functions on $\mathcal{U}$, called the *characteristic speeds* of the system. The characteristic spaces of the tensor form a basis of 1-dimensional *characteristic distributions* on $\mathcal{U}$. Conversely, fix distinct real functions $\lambda_1, \ldots, \lambda_n$, and independent distributions $\Delta_1, \ldots, \Delta_n$. If $\mathbf{X}_\alpha$ is a basis for Δ_α and $\mathbf{\Xi}^1, \ldots, \mathbf{\Xi}^n$ are the dual 1-forms then

$$\mathbf{A} = \sum_\alpha \lambda_\alpha \mathbf{X}_\alpha \otimes \mathbf{\Xi}^\alpha$$

is a (1,1)-tensor. Note that $\mathbf{A}$ thus defined is independent of the choice of basis, and that the system $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$ is hyperbolic, with characteristic speeds $\lambda_1, \ldots, \lambda_n$ and characteristic distributions $\Delta_1, \ldots, \Delta_n$. The *symmetry forms*

$$\mathbf{\Omega}^\alpha = \sum_{\beta \neq \alpha} \frac{\mathbf{X}_\beta \lambda_\alpha}{\lambda_\alpha - \lambda_\beta} \mathbf{\Xi}^\beta \tag{4.1}$$

are independent of the choice of basis, hence are intrinsically associated with the system. Any simple symmetry of a hyperbolic system is also hyperbolic, and the two systems have the same characteristic distributions. Moreover, hyperbolic systems which are symmetries of one another have the same symmetry forms, by (3.14), so that a single set of symmetry forms (4.1) is assigned to each hyperbolic class. A hyperbolic symmetry class is *nondegenerate* if its symmetry forms are independent, in which case they constitute the *symmetry coframe* of the class. The characteristic distributions of a nondegenerate symmetry class are determined by its symmetry coframe.

The simplest hyperbolic systems are those which can be diagonalized. A *Riemann invariant* for the system $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$ is a real function $r(u)$ such that

$$dr\mathbf{A} = \lambda dr$$

for some function $\lambda(u)$. If $\mathbf{u}(x,t)$ is a solution, and $(x(t), t)$ is an integral curve of the vector field

$$\partial_t - \lambda(\mathbf{u}(x,t))\partial_x$$

in the x, t plane then

$$\frac{d}{dt} r\left(\mathbf{u}(x(t), t)\right) = 0.$$

In this sense, a Riemann invariant is a constant of the motion. A quasilinear system is *diagonalizable* if the matrix of its tensor is diagonal in some coordinates, which are then

Riemann invariants of the system. Conversely, a complete set of functionally independent Riemann invariants can be used as coordinates, in which the system is diagonal. Coordinates which diagonalize a system are called *Riemann coordinates.* Throughout the discussion, the symbols $r^1, \dots, r^n$ will be used exclusively to denote Riemann coordinates. If $\Delta_1, \dots, \Delta_n$ are the characteristic distributions of a hyperbolic system then the 1-codimensional distribution

$$\Delta_1 \oplus \cdots \oplus \widehat{\Delta}_\alpha \oplus \cdots \oplus \Delta_n$$

is involutive if and only if it is locally the kernel of the differential of a Riemann invariant. This implies that the system is diagonalizable if and only if the distributions $\Delta_\alpha + \Delta_\beta$ are involutive for all α, β. This condition is trivially satisfied if $n = 2$, so 2-component hyperbolic systems are always diagonalizable. A symmetry of a simple diagonalizable system is also diagonalizable, and the two systems have the same Riemann coordinates.

Theorem 4.1 *Simple diagonalizable systems are symmetries if and only if they have the same characteristic distributions and the same symmetry forms.*

P r o o f Compare (4.1) with (3.14), and recall that the conditions (3.15) are vacuous if the distributions $\langle \mathbf{X}_\alpha, \mathbf{X}_\beta \rangle$ are involutive. □

The symmetry forms of a diagonalizable symmetry class have Riemann coordinate description

$$\boldsymbol{\Omega}^\alpha = \sum_{\beta \neq \alpha} a_{\alpha\beta}(r)\, dr^\beta. \tag{4.2}$$

If the class is nondegenerate then the structure equations of the coframe $\boldsymbol{\Omega}$ are

$$d\,\boldsymbol{\Omega}^\alpha = -\frac{1}{2} c^\alpha_{\beta\gamma}\, \boldsymbol{\Omega}^\beta \wedge \boldsymbol{\Omega}^\gamma,$$

where the functions $c^\alpha_{\beta\gamma}$ can be described in terms of the functions $a_{\alpha\beta}$. If $\mathbf{Z}_1, \dots, \mathbf{Z}_n$ are the vector fields dual to $\boldsymbol{\Omega}^1, \dots, \boldsymbol{\Omega}^n$ then

$$[\mathbf{Z}_\alpha, \mathbf{Z}_\beta] = c^\gamma_{\alpha\beta}\, \mathbf{Z}_\gamma.$$

The invariants

$$\mathbf{Z}_{\alpha_1} \circ \cdots \circ \mathbf{Z}_{\alpha_k}\, c^\alpha_{\beta\gamma}$$

give a generic characterization of the frame [14]. A nondegenerate diagonalizable symmetry class is completely characterized by its symmetry coframe, so it is reasonable to suppose that the invariants have some relation to the systems in the class. This idea is explored in Section VI for the case $n = 2$.

The class with symmetry forms (4.2) consists of the simple systems

$$r^1_t + \lambda_1 r^1_x = 0, \dots, r^n_t + \lambda_n r^n_x = 0$$

defined by the solutions $\lambda_1(r), \dots, \lambda_n(r)$ of the linear system of partial differential equations

$$\lambda_{\alpha,\beta} = a_{\alpha\beta}(r)(\lambda_\alpha - \lambda_\beta), \qquad \alpha \neq \beta. \tag{4.3}$$

The compatibility conditions of (4.3) are

$$a_{\alpha\beta,\gamma} = a_{\alpha\beta}a_{\alpha\gamma} - a_{\alpha\beta}a_{\beta\gamma} - a_{\alpha\gamma}a_{\gamma\beta}, \qquad \alpha, \beta, \gamma \text{ distinct.} \tag{4.4}$$

Serre [9] proves that the solution space of a compatible system (4.3) is parametrized by n arbitrary functions of a single variable, and that the family of compatible systems is parametrized by $n^2 - n$ arbitrary functions of two variables. If (4.4) holds then

$$a_{\alpha\beta,\gamma} = a_{\alpha\gamma,\beta}, \qquad \beta, \gamma \neq \alpha, \tag{4.5}$$

and if (4.5) holds, where

$$a_{\alpha\beta} = \frac{\lambda_{\alpha,\beta}}{\lambda_\alpha - \lambda_\beta},$$

then direct calculation proves (4.4).

Proposition Fix a diagonalizable symmetry class with characteristic distributions $\Delta_1, \ldots, \Delta_n$ and symmetry forms $\mathbf{\Omega}^1, \ldots, \mathbf{\Omega}^n$. Denote the dual distributions by $\Delta^1, \ldots, \Delta^n$. The defining equations for the class are compatible if and only if

$$d\mathbf{\Omega}^\alpha \equiv 0 \text{ mod } \Delta^\alpha. \tag{4.6}$$

Proof In Riemann coordinates the symmetry forms are (4.2), and the conditions (4.6) are equivalent to

$$dr^\alpha \wedge d\mathbf{\Omega}^\alpha = 0,$$

or

$$\sum_{\beta,\gamma} a_{\alpha\beta,\gamma}\, dr^\alpha \wedge dr^\beta \wedge dr^\gamma = 0,$$

or

$$a_{\alpha\beta,\gamma} = a_{\alpha\gamma,\beta}. \quad \square$$

A diagonalizable symmetry class is *complete* if its symmetry forms satisfy (4.6). All hyperbolic symmetry classes are complete in the case $n = 2$. If $\mathbf{A}$ is the tensor of a system in the class defined by (4.3) then the condition

$$d\,(dT\mathbf{A}) = 0$$

which defines the zeroth order conserved densities $T(r)$ is

$$T_{,\alpha\beta} + a_{\alpha\beta}T_{,\alpha} + a_{\beta\alpha}T_{,\beta} = 0. \tag{4.7}$$

Note that these equations depend only on the functions $a_{\alpha\beta}$, so the systems in the class have the same zeroth order conserved densities, as proved in Theorem 3.3. The conditions (4.4) are the compatibility conditions of (4.7). Hence the equations (4.7) are compatible if and only if the symmetry class is complete. Serre [9] proves that the solution space of a compatible system (4.7) is parametrized by n arbitrary functions of a single variable.

Example 4.1 Consider a diagonalizable class with symmetry forms

$$\mathbf{\Omega}^\alpha = \sum_{\beta\neq\alpha} \frac{\partial F}{\partial r^\beta}\, dr^\beta, \tag{4.8}$$

where $F(r)$ is an arbitrary function. In the case $n = 2$, a class is of this type if and only if

$$d(\mathbf{\Omega}^1 + \mathbf{\Omega}^2) = 0.$$

This case is examined in Section VI. Assume $n \geq 3$. The completeness condition (4.4) is then

$$\frac{\partial^2 e^F}{\partial r^\alpha \partial r^\beta} = 0, \qquad \alpha \neq \beta,$$

which holds if and only if

$$F = \ln|f^1(r^1) + \cdots + f^n(r^n)|$$

for some functions $f^1, \ldots, f^n$. Nondegeneracy implies $f^{1\,\prime}, \ldots, f^{n\,\prime} \neq 0$, so that

$$F = \ln|\sigma|, \qquad \sigma = r^1 + \cdots + r^n, \tag{4.9}$$

in suitable Riemann coordinates. Conversely, the 1-forms (4.8) defined by the function (4.9) constitute the symmetry coframe of a nondegenerate complete symmetry class, because the coefficients

$$a_{\alpha\beta} = \frac{1 - \delta_{\alpha\beta}}{\sigma}$$

satisfy (4.4), and

$$\det a = (-1)^{n-1}\frac{n-1}{\sigma^n} \neq 0.$$

The symmetry forms are

$$\mathbf{\Omega}^\alpha = \frac{1}{\sigma}\sum_{\beta\neq\alpha} dr^\beta,$$

and the structure equations are

$$d\mathbf{\Omega}^\alpha = \frac{1}{n-1}\sum_\beta \mathbf{\Omega}^\alpha \wedge \mathbf{\Omega}^\beta.$$

The structure functions are the constants

$$c^\alpha_{\beta\gamma} = \frac{1}{n-1}(\delta^\alpha_\gamma - \delta^\alpha_\beta),$$

hence $\mathbf{\Omega}$ is a Maurer-Cartan coframe. □

If a diagonalizable symmetry class is Hamiltonian then Riemann coordinates for the class are orthogonal coordinates for the metric of the Poisson bivector, by Theorem 3.4. The Christoffel coefficients of a diagonal metric

$$\mathbf{g} = g_\alpha (dr^\alpha)^2 \tag{4.10}$$

are

$$\begin{aligned} \Gamma^\alpha_{\alpha\beta} &= [g_\alpha]_{,\beta}, \\ \Gamma^\beta_{\alpha\alpha} &= -\frac{g_\alpha}{g_\beta}[g_\alpha]_{,\beta}, \qquad \alpha \neq \beta, \\ \Gamma^\gamma_{\alpha\beta} &= 0, \qquad \alpha, \beta, \gamma \text{ distinct}, \end{aligned} \tag{4.11}$$

where

$$[g_\alpha] = \ln |g_\alpha|^{\frac{1}{2}}.$$

The only curvature coefficients which are possibly nonzero are

$$\begin{aligned} R^\alpha_{\beta\alpha\beta} = g_\beta\Big\{ &\frac{1}{g_\beta}[g_\alpha]_{,\beta\beta} + \frac{1}{g_\alpha}[g_\beta]_{,\alpha\alpha} + \frac{1}{g_\beta}[g_\alpha]_{,\beta}[g_\alpha]_{,\beta} + \frac{1}{g_\alpha}[g_\beta]_{,\alpha}[g_\beta]_{,\alpha} - \\ &\frac{1}{g_\beta}[g_\alpha]_{,\beta}[g_\beta]_{,\beta} - \frac{1}{g_\alpha}[g_\alpha]_{,\alpha}[g_\beta]_{,\alpha} + \sum_{\gamma \neq \alpha,\beta} \frac{1}{g_\gamma}[g_\alpha]_{,\gamma}[g_\beta]_{,\gamma}\Big\}, \quad \alpha \neq \beta, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} R^\alpha_{\beta\alpha\gamma} = \frac{g_\beta}{g_\alpha} R^\beta_{\alpha\gamma\alpha} = [g_\alpha]_{,\beta\gamma} + [g_\alpha]_{,\beta}[g_\alpha]_{,\gamma} - [g_\alpha]_{,\beta}[g_\beta]_{,\gamma} - [g_\gamma]_{,\beta}[g_\alpha]_{,\gamma}, \\ \alpha, \beta, \gamma \text{ distinct.} \end{aligned} \tag{4.13}$$

Theorem 4.2 was proved by Tsarev [4].

Theorem 4.2
i) *A diagonalizable Hamiltonian symmetry class is complete.*
ii) *Fix a system of orthogonal coordinates for a flat metric. There is a unique symmetry class which is diagonal in these coordinates and is Hamiltonian with respect to the Poisson bivector of the metric.*

P r o o f A (1,1)-tensor $\mathbf{A}$ is Hessian with respect to a given flat metric if and only if the associated (0,2)-tensor $\mathbf{A}_\#$ is symmetric and the (1,2)-tensor $\nabla\mathbf{A}$ is symmetric in its covariant arguments. If $\mathbf{A}$ is the tensor of the system

$$r^1_t + \lambda_1 r^1_x = 0, \ldots, r^n_t + \lambda_n r^n_x = 0$$

and $\mathbf{g}$ is the metric (4.10) then $\mathbf{A}_\#$ is symmetric. Moreover, the formulas (4.11) imply

$$\begin{aligned} \langle \nabla\mathbf{A}; dr^\gamma, \partial_{r^\alpha}, \partial_{r^\beta}\rangle - \langle \nabla\mathbf{A}; dr^\gamma, \partial_{r^\beta}, \partial_{r^\alpha}\rangle = \delta^\gamma_\alpha(\lambda_{\alpha,\beta} + (\lambda_\alpha - \lambda_\beta)[g_\alpha]_{,\beta}) - \\ \delta^\gamma_\beta(\lambda_{\beta,\alpha} + (\lambda_\beta - \lambda_\alpha)[g_\beta]_{,\alpha}), \end{aligned}$$

so $\nabla\mathbf{A}$ is symmetric if and only if

$$a_{\alpha\beta} = -[g_\alpha]_{,\beta}, \qquad \alpha \neq \beta, \tag{4.14}$$

where

$$a_{\alpha\beta} = \frac{\lambda_{\alpha,\beta}}{\lambda_\alpha - \lambda_\beta}.$$

If the symmetry class of the system is Hamiltonian then (4.14) implies (4.5), so the class is complete. If a flat diagonal metric is given, then the curvature conditions (4.13) imply that the coefficients $a_{\alpha\beta}$ given by (4.14) define a complete symmetry class. □

This describes a diagonalizable Hamiltonian symmetry class in terms of its Riemann invariants and the metric of the Poisson bivector. It does not determine which complete symmetry classes are Hamiltonian. The conditions (4.5) hold if and only if there are functions $G_1(r), \dots, G_n(r)$ such that

$$a_{\alpha\beta} = -\frac{G_{\alpha,\beta}}{G_\alpha}.$$

Note that G_α is defined up to multiplication by a nonzero function of r^α. The diagonal metric (4.10) satisfies (4.14) if and only if

$$g_\alpha = \frac{G_\alpha^{\,2}}{h_\alpha(r^\alpha)}$$

for some functions $h_1, \dots, h_n$, in which case the conditions (4.4) imply that the curvature coefficients (4.13) are zero. The curvature coefficients (4.12) are zero if and only if

$$h_\alpha{}' \frac{G_{\beta,\alpha}}{G_\alpha} + h_\beta{}' \frac{G_{\alpha,\beta}}{G_\beta} + 2h_\alpha \left(\frac{G_{\beta,\alpha}}{G_\alpha}\right)_{,\alpha} + 2h_\beta \left(\frac{G_{\alpha,\beta}}{G_\beta}\right)_{,\beta} +$$
$$2 \sum_{\gamma \neq \alpha, \beta} h_\gamma \frac{G_{\alpha,\gamma}}{G_\gamma} \frac{G_{\beta,\gamma}}{G_\gamma} = 0, \qquad \alpha \neq \beta. \tag{4.15}$$

Note that these equations are linear in h. This implies the following result, which has also recently been reported by Tsarev [5]. See Olver [12] for a discussion of compatibility of Poisson bivectors, and of bi-Hamiltonian systems.

Theorem 4.3 *The bivectors which generate the Hamiltonian structures of a complete symmetry class are compatible.*

P r o o f If h, k are nondegenerate solutions of (4.15) then $h + sk$ is a nondegenerate solution for small s. □

Example 4.2 The system

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t + \begin{pmatrix} v & u & 0 \\ p_u/u & v & p_w/u \\ 0 & 0 & v \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = 0 \tag{4.16}$$

describes the dynamics of a 1-dimensional gas. The field variables u, v, w denote the density, velocity, and entropy of the gas, and the pressure is $p(u, w)$. The propagation speed is $n(u, w) = \sqrt{p_u}$. A Hamiltonian structure was discovered by Verosky [2]. The system is hyperbolic, with characteristic speeds

$$\lambda_\pm = v \pm n, \quad \lambda = v,$$

and corresponding characteristic vector fields

$$\mathbf{X}_\pm = nu\partial_u \pm p_u\partial_v, \qquad \mathbf{X} = p_w\partial_u - p_u\partial_w.$$

The fact that any function $s(w)$ is a Riemann invariant corresponds to involutivity of the distribution $\langle \mathbf{X}_+, \mathbf{X}_- \rangle$. The existence of Riemann coordinates is equivalent to involutivity of the distributions $\langle \mathbf{X}_\pm, \mathbf{X} \rangle$, which holds if and only if

$$nup_{uw} = 2(nu)_u p_w. \tag{4.17}$$

The condition (4.17) is satisfied for any pressure $p(u)$, in which case (4.16) is an independent 2-component hyperbolic system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} v & u \\ p_u/u & v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0, \tag{4.18}$$

augmented by the equation

$$w_t + v w_x = 0.$$

The system (4.18) is discussed in Section VI. If $p_w \neq 0$ then the general solution of (4.17) is

$$p(u, w) = F\left(s(w) - \frac{1}{u}\right),$$

where $s' \neq 0$, $F' \neq 0$. The propagation speed is entropy independent if and only if $F'' = 0$, i.e.,

$$p = s(w) - \frac{1}{u},$$

up to constant multiple, so that $n = 1/u$. In this case, (4.16) is

$$\begin{pmatrix} u \\ v \\ s \end{pmatrix}_t + \begin{pmatrix} v & u & 0 \\ 1/u^3 & v & 1/u \\ 0 & 0 & v \end{pmatrix} \begin{pmatrix} u \\ v \\ s \end{pmatrix}_x = 0, \tag{4.19}$$

or

$$q_t + (s - r)q_x = 0, \qquad r_t + (q - s)r_x = 0, \qquad s_t + \frac{1}{2}(q - r)s_x = 0, \tag{4.20}$$

in Riemann coordinates

$$q = v + s - \frac{1}{u}, \qquad r = -v + s - \frac{1}{u}, \qquad s = s(w).$$

The symmetry forms of the class of (4.20) are

$$\mathbf{\Omega}^1 = -\frac{1}{2}u\,dr + u\,ds, \qquad \mathbf{\Omega}^2 = -\frac{1}{2}u\,dq + u\,ds, \qquad \mathbf{\Omega}^3 = -\frac{1}{2}u\,dq - \frac{1}{2}u\,dr.$$

The class is nondegenerate and complete, and $G_1 = G_2 = G_3 = u$. Note that this is the symmetry class of Example 4.1, in the case $n = 3$. It consists of the simple systems

$$q_t + \lambda_1 q_x = 0, \qquad r_t + \lambda_2 r_x = 0, \qquad s_t + \lambda_3 s_x = 0,$$

where

$$\begin{aligned} \lambda_1 &= \xi(q) + \eta(r) + \zeta(s) + \tfrac{2}{u}\xi'(q), \\ \lambda_2 &= \xi(q) + \eta(r) + \zeta(s) + \tfrac{2}{u}\eta'(r), \\ \lambda_3 &= \xi(q) + \eta(r) + \zeta(s) - \tfrac{1}{u}\zeta'(s). \end{aligned}$$

The zeroth order conserved densities are

$$T = u(f_1(q) + f_2(r) + f_3(s)). \tag{4.21}$$

The solution to (4.15) is

$$h_1(q) = a + 2cq, \qquad h_2(r) = b + 2cr, \qquad h_3(s) = -\frac{1}{4}(a+b) - cs,$$

where a, b, c are arbitrary constants. Hence the Hamiltonian structure of the symmetry class is generated by the 3-dimensional space of Poisson bivectors with flat metrics

$$u^2\left(\frac{dq^2}{a+2cq} + \frac{dr^2}{b+2cr} - \frac{ds^2}{\frac{1}{4}(a+b)+cs}\right). \tag{4.22}$$

These bivectors are compatible, by Theorem 4.3. In the field variables u, v, s, the 3-dimensional space of Poisson operators has basis

$$\mathcal{D}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -s_x/u \\ 0 & s_x/u & 0 \end{pmatrix},$$

$$\mathcal{D}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1/u^2 & 0 \\ 1 & 0 & -1/u^2 \end{pmatrix} \mathrm{D}_x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -u_x/u^3 & v_x/u \\ 0 & -v_x/u & u_x/u^3 \end{pmatrix},$$

$$\mathcal{D}_3 = 2\begin{pmatrix} -u & v & s \\ v & (s-1/u)/u^2 & 0 \\ s & 0 & -s/u^2 \end{pmatrix} \mathrm{D}_x +$$

$$\begin{pmatrix} -u_x & 3v_x & 3s_x \\ -v_x & (3/u-2s)u_x/u^3 + s_x/u^2 & 2(sv_x - vs_x)/u \\ -s_x & 2(vs_x - sv_x)/u & 2su_x/u^3 - s_x/u^2 \end{pmatrix}.$$

The corresponding densities which generate (4.19) are

$$H_1 = -\frac{1}{2}u(v^2+p^2), \qquad H_2 = -uvp, \qquad H_3 = -uv.$$

The exponential coordinates for a flat metric (4.22) are Casimirs for the corresponding bivector, hence are conserved densities for the symmetry class. A conserved density (4.21) is a Casimir for the bivector with metric (4.22) if and only if

$$\begin{aligned} T_{qq} &= \frac{g_{1q}}{2g_1}T_q - \frac{ug_1}{2g_2}T_r + \frac{ug_1}{g_3}T_s, \\ T_{rr} &= \frac{g_{2r}}{2g_2}T_r - \frac{ug_2}{2g_1}T_q + \frac{ug_2}{g_3}T_s, \\ T_{ss} &= \frac{g_{3s}}{2g_3}T_s - \frac{ug_3}{2g_1}T_q - \frac{ug_3}{2g_2}T_r, \end{aligned} \tag{4.23}$$

where

$$g_1 = \frac{u^2}{h_1}, \qquad g_2 = \frac{u^2}{h_2}, \qquad g_3 = \frac{u^2}{h_3}.$$

The equations (4.23) are equivalent to

$$2h_1 f_1{}'' + h_1{}' f_1{}' = 2h_2 f_2{}'' + h_2{}' f_2{}' = 2h_3 f_3{}'' + h_3{}' f_3{}' = \\ cT - u(h_1 f_1{}' + h_2 f_2{}' - 2h_3 f_3{}').$$

If $c = 0$ then the solution space is spanned by

$$\tilde{u} = u, \qquad \tilde{v} = u(bq - ar), \qquad \tilde{w} = u\left(bq^2 + ar^2 - \frac{4ab}{a+b}\, s^2\right),$$

and $T = 1$. If $c \neq 0$ then the solution space is spanned by

$$\tilde{u} = u\sqrt{|h_1|}, \qquad \tilde{v} = u\sqrt{|h_2|}, \qquad \tilde{w} = u\sqrt{|h_3|},$$

and $T = 1$. In any case, the functions $\tilde{u}, \tilde{v}, \tilde{w}$ are exponential coordinates for the metric. □

V Higher Order Conservation Laws

The systems in a symmetry class do not generally have the same higher order conserved densities. We now describe the first order conserved densities for systems in a complete class.

Theorem 5.1 *Fix a system*

$$r_t^1 + \lambda_1 r_x^1 = 0, \ldots, r_t^n + \lambda_n r_x^n = 0$$

in a complete symmetry class, i.e.,

$$\frac{\lambda_{\alpha,\beta}}{\lambda_\alpha - \lambda_\beta} = -\frac{G_{\alpha,\beta}}{G_\alpha} \tag{5.1}$$

for some functions $G_1(r), \ldots, G_n(r)$. Assume

$$\lambda_{\alpha,\alpha} \neq 0, \qquad \alpha = 1, \ldots, m, \tag{5.2}$$

and

$$\lambda_{\alpha,\alpha} = 0, \qquad \alpha = m+1, \ldots, n. \tag{5.3}$$

Then

$$\sum_\alpha G_\alpha F_\alpha(r^\alpha, r_x^\alpha / G_\alpha)$$

is a conserved density if and only if

$$F_\alpha(y, z) = \frac{f_\alpha(y)}{z}, \qquad \alpha = 1, \ldots, m,$$

where $\sum_{\alpha=1}^{m} \lambda_{\alpha,\alpha} G_\alpha^{\,2} f_\alpha(r^\alpha)$ *is constant, in which case the associated flux is*

$$\sum_\alpha \lambda_\alpha G_\alpha F_\alpha(r^\alpha, r_x^\alpha/G_\alpha) - 2x \sum_{\alpha=1}^{m} \lambda_{\alpha,\alpha} G_\alpha^{\,2} f_\alpha(r^\alpha).$$

Each conserved density depending only on r, r_x *is equivalent modulo the space of total derivatives to the sum of a density of this form and a zeroth order conserved density.*

P r o o f The evolutionary vector field which represents the system is

$$\mathbf{V} = -\sum_\alpha \lambda_\alpha r_x^\alpha \partial_{r^\alpha}.$$

If $T = T(r, r_x)$ and $\frac{\delta}{\delta r}\,(\mathbf{V}T) = 0$ then

$$0 = \frac{\partial}{\partial r_{xxx}^\beta} \frac{\delta}{\delta r^\alpha}(\mathbf{V}T) = (\lambda_\beta - \lambda_\alpha) \frac{\partial^2 T}{\partial r_x^\alpha \partial r_x^\beta},$$

so that

$$T = \sum_\alpha T_\alpha(r, r_x^\alpha).$$

Therefore

$$0 = \frac{\partial}{\partial r_{xx}^\beta} \frac{\delta}{\delta r^\alpha}(\mathbf{V}T) = (\lambda_\beta - \lambda_\alpha) \frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r^\beta} + \lambda_{\alpha,\beta} \frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha} r_x^\alpha +$$
$$(\lambda_\alpha - \lambda_\beta) \frac{\partial^2 T_\beta}{\partial r_x^\beta \partial r^\alpha} + \lambda_{\beta,\alpha} \frac{\partial^2 T_\beta}{\partial r_x^\beta \partial r_x^\beta} r_x^\beta, \qquad \alpha \neq \beta,$$

or

$$\frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r^\beta} + \frac{G_{\alpha,\beta}}{G_\alpha}\, \frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha} r_x^\alpha = \frac{\partial^2 T_\beta}{\partial r_x^\beta \partial r^\alpha} + \frac{G_{\beta,\alpha}}{G_\beta}\, \frac{\partial^2 T_\beta}{\partial r_x^\beta \partial r_x^\beta}\, r_x^\beta, \tag{5.4}$$

using (5.1). If $\alpha \neq \beta$ then the right side of (5.4) is independent of r_x^α, hence

$$\frac{\partial}{\partial r_x^\alpha} \left(G_\alpha \frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r^\beta} + G_{\alpha,\beta} \frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha} r_x^\alpha \right) = 0, \qquad \alpha \neq \beta,$$

or

$$\left(G_\alpha \frac{\partial}{\partial r^\beta} + G_{\alpha,\beta}\, r_x^\alpha \frac{\partial}{\partial r_x^\alpha} \right) \left(\frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha} r_x^\alpha \right) = 0, \quad \alpha \neq \beta.$$

This implies that

$$\frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha}\, r_x^\alpha$$

is a function of r^α and r_x^α/G_α, so

$$T_\alpha = G_\alpha F_\alpha(r^\alpha, r_x^\alpha/G_\alpha) + R_\alpha(r) r_x^\alpha + S_\alpha(r)$$

for some functions F_α, R_α, S_α. The conditions (5.4) imply

$$R_{\alpha,\beta} = R_{\beta,\alpha},$$

or

$$\sum_{\alpha} R_\alpha r_x^\alpha = \mathrm{D}_x R$$

for some function $R(r)$. Now,

$$0 = \frac{\partial}{\partial r_{xx}^\alpha}\frac{\delta}{\delta r^\alpha}(\mathbf{V}T) = \lambda_{\alpha,\alpha}\left(\frac{\partial^3 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha \partial r_x^\alpha} r_x^\alpha + 3\frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha}\right) r_x^\alpha +$$

$$\sum_{\beta\neq\alpha}(\lambda_\beta - \lambda_\alpha)\frac{\partial}{\partial r_x^\alpha}\left(\frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r^\beta} + \frac{G_{\alpha,\beta}}{G_\alpha}\frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha} r_x^\alpha\right) r_x^\beta =$$

$$\lambda_{\alpha,\alpha}\left(\frac{\partial^3 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha \partial r_x^\alpha} r_x^\alpha + 3\frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha}\right) r_x^\alpha,$$

hence

$$\frac{\partial^3 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha \partial r_x^\alpha} r_x^\alpha + 3\frac{\partial^2 T_\alpha}{\partial r_x^\alpha \partial r_x^\alpha} = 0, \qquad \alpha = 1, \ldots, m,$$

because of (5.2). This implies

$$z\frac{\partial^3 F_\alpha}{\partial z^3} + 3\frac{\partial^2 F_\alpha}{\partial z^2} = 0,$$

or

$$F_\alpha(y,z) = \frac{f_\alpha(y)}{z} + a(y) + b(y)z,$$

so that

$$G_\alpha F_\alpha(r^\alpha, r_x^\alpha/G_\alpha) = \frac{G_\alpha^{\,2} f_\alpha(r^\alpha)}{r_x^\alpha} + a(r^\alpha)G_\alpha + b(r^\alpha)r_x^\alpha, \quad \alpha = 1, \ldots, m.$$

The term aG_α can be absorbed into S_α, and the term br_x^α is a total derivative. This proves that the conserved density T is equivalent to a density of the form

$$\sum_\alpha G_\alpha F_\alpha\left(r^\alpha, r_x^\alpha/G_\alpha\right) + S(r),$$

where

$$F_\alpha(y,z) = \frac{f_\alpha(y)}{z}, \qquad \alpha = 1, \ldots, m.$$

Differentiation proves that

$$\sum_{\alpha=m+1}^{n} G_\alpha F_\alpha(r^\alpha, r_x^\alpha/G_\alpha)$$

is a conserved density with flux

$$\sum_{\alpha=m+1}^{n} \lambda_\alpha G_\alpha F_\alpha(r^\alpha, r_x^\alpha/G_\alpha),$$

using (5.1), (5.3), for arbitrary functions F_α. Finally,

$$\mathbf{V}\left\{\sum_{\alpha=1}^{m}\frac{G_\alpha^{\,2} f_\alpha(r^\alpha)}{r_x^\alpha} + S(r)\right\} =$$

$$-\left(\sum_{\alpha=1}^{m}\frac{\lambda_\alpha G_\alpha^{\,2} f_\alpha}{r_x^\alpha}\right)_x + 2\sum_{\alpha=1}^{m}\lambda_{\alpha,\alpha}G_\alpha^{\,2}f_\alpha - \sum_\alpha \lambda_\alpha S_{,\alpha}r_x^\alpha +$$

$$2\sum_{\alpha=1}^{m}\sum_{\beta\neq\alpha}\{(\lambda_\alpha-\lambda_\beta)G_{\alpha,\beta}+\lambda_{\alpha,\beta}G_\alpha\}G_\alpha f_\alpha\frac{r_x^\beta}{r_x^\alpha} =$$

$$-\left(\sum_{\alpha=1}^{m}\frac{\lambda_\alpha G_\alpha^{\,2} f_\alpha}{r_x^\alpha}\right)_x + 2\sum_{\alpha=1}^{m}\lambda_{\alpha,\alpha}G_\alpha^{\,2}f_\alpha - \sum_\alpha \lambda_\alpha S_{,\alpha}r_x^\alpha,$$

hence

$$\sum_{\alpha=1}^{m}\frac{G_\alpha^{\,2}f_\alpha}{r_x^\alpha} + S(r)$$

is a conserved density if and only if

$$\frac{\delta}{\delta r}\left(2\sum_{\alpha=1}^{m}\lambda_{\alpha,\alpha}G_\alpha^{\,2}f_\alpha - \sum_\alpha\lambda_\alpha S_{,\alpha}r_x^\alpha\right) = 0.$$

The latter holds if and only if $\sum_{\alpha=1}^{m}\lambda_{\alpha,\alpha}G_\alpha^{\,2}f_\alpha$ is constant and S is a zeroth order conserved density, in which case

$$\sum_{\alpha=1}^{m}\frac{G_\alpha^{\,2}f_\alpha}{r_x^\alpha}$$

is a conserved density with flux

$$\sum_{\alpha=1}^{m}\frac{\lambda_\alpha G_\alpha^{\,2}f_\alpha}{r_x^\alpha} - 2x\sum_{\alpha=1}^{m}\lambda_{\alpha,\alpha}G_\alpha^{\,2}f_\alpha. \qquad \square$$

A less general result has recently been reported by Tsarev [5].

A characteristic speed of a hyperbolic system is *degenerate* if it is constant along its characteristic foliation. In this case, there are solutions with values constrained to an arbitrary leaf of the foliation, with arbitrary initial conditions, and these simple solutions propagate along the x-axis with constant speed, for all time. A hyperbolic system is *genuinely nonlinear* if the derivative of each speed is nonzero along its characteristic foliation.

Example 5.1 Fix $F(r^1, r^2)$ such that $F_{,1} \neq 0$, $F_{,2} \neq 0$. The diagonal system

$$r_t^1 + e^{2F}r_x^1 = 0, \qquad r_t^2 - e^{2F}r_x^2 = 0$$

is genuinely nonlinear. It will be referred to in Section VI. The symmetry forms of the class of the system are

$$\mathbf{\Omega}^1 = F_{,2}\,dr^2, \qquad \mathbf{\Omega}^2 = F_{,1}\,dr^1,$$

and $G_1 = G_2 = \mathrm{e}^{-F}$. The first order conserved densities for the system are

$$T = e^{-2F}\left(\frac{f_1(r^1)}{r_x^1} - \frac{f_2(r^2)}{r_x^2}\right) + S(r),$$

where $f_1F_{,1}+f_2F_{,2}$ is constant and S is zeroth order conserved density. □

Corollary to Theorem 5.1 If the characteristic speeds of the system in Theorem 5.1 are degenerate then the first order conservation laws are

$$T=\sum_\alpha G_\alpha F_\alpha(r^\alpha, r^\alpha_x/G_\alpha), \qquad X=\sum_\alpha \lambda_\alpha G_\alpha F_\alpha(r^\alpha, r^\alpha_x/G_\alpha),$$

where the functions $F_1,\dots,F_n$ are arbitrary.

P r o o f It suffices to show that every zeroth order conservation law has the form

$$T=\sum_\alpha G_\alpha f_\alpha(r^\alpha), \qquad X=\sum_\alpha \lambda_\alpha G_\alpha f_\alpha(r^\alpha)$$

for some functions $f_1,\dots,f_n$. This follows from the fact that the solutions of (4.7) are uniquely determined by their values on the coordinate axes, as proved in Serre [9]. □

Example 5.2 The gas dynamics system (4.20) has degenerate characteristic speeds. The first order conservation laws are

$$\begin{aligned} T &= u\Big(F_1(q,q_x/u)+F_2(r,r_x/u)+F_3(s,s_x/u)\Big),\\ X &= u\Big((s-r)F_1(q,q_x/u)+(q-s)F_2(r,r_x/u)+\frac{1}{2}(q-r)F_3(s,s_x/u)\Big), \end{aligned}$$

by the Corollary. Compare with the result of Verosky [2]. Theorem 5.2 now proves that this system has conservation laws of arbitrarily high order. These conservation laws generate higher order symmetries of the system via the Poisson bivectors described in Example 4.2. □

Theorem 5.2 *If a characteristic speed of a system in a complete symmetry class is degenerate then the system has nontrivial conserved densities of arbitrarily high order.*

P r o o f If $\lambda_{\alpha,\alpha}=0$ then

$$\mathbf{V}\,G_\alpha+\mathrm{D}_x(\lambda_\alpha G_\alpha)=0. \tag{5.5}$$

Hence if

$$(\mathbf{V}+\lambda_\alpha \mathrm{D}_x)P[r]=0$$

then

$$\mathbf{V}\,(G_\alpha P)+\mathrm{D}_x(\lambda_\alpha G_\alpha P)=P\Big(\mathbf{V}\,G_\alpha+\mathrm{D}_x(\lambda_\alpha G_\alpha)\Big)+G_\alpha(\mathbf{V}+\lambda_\alpha\mathrm{D}_x)P=0,$$

i.e.,

$$T=G_\alpha P, \qquad X=\lambda_\alpha G_\alpha P \tag{5.6}$$

is a conservation law. For example, $(\mathbf{V}+\lambda_\alpha\mathrm{D}_x)f=0$ for any function $f(r^\alpha)$. Now, (5.5) implies

$$[\mathbf{V}+\lambda_\alpha\mathrm{D}_x, G_\alpha^{-1}\mathrm{D}_x]=0,$$

hence $G_\alpha^{-1}\mathrm{D}_x$ stabilizes $ker(\mathbf{V}+\lambda_\alpha \mathrm{D}_x)$. Therefore

$$\frac{1}{G_\alpha} f(r^\alpha)_x, \qquad \frac{1}{G_\alpha}\left(\frac{1}{G_\alpha} f(r^\alpha)_x\right)_x, \ldots$$

are differential functions of arbitrarily high order contained in $ker(\mathbf{V}+\lambda_\alpha \mathrm{D}_x)$. The conservation laws (5.6) associated with these functions are trivial, in that T is a total derivative in each case. But note that if $P_0, \ldots, P_k$ are functions in $ker(\mathbf{V}+\lambda_\alpha \mathrm{D}_x)$ then

$$(\mathbf{V}+\lambda_\alpha \mathrm{D}_x) F(P_0, \ldots, P_k) = 0$$

for any function F. $\square$

The technique of Theorem 5.2 was used by Serre [8] to prove the existence of conserved densities of arbitrarily high order for the gas dynamics system (4.16).

Example 5.3 If the characteristic speeds of a 2-component hyperbolic system are degenerate but regular then they may be used as Riemann coordinates, in which the system has the form

$$r^1_t + r^2 r^1_x = 0, \qquad r^2_t + r^1 r^2_x = 0,$$

which is equivalent to the Born-Infeld equation [10]. The symmetry class is defined by the functions

$$G_1 = \frac{1}{r^1 - r^2}, \qquad G_2 = \frac{1}{r^2 - r^1}.$$

The first order conservation laws are

$$T = \frac{F_1(r^1, (r^1 - r^2) r^1_x)}{r^1 - r^2} + \frac{F_2(r^2, (r^2 - r^1) r^2_x)}{r^2 - r^1}$$

$$X = \frac{r^2 F_1(r^1, (r^1 - r^2) r^1_x)}{r^1 - r^2} + \frac{r^1 F_2(r^2, (r^2 - r^1) r^2_x)}{r^2 - r^1},$$

by the Corollary to Theorem 5.1. The second order conservation laws provided by Theorem 5.2 are

$$T = \frac{F_1\Big(r^1, (r^1 - r^2) r^1_x, (r^1 - r^2)((r^1 - r^2) r^1_x)_x\Big)}{r^1 - r^2} + \frac{F_2\Big(r^2, (r^2 - r^1) r^2_x, (r^2 - r^1)((r^2 - r^1) r^2_x)_x\Big)}{r^2 - r^1},$$

$$X = \frac{r^2 F_1\Big(r^1, (r^1 - r^2) r^1_x, (r^1 - r^2)((r^1 - r^2) r^1_x)_x\Big)}{r^1 - r^2} + \frac{r^1 F_2\Big(r^2, (r^2 - r^1) r^2_x, (r^2 - r^1)((r^2 - r^1) r^2_x)_x\Big)}{r^2 - r^1}. \quad \square$$

VI 2-Component Systems

We now examine symmetry classes of 2-component hyperbolic systems. This is the case $n = 2$. All 2-component hyperbolic symmetry classes are diagonalizable and complete. A hyperbolic symmetry class is characterized by its characteristic distributions and its symmetry forms $\mathbf{\Psi} = \mathbf{\Omega}^1, \mathbf{\Pi} = \mathbf{\Omega}^2$. The sum $\mathbf{\Omega} = \mathbf{\Pi} + \mathbf{\Psi}$ of the symmetry forms is the *total form* of the class, and a hyperbolic class is *exact* if its total form is exact. If $\mathbf{X}$,$\mathbf{Y}$ are tangent to the characteristic distributions and $\mathbf{\Xi}, \mathbf{\Upsilon}$ are the dual 1-forms then the class consists of the systems $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$ with tensors

$$\mathbf{A} = \lambda\, \mathbf{X} \otimes \mathbf{\Xi} + \mu\, \mathbf{Y} \otimes \mathbf{\Upsilon}$$

such that

$$\frac{\mathbf{X}\mu}{\mu - \lambda}\, \mathbf{\Xi} = \mathbf{\Pi}, \qquad \frac{\mathbf{Y}\lambda}{\lambda - \mu}\, \mathbf{\Upsilon} = \mathbf{\Psi}.$$

Proposition 6.1 A hyperbolic symmetry class is exact if and only if it contains a system with characteristic speeds of equal magnitude and opposite sign. The system is unique, up to constant multiple, and is called the *canonical representative.*

P r o o f If $\mu = -\lambda$ then

$$\mathbf{\Omega} = \frac{\mathbf{X}\lambda}{2\lambda}\, \mathbf{\Xi} + \frac{\mathbf{Y}\lambda}{2\lambda}\, \mathbf{\Upsilon} = dF,$$

where $F = \frac{1}{2} \ln |\lambda|$. Conversely, if $\mathbf{\Omega} = dF$ for some function F then

$$\mathbf{A} = e^{2F}\, \mathbf{X} \otimes \mathbf{\Xi} - e^{2F}\, \mathbf{Y} \otimes \mathbf{\Upsilon}$$

is the tensor of a system in the class. □

The symbols r, s are used exclusively to denote Riemann coordinates, in which the symmetry forms are

$$\mathbf{\Pi} = a\, dr, \qquad \mathbf{\Psi} = b\, ds, \tag{6.1}$$

the total form is

$$\mathbf{\Omega} = a\, dr + b\, ds, \tag{6.2}$$

and the symmetry class consists of the simple systems

$$r_t + \lambda\, r_x = 0, \qquad s_t + \mu\, s_x = 0,$$

where λ, μ is a solution of

$$\mu_r = a(r, s)(\mu - \lambda), \qquad \lambda_s = b(r, s)(\lambda - \mu).$$

The solution space of the latter system is parametrized by two arbitrary functions of a single variable, for any a, b.

The Hamiltonian structures of the hyperbolic symmetry class with Riemann coordinates r, s, symmetry forms (6.1), and total form (6.2) are generated by the Poisson bivectors with flat metrics

$$g\, dr^2 + h\, ds^2$$

such that

$$(\ln|h|)_r = -2a, \qquad (\ln|g|)_s = -2b, \tag{6.3}$$

as in Theorem 4.2.

Proposition 6.2 A hyperbolic Hamiltonian symmetry class is exact if and only if there exist Riemann coordinates which are isothermal for the metric of the Poisson bivector.

P r o o f If the coordinates are isothermal, so that

$$|g| = |h| = e^{-2F}$$

for some function F, then the conditions (6.3) imply $\mathbf{\Omega} = \mathbf{dF}$. Conversely if the class is exact then

$$a = F_r, \qquad b = F_s$$

for some function F, so that

$$g = \frac{e^{-2F}}{p(r)}, \quad h = \frac{e^{-2F}}{q(s)}$$

for some functions p, q, by (6.3). Then the Riemann coordinates

$$\tilde{r}(r) = \int^r |p(r')|^{-\frac{1}{2}} dr', \qquad \tilde{s}(s) = \int^s |q(s')|^{-\frac{1}{2}} ds'$$

are isothermal. □

Proposition 6.3 An exact symmetry class with total form $\mathbf{\Omega} = dF$ is Hamiltonian if and only if there are Riemann coordinates r, s such that

$$F_{rr} \pm F_{ss} = 0, \tag{6.4}$$

in which case the class is Hamiltonian with respect to the Poisson bivector with flat metric

$$e^{-2F}(dr^2 \pm ds^2). \tag{6.5}$$

P r o o f If the class is Hamiltonian then the metric of the bivector is (6.5) in isothermal Riemann coordinates, by (6.3). Hence (6.4) holds, because the metric is flat. Conversely, if (6.4) holds then the metric (6.5) is flat. This metric also satisfies the conditions (6.3), so the class is Hamiltonian with respect to the associated bivector. □

Note that this does not exclude the possibility that the exact class is also Hamiltonian with respect to other bivectors. See Theorem 6.5.

A 2-component hyperbolic symmetry class is nondegenerate if its symmetry forms are nonzero. The characteristic distributions are then given by the symmetry forms. The symmetry coframe provides a local one-to-one correspondence between nondegerate classes and coframes on $\mathbb{R}^2$. See Gardner [14] for details of the equivalence method. Only the following facts are needed here. Coframes are equivalent if they are related by a diffeomorphism. An equivalence identifies the characteristic distributions, hence the Riemann coordinates of the symmetry classes. An equivalence may therefore be viewed as a change of Riemann coordinates. A vector field $\mathbf{W}$ is an infinitesimal automorphism of a coframe if its flow is a 1-parameter family of self-equivalences, which holds if and only if

$$L_{\mathbf{W}}\mathbf{\Pi} = L_{\mathbf{W}}\mathbf{\Psi} = 0.$$

The automorphisms are the vector fields with Riemann coordinate description

$$\mathbf{W} = f(r)\partial_r + g(s)\partial_s$$

such that

$$L_{\mathbf{W}}\mathbf{\Omega} = \mathbf{0}.$$

The structure equations of a coframe $\binom{\mathbf{\Pi}}{\mathbf{\Psi}}$ are

$$d\mathbf{\Pi} = I\ \mathbf{\Psi}\wedge\mathbf{\Pi}, \qquad d\mathbf{\Psi} = J\ \mathbf{\Pi}\wedge\mathbf{\Psi},$$

where the functions I, J are the zeroth order invariants. The symmetry class is exact if and only if $I = J$. The first order invariants are the functions I_1, I_2, J_1, J_2 defined by

$$dI = I_1\,\mathbf{\Pi} + I_2\,\mathbf{\Psi}, \qquad dJ = J_1\,\mathbf{\Pi} + J_2\,\mathbf{\Psi}.$$

Higher order invariants are defined analogously. Generic coframes are locally equivalent if and only if their invariants satisfy the same functional relations. The rank of a coframe is the number of functionally independent invariants. The dimension of the automorphism algebra of a coframe is equal to the codimension of the rank, e.g., the automorphism algebra of a rank 0 coframe is 2-dimensional. Proposition 6.4 is given here as an example of a coframe invariant. The result is used in Theorem 6.5.

Proposition 6.4 The Gaussian curvature of the metric $\mathbf{\Pi}^2 + \mathbf{\Psi}^2$ defined by a coframe $\binom{\mathbf{\Pi}}{\mathbf{\Psi}}$ is

$$K = -(I_2 + J_1 + I^2 + J^2).$$

P r o o f If $\mathbf{\Pi} = a\,dr$, $\mathbf{\Psi} = b\,ds$ then

$$d\mathbf{\Pi} = \frac{a_s}{ab}\,\mathbf{\Psi}\wedge\mathbf{\Pi}, \qquad d\mathbf{\Psi} = \frac{b_r}{ab}\,\mathbf{\Pi}\wedge\mathbf{\Psi},$$

hence

$$I = \frac{a_s}{ab}, \qquad J = \frac{b_r}{ab}.$$

The Gaussian curvature of the metric $a^2\,dr^2 + b^2\,ds^2$ is

$$\frac{R_{1221}}{a^2b^2} = -\frac{1}{b^2}\left(\frac{a_s}{a}\right)_s - \frac{1}{a^2}\left(\frac{b_r}{b}\right)_r - \frac{1}{b^2}\frac{a_s}{a}\frac{a_s}{a} - \frac{1}{a^2}\frac{b_r}{b}\frac{b_r}{b} + \frac{1}{b^2}\frac{a_s}{a}\frac{b_s}{b} + \frac{1}{a^2}\frac{a_r}{a}\frac{b_r}{b} =$$

$$-\frac{1}{b}\left(\frac{a_s}{ab}\right)_s - \frac{1}{a}\left(\frac{b_r}{ab}\right)_r - \frac{a_s}{ab}\frac{a_s}{ab} - \frac{b_r}{ab}\frac{b_r}{ab} = -(I_2 + J_1 + I^2 + J^2),$$

using (4.12). □

In the case $n = 2$, a 1-weight Poisson bivector is either definite or indefinite, i.e., the corresponding flat metric on $\mathcal{U}$ is definite or indefinite. A definite bivector is described in appropriate Darboux coordinates u, v by the operator

$$\mathcal{D} = \begin{pmatrix} \mathrm{D}_x & 0 \\ 0 & \mathrm{D}_x \end{pmatrix},$$

and the Hamiltonian system generated by a function $H(u, v)$ is

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} H_{uu} & H_{uv} \\ H_{uv} & H_{vv} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x.$$

Theorem 6.1 *The total form of the hyperbolic Hamiltonian system*

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} H_{uu} & H_{uv} \\ H_{uv} & H_{vv} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x, \qquad H_{uv} \neq 0,$$

is

$$\mathbf{\Omega} = \frac{P_v\, du - P_u\, dv}{2(P^2+1)}, \qquad P = \frac{H_{uu} - H_{vv}}{2H_{uv}}.$$

The symmetry class of the system is exact if and only if

$$(\tan^{-1} P)_{uu} + (\tan^{-1} P)_{vv} = 0.$$

The class is nondegenerate if and only if

$$P_u P_u - 2P P_u P_v - P_v P_v \neq 0,$$

in which case the zeroth order invariants of the coframe are

$$I - J = 4(P^2+1)^{\frac{5}{2}} \, \frac{(\tan^{-1} P)_{uu} + (\tan^{-1} P)_{vv}}{P_u P_u - 2P P_u P_v - P_v P_v},$$

$$I + J = 4(P^2+1)^{\frac{5}{2}} \, \frac{(P^2+1)^{-\frac{1}{2}}{}_{uu} - (P^2+1)^{-\frac{1}{2}}{}_{vv}}{P_u P_u - 2P P_u P_v - P_v P_v} + \frac{24 P P_u P_v - 8(P^2+1) P_{uv}}{P_u P_u - 2P P_u P_v - P_v P_v}.$$

P r o o f A tedious but direct calculation. See Doyle [15]. □

The symmetry class is exact if and only if $P = f_1/f_2$ for some holomorphic function $f_1(u, v) - i f_2(u, v)$, in which case

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} f_1 & f_2 \\ f_2 & -f_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x$$

is the canonical representative.

An indefinite bivector is described in appropriate Darboux coordinates u, v by the operator

$$\mathcal{D} = \begin{pmatrix} 0 & \mathrm{D}_x \\ \mathrm{D}_x & 0 \end{pmatrix},$$

and the Hamiltonian system generated by a function $H(u,v)$ is

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} H_{uv} & H_{vv} \\ H_{uu} & H_{uv} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x . \tag{6.6}$$

Theorem 6.2 *The total form of the hyperbolic Hamiltonian system*

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} H_{uv} & H_{vv} \\ H_{uu} & H_{uv} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x , \qquad H_{uu}, H_{vv} > 0,$$

is

$$\mathbf{\Omega} = \frac{P_u\, du - P_v\, dv}{2P}, \qquad P = \sqrt{\frac{H_{uu}}{H_{vv}}}.$$

The symmetry class of the system is exact if and only if

$$(\ln P)_{uv} = 0.$$

The class is nondegenerate if and only if

$$P{P^{-1}}_u{P^{-1}}_u \neq P^{-1}P_vP_v,$$

in which case the zeroth order invariants of the coframe are

$$I - J = \frac{8(\ln P)_{uv}}{P^{-1}P_vP_v - P{P^{-1}}_u{P^{-1}}_u}, \qquad I + J = \frac{4({P^{-1}}_{uu} - P_{vv})}{P^{-1}P_vP_v - P{P^{-1}}_u{P^{-1}}_u}.$$

P r o o f Again, a tedious calculation. See Doyle [15]. □

The symmetry class is exact if and only if $P = m_1/m_2$ for some functions $m_1(u)$, $m_2(v) > 0$, in which case

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & {m_2}^2 \\ {m_1}^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x$$

is the canonical representative. If the class is nondegenerate then the zeroth order invariant is

$$I = 2\ \frac{\frac{1}{m_1}\left(\frac{1}{m_1}\right)'' - \frac{1}{m_2}\left(\frac{1}{m_2}\right)''}{\left(\frac{1}{m_2}\right)'\left(\frac{1}{m_2}\right)' - \left(\frac{1}{m_1}\right)'\left(\frac{1}{m_1}\right)'}.$$

Example 6.1 The elasticity equation

$$\psi_{tt} = m(\psi_x)^2\psi_{xx}$$

with propagation speed $m > 0$ is equivalent to

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ m(u)^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x , \tag{6.7}$$

via $u = \psi_x$, $v = \psi_t$. The system (6.7) is a hyperbolic Hamiltonian system (6.6), with $H(u,v) = M(u) + \frac{1}{2}v^2$, $M'' = m^2$, and is the canonical representative of its symmetry class. The system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} v & u \\ n(u)^2/u & v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0 \tag{6.8}$$

describes the dynamics of a 1-dimensional isentropic gas. The field variables u, v denote the density and velocity of the gas, and the propagation speed is $n(u) > 0$. The system (6.8) is a hyperbolic Hamiltonian system (6.6), with $H(u,v) = -N(u) - \frac{1}{2}uv^2$, $N'' = n^2/u$. The systems (6.7), (6.8) are symmetries if and only if $n^2 = u^2m^2$. The function which describes the symmetry structure is $P = m$. The symmetry class is exact, and is nondegenerate if and only if $m' \neq 0$, in which case the zeroth order invariant is

$$I = -2\,\frac{\frac{1}{m}\left(\frac{1}{m}\right)''}{\left(\frac{1}{m}\right)'\left(\frac{1}{m}\right)'}.$$

If I is constant, then this equation can be solved for m :

$$I \neq -2 \quad : \quad m = (c_1u + c_2)^{-2/(I+2)},$$

$$I = -2 \quad : \quad m = \exp(c_1u + c_2).$$

The polytropic gas dynamics systems thus correspond to the case of a constant invariant $I \neq -2$. □

The symmetry forms of a nondegenerate exact symmetry class with total form $\mathbf{\Omega} = dF$ are $\mathbf{\Pi} = F_r\,dr$, $\mathbf{\Psi} = F_s\,ds$, in Riemann coordinates r, s, where $F_r, F_s \neq 0$. The zeroth order invariant is

$$I = \frac{F_{rs}}{F_rF_s},$$

and the first order invariants are

$$I_1 = \frac{I_r}{F_r}, \qquad I_2 = \frac{I_s}{F_s}.$$

There are three generic possibilities:

$$\text{rank } 0 \quad : \quad dI = 0,$$

$$\text{rank } 1 \quad : \quad dI \neq 0, \quad dI_1 \wedge dI = dI_2 \wedge dI = 0,$$

$$\text{rank } 2 \quad : \quad dI_1 \wedge dI \neq 0 \quad \text{or} \quad dI_2 \wedge dI \neq 0.$$

The automorphisms are the vector fields

$$\mathbf{W} = f(r)\partial_r + g(s)\partial_s$$

such that

$$L_{\mathbf{W}}\,dF = 0,$$

i.e.,

$$d(L_{\mathbf{W}}\,F) = 0.$$

Theorem 6.3 The first order conserved densities for the canonical representative of a nondegenerate exact symmetry class correspond to the infinitesimal automorphisms of the coframe of the class.

P r o o f The canonical system is

$$r_t + e^{2F} r_x = 0, \qquad s_t - e^{2F} s_x = 0.$$

Compare Example 5.1. □

We now use the automorphisms to extract the normal forms for the rank 0 and rank 1 exact symmetry classes. Suppose that the coframe has a nonzero automorphism

$$(f(r)\partial_r + g(s)\partial_s)F = c,$$

where c is constant. If $f = 0$ or $g = 0$ then $F_{rs} = 0$, i.e., $I = 0$. If $f, g \neq 0$ then

$$(\partial_r + \partial_s)F = 2c,$$

or

$$F(r,s) = \Phi(r-s) + c(r+s)$$

in suitable Riemann coordinates. In these coordinates, I, I_1, I_2 are functions of $r-s$, hence the coframe has rank 0 or rank 1. The coframe has rank 0 if and only if I is constant. If $I = 0$ then $F_{rs} = 0$. This implies

$$F(r,s) = r - s$$

in suitable Riemann coordinates. In these coordinates, the automorphism algebra is the 2-dimensional abelian algebra of constant vector fields, reflecting the invariance of

$$\mathbf{\Omega} = dr - ds$$

under arbitrary translations of r, s. If $I = -1/\varepsilon$ then $(e^{F/\varepsilon})_{rs} = 0$. This implies

$$F(r,s) = \varepsilon \ln|r-s|$$

in suitable Riemann coordinates. In these coordinates, the automorphism algebra is the 2-dimensional nonabelian algebra spanned by the vector fields $r\,\partial_r + s\,\partial_s$, $\partial_r + \partial_s$, reflecting the invariance of

$$\mathbf{\Omega} = \frac{\varepsilon}{r-s}(dr - ds)$$

under dilations and equal translations of r, s. If $dI \wedge \mathbf{\Omega} = 0$, then the higher order invariants are also functionally dependent on F, so the coframe has rank 0 or rank 1. Note that $dI \wedge \mathbf{\Omega} = \mathbf{0}$ if and only if $I_1 = I_2$, because

$$dI \wedge \mathbf{\Omega} = (I_1 - I_2)\,\mathbf{\Pi} \wedge \mathbf{\Psi}.$$

The condition $I_1 = I_2$ is equivalent to

$$\frac{F_{rsr}}{F_r} - \frac{F_{rs}F_{rr}}{{F_r}^2} = \frac{F_{srs}}{F_s} - \frac{F_{sr}F_{ss}}{{F_s}^2},$$

or

$$\left(\ln\left|\frac{F_r}{F_s}\right|\right)_{rs} = 0,$$

or

$$(f(r)\partial_r + g(s)\partial_s)F = 0$$

for some $f, g \neq 0$, which holds if and only if

$$(\partial_r + \partial_s)F = 0,$$

or

$$F(r,s) = \Phi(r-s)$$

in suitable Riemann coordinates. In these coordinates, the automorphism $\partial_r + \partial_s$ represents the invariance of

$$\mathbf{\Omega} = \Phi'(r-s)(dr - ds)$$

under equal translations of r, s. There is an independent automorphism if and only if

$$\Phi'(r-s) = \frac{1}{f(r) - g(s)}$$

for some $f \neq g$, in which case

$$I = -f' = -g',$$

so that I is constant, i.e., the coframe has rank 0. Finally, suppose that the coframe has rank 1, and that $dI \wedge \mathbf{\Omega} \neq 0$, i.e., $I_1 \neq I_2$. If $I_r = 0$ then $I_s \neq 0$. Therefore $I_{2\,r} = 0$, because $dI_2 \wedge dI = 0$. But then $F_{rs} = 0$, contradicting $dI \neq 0$. This proves $I_1 \neq 0$; similarly $I_2 \neq 0$. The conditions $dI_1 \wedge dI = dI_2 \wedge dI = 0$ imply

$$I_1 = h(I), \qquad I_2 = k(I), \tag{6.9}$$

and the argument just given shows $h \neq 0$, $k \neq 0$. Fix H, K such that $H' = 1/h$, $K' = 1/k$. Then

$$H(I)_r = F_r\,, \qquad K(I)_s = F_s\,,$$

so that

$$H(I) = F + g(s), \qquad K(I) = F + f(r). \tag{6.10}$$

The condition $dI \wedge dF \neq 0$ implies $f' \neq 0$, $g' \neq 0$. Differentiating (6.10) gives

$$I_1 = k(I)\left(\frac{F_r + f'(r)}{F_r}\right), \qquad I_2 = h(I)\left(\frac{F_s + g'(s)}{F_s}\right). \tag{6.11}$$

The conditions (6.9), (6.11) imply

$$\left(\frac{F_r + f'(r)}{F_r}\right)\left(\frac{F_s + g'(s)}{F_s}\right) = 1,$$

or

$$\left(\frac{1}{f'(r)}\partial_r + \frac{1}{g'(s)}\partial_s\right) F = -1,$$

so that

$$(\partial_r + \partial_s)F = 2,$$

or

$$F(r, s) = \Phi(r - s) + r + s$$

in suitable Riemann coordinates. In these coordinates, the automorphism $\partial_r + \partial_s$ represents the invariance of

$$\mathbf{\Omega} = \Phi'(r - s)(dr - ds) + dr + ds$$

under equal translations of r, s. There is an independent automorphism if and only if

$$f(r)(1 + \Phi'(r - s)) + g(s)(1 - \Phi'(r - s)) = 0$$

for some nonzero functions $f \neq g$. This implies

$$\Phi'(r - s) = \frac{g(s) + f(r)}{g(s) - f(r)}.$$

But then

$$I = \frac{f'}{2f} = \frac{g'}{2g},$$

so that I is constant, contradicting $dI \neq 0$.

Theorem 6.4 *A nondegenerate exact symmetry class with rank 0 or rank 1 coframe is generically equivalent to one of the normal forms*

$$\text{rank } 0,\ I = 0 \quad : \quad \mathbf{\Omega} = dr - ds, \tag{6.12a}$$

$$\text{rank } 0,\ I = -\tfrac{1}{\varepsilon} \quad : \quad \mathbf{\Omega} = \tfrac{\varepsilon}{r - s}(dr - ds), \tag{6.12b}$$

$$\text{rank } 1,\ I_1 = I_2 \quad : \quad \mathbf{\Omega} = \phi(r - s)(dr - ds), \qquad \left(\frac{1}{\phi}\right)'' \neq 0, \tag{6.12c}$$

$$\text{rank } 1,\ I_1 \neq I_2 \quad : \quad \mathbf{\Omega} = \phi(r - s)(dr - ds) + dr + ds, \tag{6.12d}$$

$$\left(\ln\left|\frac{\phi - 1}{\phi + 1}\right|\right)'' \neq 0,$$

where r, s are Riemann coordinates. Two coframes (6.12c) defined by functions $\phi, \tilde{\phi}$ are equivalent if and only if

$$\tilde{\phi}(y) = c\phi(cy + d) \tag{6.13}$$

for some constants $c \neq 0$ and d. Two coframes (6.12d) defined by functions $\phi, \tilde{\phi}$ are equivalent if and only if

$$\tilde{\phi}(y) = \phi(y + d) \tag{6.14}$$

for some constant d. There are no other redundancies in (6.12).

P r o o f It has already been shown that each rank 0 or rank 1 exact symmetry class is generically equivalent to one of the normal forms (6.12). A coframe (6.12) is exact, and has rank 0 or rank 1, because all of its invariants are functions of $r-s$. The coframes (6.12a,b) are not equivalent, because they have distinct constant zeroth order invariants. The condition $(1/\phi)'' \neq 0$ ensures that (6.12c) has rank 1. The condition $(\ln|\phi - 1/\phi + 1|)'' \neq 0$ ensures that the invariants of (6.12d) satisfy $I_1 \neq I_2$. Now, assume that the coframes

$$\begin{pmatrix} \mathbf{\Pi} & = & \phi(r-s)\,dr \\ \mathbf{\Psi} & = & -\phi(r-s)\,ds \end{pmatrix}, \quad \begin{pmatrix} \widetilde{\mathbf{\Pi}} & = & \tilde{\phi}(\tilde{r}-\tilde{s})\,d\tilde{r} \\ \widetilde{\mathbf{\Psi}} & = & -\tilde{\phi}(\tilde{r}-\tilde{s})\,d\tilde{s} \end{pmatrix}$$

are equivalent via

$$\varphi(\tilde{r},\tilde{s}) = (r(\tilde{r}), s(\tilde{s})),$$

i.e.,

$$\varphi^*\mathbf{\Pi} = \widetilde{\mathbf{\Pi}}, \qquad \varphi^*\mathbf{\Psi} = \widetilde{\mathbf{\Psi}}.$$

Then

$$\phi(r-s)r'(\tilde{r}) = \tilde{\phi}(\tilde{r}-\tilde{s}), \qquad \phi(r-s)s'(\tilde{s}) = \tilde{\phi}(\tilde{r}-\tilde{s}).$$

This implies $r' = s'$, so that

$$r = c\tilde{r} + d_1, \qquad s = c\tilde{s} + d_2,$$

and

$$c\phi(r-s) = \tilde{\phi}(\tilde{r}-\tilde{s}).$$

Therefore (6.13) holds. Conversely, (6.13) implies that the coframes (6.12c) defined by $\phi, \tilde{\phi}$ are equivalent. Finally, assume that the coframes

$$\begin{pmatrix} \mathbf{\Pi} & = & (1+\phi(r-s))\,dr \\ \mathbf{\Psi} & = & (1-\phi(r-s))\,ds \end{pmatrix}, \qquad \left(\ln\left|\frac{\phi-1}{\phi+1}\right|\right)'' \neq 0,$$

and

$$\begin{pmatrix} \widetilde{\mathbf{\Pi}} & = & (1+\tilde{\phi}(\tilde{r}-\tilde{s}))\,d\tilde{r} \\ \widetilde{\mathbf{\Psi}} & = & (1-\tilde{\phi}(\tilde{r}-\tilde{s}))\,d\tilde{s} \end{pmatrix}$$

are equivalent via

$$\varphi(\tilde{r},\tilde{s}) = (r(\tilde{r}), s(\tilde{s})),$$

i.e.,

$$\varphi^*\mathbf{\Pi} = \widetilde{\mathbf{\Pi}}, \qquad \varphi^*\mathbf{\Psi} = \widetilde{\mathbf{\Psi}}.$$

Then

$$\begin{aligned} (1+\phi(r-s))r'(\tilde{r}) &= 1+\tilde{\phi}(\tilde{r}-\tilde{s}), \\ (1-\phi(r-s))s'(\tilde{s}) &= 1-\tilde{\phi}(\tilde{r}-\tilde{s}), \end{aligned} \tag{6.15}$$

so that

$$(1+\phi(r-s))r' + (1-\phi(r-s))s' = 2. \tag{6.16}$$

If $r' \neq s'$ then differentiating (6.16) by $\tilde{r}, \tilde{s}$ implies

$$\Big(\ln|(\phi(r-s)+1)(r'-s')|\Big)_{\tilde{r}} = \Big(\ln|(\phi(r-s)-1)(r'-s')|\Big)_{\tilde{s}} = 0,$$

hence

$$(\phi(r-s)-1)(r'-s') = \tilde{f}(\tilde{r}), \qquad (\phi(r-s)+1)(r'-s') = \tilde{g}(\tilde{s}),$$

or

$$\left(\frac{\phi-1}{\phi+1}\right)(r-s) = \frac{f(r)}{g(s)},$$

so that

$$\left(\ln\left|\frac{\phi-1}{\phi+1}\right|\right)'' = 0,$$

contradicting the contrary assumption. Therefore $r' = s'$, or

$$r = \tilde{r} + d_1, \qquad s = \tilde{s} + d_2,$$

because of (6.16). The relation (6.14) now follows from (6.15). Conversely, (6.14) implies that the coframes (6.12d) defined by $\phi, \tilde{\phi}$ are equivalent. □

Corollary A nondegenerate exact symmetry class with rank 0 or rank 1 coframe has (generically) an indefinite Hamiltonian structure.

P r o o f This follows from Theorem 6.4 and Proposition 6.3. □

The automorphisms of the normal forms were described in the discussion leading to Theorem 6.4. The first order conservation laws for the canonical systems are now provided by Theorem 6.3. The canonical representative of the symmetry class (6.12a) has two independent first order conservation laws

$$T = \frac{e^{2(s-r)}}{r_x}, \qquad X = \frac{1}{r_x} - 4x$$

and

$$T = -\frac{e^{2(s-r)}}{s_x}, \qquad X = \frac{1}{s_x} + 4x.$$

The canonical representative of the the symmetry class (6.12b) also has two independent first order conservation laws

$$T = (r-s)^{-2\varepsilon}\left(\frac{1}{r_x} - \frac{1}{s_x}\right), \qquad X = \frac{1}{r_x} + \frac{1}{s_x}$$

and

$$T = (r-s)^{-2\varepsilon}\left(\frac{r}{r_x} - \frac{s}{s_x}\right), \qquad X = \frac{r}{r_x} + \frac{s}{s_x} - 4\varepsilon x.$$

The canonical representative of the symmetry class (6.12c) has one first order conservation law

$$T = e^{-2\Phi(r-s)}\left(\frac{1}{r_x} - \frac{1}{s_x}\right), \qquad X = \frac{1}{r_x} + \frac{1}{s_x},$$

and the canonical representative of the symmetry class (6.12d) has one first order conservation law

$$T = e^{-2\left(\Phi(r-s)+r+s\right)}\left(\frac{1}{r_x} - \frac{1}{s_x}\right), \qquad X = \frac{1}{r_x} + \frac{1}{s_x} - 8x,$$

where $\Phi' = \phi$.

A nondegenerate exact symmetry class with total form $\mathbf{\Omega} = dF$ is *simple* if $dI \wedge \mathbf{\Omega} = 0$, i.e., $I_1 = I_2$. The normal forms for the simple classes are (6.12a,b,c). The following arguments show that the simple classes are the elasticity/gas dynamics classes of Example 6.1.

Example 6.2 If the function P which describes the symmetry structure of a nondegenerate class of hyperbolic Hamiltonian systems (6.6) has the form $P = P(u)$ then the class is an elasticity/gas dynamics class with canonical representative (6.7), where $m = P$. Conversely, the function P of the elasticity/gas dynamics class with canonical representative (6.7) has the form $P = P(u)$. The total form is

$$\mathbf{\Omega} = \frac{P'}{2P}\, du,$$

and the zeroth order invariant is

$$I(u) = -2\,\frac{\frac{1}{P}\left(\frac{1}{P}\right)''}{\left(\frac{1}{P}\right)'\left(\frac{1}{P}\right)'},$$

hence the class is simple. The case $P = P(v)$ reduces to the previous case by interchange of u, v. $\square$

There are Riemann coordinates r, s in which

$$F(r,s) = \Phi(r-s),$$

so that

$$\mathbf{\Omega} = \phi(r-s)(dr - ds),$$

where $\phi = \Phi'$. The symmetry class consists of the simple systems

$$r_t + \lambda\, r_x = 0, \quad s_t + \mu\, s_x = 0,$$

where λ, μ is a solution of

$$\lambda_s = -\phi(r-s)(\lambda - \mu), \quad \mu_r = \phi(r-s)(\mu - \lambda). \tag{6.17}$$

The solutions of (6.17) are $\lambda = f_r$, $\mu = f_s$, where f satisfies the Euler-Poisson-Darboux equation

$$f_{zz} = f_{yy} - 2\phi(y) f_y, \tag{6.18}$$

where $y = r - s$, $z = r + s$. The Riemann coordinates in which $\mathbf{\Omega}$ has the prescribed form are unique up to independent translations, equal dilations, and interchange. If

$$\mathbf{\Omega} = \tilde{\phi}(\tilde{r} - \tilde{s})(d\tilde{r} - d\tilde{s})$$

then f satisfies (6.18) if and only if

$$f_{\tilde z\tilde z} = f_{\tilde y\tilde y} - 2\tilde{\phi}(\tilde y) f_{\tilde y},$$

where $\tilde y = \tilde r - \tilde s$, $\tilde z = \tilde r + \tilde s$. Moreover, $f_{yz} = 0$ if and only if $f_{\tilde y\tilde z} = 0$. Hence there is an intrinsic space of systems generated by the solutions of (6.18) with the form $f = f_1(y) + f_2(z)$. This space is 3-dimensional. The solutions $f = z$ and $f = \int^y e^{2\Phi(y')}dy'$ produce the translation generator and the canonical system, and the solution

$$f = \frac{1}{4} z^2 + \frac{1}{2}\int^y e^{2\Phi(y')}\int^{y'} e^{-2\Phi(y'')}dy''\, dy'$$

produces the system

$$\begin{aligned} r_t + \Big(\tfrac{1}{2}(r+s) + \nu(r-s)\Big)\, r_x &= 0, \\ s_t + \Big(\tfrac{1}{2}(r+s) - \nu(r-s)\Big)\, s_x &= 0, \end{aligned} \tag{6.19}$$

where

$$\nu(y) = \frac{1}{2} e^{2\Phi(y)}\int^y e^{-2\Phi(y')}dy' \ .$$

The symmetry class is Hamiltonian with respect to the Poisson bivector with flat metric

$$e^{-2F}(dr^2 - ds^2),$$

by Proposition 6.3. The coordinates in which the bivector is described by the operator

$$\mathcal{D} = \begin{pmatrix} 0 & \mathrm{D}_x \\ \mathrm{D}_x & 0 \end{pmatrix}$$

are

$$u = a\tilde u + b\tilde v, \quad v = c\tilde u + d\tilde v,$$

where

$$\tilde u(r,s) = \int^{r-s} \mathrm{e}^{-2\Phi(y)}dy, \qquad \tilde v(r,s) = \frac{r+s}{2},$$

and where a, b, c, d are constants such that $ad \neq bc$, and

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This implies $F = F(u)$ or $F = F(v)$. But then $P = P(u)$ or $P = P(v)$, as in Example 6.2. This proves that every simple class is an elasticity/gas dynamics class. The system (6.19) is described in the coordinates $u = \tilde u$, $v = \tilde v$ by the gas dynamics system (6.8), with $n(u) = \nu((r-s)(u))$.

Example 6.3 The system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1/v^2 \\ 4/u^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x$$

is a hyperbolic Hamiltonian system (6.6). The symmetry class is exact and nondegenerate. The zeroth order invariant is $I = 0$, so the class is simple. However, $P = 2v/u$ is neither a function of u only nor of v only. □

Comparison of Examples 6.2 and 6.3 may seem to contradict the claim that each simple class is an elasticity/gas dynamics class. The symmetry class of Example 6.3 has a second Hamiltonian structure, however, and assumes the form of Example 6.2 in Darboux coordinates for the second Poisson bivector.

Theorem 6.5 *A simple symmetry class is multi-Hamiltonian if and only if the metric of its coframe has constant Gaussian curvature K, in which case the space of Poisson bivectors for the class is 3-dimensional. Each simple tri-Hamiltonian symmetry class is equivalent to one and only one of the normal forms*

$$K = 0,\ I = 0 \quad : \quad \mathbf{\Omega} = dr - ds, \tag{6.20a}$$

$$K = 0,\ I > 0 \quad : \quad \mathbf{\Omega} = e^{r-s}(dr - ds), \tag{6.20b}$$

$$K = 0,\ I < 0 \quad : \quad \mathbf{\Omega} = e^{s-r}(dr - ds), \tag{6.20c}$$

$$K = -\frac{2}{\varepsilon^2},\ I = -\frac{1}{\varepsilon} \quad : \quad \mathbf{\Omega} = \frac{\varepsilon}{r-s}\,(dr - ds), \tag{6.20d}$$

$$K = -\frac{2}{\varepsilon^2},\ I' > 0 \quad : \quad \mathbf{\Omega} = \frac{\varepsilon}{\cos(r-s)}\,(dr - ds), \qquad \varepsilon > 0, \tag{6.20e}$$

$$K = -\frac{2}{\varepsilon^2},\ I' < 0, I > 0 \quad : \quad \mathbf{\Omega} = \frac{\varepsilon}{\sin h(r-s)}\,(dr - ds), \qquad \varepsilon < 0, \tag{6.20f}$$

$$K = -\frac{2}{\varepsilon^2},\ I' < 0, I < 0 \quad : \quad \mathbf{\Omega} = \frac{\varepsilon}{\sin h(r-s)}\,(dr - ds), \qquad \varepsilon > 0, \tag{6.20g}$$

$$K = \frac{2}{\varepsilon^2} \quad : \quad \mathbf{\Omega} = \frac{\varepsilon}{\cos h(r-s)}\,(dr - ds), \qquad \varepsilon > 0, \tag{6.20h}$$

where r, s are Riemann coordinates, and I' is the first order invariant.

P r o o f The zeroth order invariant of the class is

$$I(r,s) = -\left(\frac{1}{\phi}\right)'(r-s),$$

and the common value of the first order invariants is

$$I'(r,s) = -\left(\frac{1}{\phi}\left(\frac{1}{\phi}\right)''\right)(r-s). \tag{6.21}$$

The Gaussian curvature is $K = -2(I' + I^2)$, by Proposition 6.4, or

$$K(r,s) = 2\left(\frac{1}{\phi}\left(\frac{1}{\phi}\right)'' - \left(\frac{1}{\phi}\right)'\left(\frac{1}{\phi}\right)'\right)(r-s). \tag{6.22}$$

The curvature is constant if and only if

$$\left(\phi\left(\frac{1}{\phi}\right)''\right)' = 0, \tag{6.23}$$

because
$$\left(\frac{1}{\phi}\left(\frac{1}{\phi}\right)''-\left(\frac{1}{\phi}\right)'\left(\frac{1}{\phi}\right)'\right)'=\frac{1}{\phi^2}\left(\phi\left(\frac{1}{\phi}\right)''\right)'.$$
The Poisson bivectors which generate the Hamiltonian structure of the class correspond to the flat metrics
$$e^{-2F}\left(\frac{dr^2}{p(r)}+\frac{ds^2}{q(s)}\right),$$
by (6.3). ¿From (4.15), such a metric is flat if and only if
$$2pF_{rr}+p'F_r+2qF_{ss}+q'F_s=0, \tag{6.24}$$
or
$$((p+q)\phi)_r=((p+q)\phi)_s, \tag{6.25}$$
i.e., if and only if
$$p(r)+q(s)=\frac{\psi(r+s)}{\phi(r-s)}$$
for some function $\psi(z)$. The solution $p=1$, $q=-1$, $\psi=0$ corresponds to the indefinite Hamiltonian structure provided by Proposition 6.3. There is a solution p,q to (6.24) with $p+q\neq 0$ if and only if there is a non-zero function ψ such that
$$\left(\frac{\psi(r+s)}{\phi(r-s)}\right)_{rs}=0, \tag{6.26}$$
or
$$\left(\phi\left(\frac{1}{\phi}\right)''\right)(r-s)=\left(\frac{1}{\psi}\,\psi''\right)(r+s), \tag{6.27}$$
which is the case if and only if
$$\phi\left(\frac{1}{\phi}\right)''=\kappa$$
for some constant κ, i.e. if and only if (6.23) holds. Then (6.27) implies
$$\psi''=\kappa\,\psi,$$
so there is a 2-dimensional space of solutions ψ to (6.26). There is thus a 2-dimensional space of solutions $p+q=\psi/\phi$ to (6.25), and a 3-dimensional space of solutions p,q to (6.24). If $\kappa=0$ then
$$\phi(y)=\frac{1}{cy+d},$$
in which case the invariant of the corresponding coframe is constant, so the normal forms are (6.20a,d), as in Theorem 6.4. If $\kappa<0$ then
$$\phi(y)=\frac{1}{c\cos(ky+\delta)},$$
where $k^2=-\kappa$, and the coframe is equivalent to (6.20e), by Theorem 6.4. If $\kappa>0$ then
$$\phi(y)=c\,e^{\pm ky},$$

or

$$\phi(y) = \frac{1}{c \sin h(ky + \delta)}$$

or

$$\phi(y) = \frac{1}{c \cosh{(ky + \delta)}},$$

where $k^2 = \kappa$, and the coframe is equivalent to one of (6.20b,c,f,g,h), by Theorem 6.4. The values of I', K in (6.20) follow from (6.21), (6.22). Note that these imply that the listed normal forms are inequivalent. □

The tri-Hamiltonian structure of the rank 0 exact symmetry classes was first described by Nutku [6].

Only first order Poisson operators (2.5) have been considered thus far in our analysis of the Hamiltonian structure of quasilinear systems. The Hamiltonian structure of 2-component systems with respect to third order Poisson operators was studied by Olver and Nutku [7]. Theorem 6.6 is one of their results, given here in invariant form.

Theorem 6.6 *All of the systems of an exact hyperbolic symmetry class with indefinite Hamiltonian structure are also Hamiltonian with respect to a third order Poisson bivector. The first and third order bivectors are compatible.*

P r o o f The Darboux coordinate description of exactness given in Theorem 6.2 is the separability criterion of Olver and Nutku [7]. □

If the symmetry class is simple and tri-Hamiltonian then there is an open set of indefinite Poisson bivectors in the relevant 3-dimensional space. To each of these there corresponds a compatible third order Poisson bivector, with respect to which the symmetry class is Hamiltonian. This generalizes the results of Arik, et al. [10].

VII Remarks

It might be interesting to have an invariant description of the quasilinear systems which are Hamiltonian. This is equivalent to giving an invariant characterization of the (1,1)-tensors on $\mathbb{R}^n$ which are Hessian with respect to some flat metric, a problem which is unsolved even in the case $n = 2$. Further study of the invariants of the symmetry coframes of nondegenerate symmetry classes may prove worthwhile. Finally, it may be possible to extend Theorem 6.6 to exact hyperbolic symmetry classes with definite Hamiltonian structure.

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