Geometrical symmetries
of the Universal equation

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Abstract
It is shown that the group of geometrical symmetries of the Universal equation of
$D$-dimensional space coincides with $SL(D+1, R)$.

1 Introduction
The Universal equation was first introduced in a series of papers by D.B. Fairlie and col-
laborators [1] with the goal of constructing the example of an integrable system in the
space of arbitrary dimensions. In the simplest case ($D = 2$) this equation is exactly the
Batemann equation [2], which in its turn is closely connected with the equation of Monge
[2]. Later it was understood that this equation may be related to the problem of construc-
tion of local solutions of the bi-harmonical equation. In particular, for the case $D = 3$
it is connected with the bi-harmonical equation in usual four-dimensional space [3]. This
problem, in turn, is in closed connection with the problem of constructing the instanton
solution in four-dimensional Yang-Mills theory [4].

In this note we want to show that the Universal equation is invariant with respect to
fractional-linear transformations in $D + 1$ dimensions, and consequently that the group of
geometrical symmetries of this equation coincides with $SL(D+1, R)$.

2 General properties of the Universal equation
The original form of the Universal equation in the space of $D$ dimensions is the following

$$
Det_{D+1} \begin{pmatrix}
0 & \phi_i \\
\phi_j & \phi_{i,j}
\end{pmatrix} = 0 \quad (D \geq i, j \geq 1),
$$

(2.1)
where $\phi_i \equiv \frac{\partial \phi}{\partial x_i}$, $\phi_{i,j} \equiv \frac{\partial^2 \phi}{\partial x_i \partial x_j}$, $\phi$ is an unknown function, and $x_i$ - the coordinates of the space.

This equation possesses remarkable properties. There is an infinite number of Lagrange functions from which (2.1) arises by the known rules. There is an infinite number of conservation laws for this equation. It is possible to find the general solution of (2.1) with the help of the Legendre transformation [1].

We shall call all transformations of coordinates $x'_i = f_i(x_1, x_2, ..., x_D)$, with respect to which the equation (2.1) conserves its form, geometrical transformations.

3 The condition of invariance

Let us suppose that $x'_i$ are new coordinates. The derivatives with respect to old coordinates which are interesting for us are connected with the new ones by the usual formulae

$$
\phi_i = \sum_s \frac{\partial x'_s}{\partial x_i} \phi'_s, \quad \phi_{i,j} = \sum_{sk} \frac{\partial x'_s}{\partial x_i} \frac{\partial x'_k}{\partial x_j} \phi_{sk} + \sum_s \frac{\partial^2 x'_s}{\partial x_i \partial x_j} \phi'_s.
$$

The condition of invariance of (2.1) is equivalent to a system

$$
\frac{\partial^2 x'_s}{\partial x_i \partial x_j} = B_i \frac{\partial x'_s}{\partial x_j} + B_j \frac{\partial x'_s}{\partial x_i},
$$

where $B_i$ are some functions which must be defined from the solution of (3.2) together with $x'_s$.

In the case $i = j$ it follows immediately from (3.2)

$$
B_i = \frac{1}{2} \frac{\partial}{\partial x_i} \ln \left( \frac{\partial x'_s}{\partial x_i} \right)
$$

for all $s$, or taking the first integral we obtain

$$
\frac{\partial x'_1}{\partial x_i} = f^{(i)}_s \frac{\partial x'_s}{\partial x_i}
$$

where the symbol $f^{(i)}$ means that the function $f$ is independent of an argument $x_i$. Integrating (3.3) once more we obtain as a result

$$
x'_1 = f^{(1)}_s x'_s + F^{(i)}_s, \quad s = 2, ..., D.
$$

Substituting $B_i$ into (3.2) we obtain

$$
2 \frac{\partial x'_s}{\partial x_i} \frac{\partial x'_s}{\partial x_j} \frac{\partial^2 x'_s}{\partial x_i \partial x_j} = \left( \frac{\partial x'_s}{\partial x_i} \right)^2 \frac{\partial^2 x'_s}{\partial x_i^2} + \left( \frac{\partial x'_s}{\partial x_j} \right)^2 \frac{\partial^2 x'_s}{\partial x_j^2}.
$$

So we see that $x'_i$ is the solution of the corresponding Batemann equation with respect to each pair of arguments $i, j$. 
4 The simplest case \( D = 2 \)

In this case we have only one pair of variables \( x_1, x_2 \), and equations (3.4) take the form

\[
x_1' = \phi(2)x_2 + \Phi(2), \quad x_1' = f(1)x_2' + F(1),
\]

where \( \phi(2) \equiv \phi(x_2) \) and so on. Or

\[
x_2' = -\frac{\Phi(2) - F(1)}{\phi(2) - f(1)}, \quad x_1' = -\frac{\phi(2) - F(1)}{\phi(2) - f(1)}. \tag{4.1}
\]

Each of the functions \( x_1', x_2' \) must satisfy the Batemann equation. Let us write this equation at first for function \( x_2' \). Technically it is more convenient to introduce the function

\[
\theta = \frac{\partial x_2'}{\partial x_1} \frac{\partial x_2'}{\partial x_2},
\]

in terms of which the Batemann equation may be rewritten in the form

\[
\frac{\partial \ln \theta}{\partial x_1} = \theta \frac{\partial \ln \theta}{\partial x_2}.
\]

In our case

\[
\theta = -\frac{f_1}{\phi_2} \left[ \frac{\Phi - F - F\phi(\phi - f)}{\Phi - F - \Phi\phi(\phi - f)} \right],
\]

where \( f_1 \equiv \frac{\partial f}{\partial x_1}, \phi_2 \equiv \frac{\partial \phi}{\partial x_2} \) and functions \( F, \Phi \) are considered as the functions of arguments \( f, \phi \), respectively. The Batemanns equation takes the form

\[
\frac{f_{11} + f_1Ff}{f_1} + \frac{f_1Ff}{\Phi - F - F\phi(\phi - f)} - \frac{(\Phi - F) f_1}{\Phi - F - \Phi\phi(\phi - f)} =
\]

\[
-\frac{f_1}{\phi_2} \left[ \frac{\Phi - F - F\phi(\phi - f)}{\Phi - F - \Phi\phi(\phi - f)} \right] \left[ \frac{\phi_2}{\phi_2} + \frac{(\Phi - F) \phi_2}{\Phi - F - \Phi\phi(\phi - f)} - \frac{\Phi\phi\phi_2}{\Phi - F - \Phi\phi(\phi - f)} \right]. \tag{4.2}
\]

From (4.2) it follows immediately that \( F, \Phi \) are linear functions of their arguments

\[
F = Af + B, \quad \Phi = \alpha\phi + \beta \tag{4.3}
\]

and

\[
\frac{f_{11}(af + b)}{f_1^2} = \frac{\phi_{22}(a\phi + b)}{\phi_2^2} = Da = \text{const}, \quad a \equiv \alpha - A, \quad b \equiv \beta - B.
\]

First integrals of the last equations have the form

\[
f_1 = l(af + b)^D, \quad \phi_2 = m(a\phi + b)^D, \tag{4.4}
\]

where \( l, m \) — are the constants of integration. So (4.3) and (4.4) have solved the problem of finding \( x_2' \) from (4.1).

The function \( x_1' \) must also be the solution of the Batemann equation. Repeating step by step the above calculations we come to the following equation

\[
(\frac{1}{f})_{11}(a + bf^{-1}) = (\frac{1}{\phi})_{22}(a + b\phi^{-1})\frac{(1/2)}{(1/2)}.
\]
Keeping in mind the relations (4.4) we conclude that $D = 2$. We obtain finally

$$\frac{1}{af + b} = kx_1 + l, \quad \frac{1}{a\phi + b} = mx_2 + n,$$

where $k, l, m, n$ are the constants of integration. After substitution of all these results into (4.1) we obtain

$$x_2' = -\frac{\alpha kx_1 - Amx_2 + al - nA}{kx_1 - mx_2 + l - n}, \quad x_1' = -\frac{\beta kx_1 - Bmx_2 + 1 - Bn + \beta l}{kx_1 - mx_2 + l - n}. \quad (4.5)$$

5 The general case

All computations of the previous section are correct with respect to each pair of variables $(x_i, x_j)$, and so the general solution of the system (3.5) is the fractional-linear transformation with respect to all the variables of the problem. So we obtain

$$x_i' = \left(\frac{(Ax)_i + B_i}{\sum l_k x_k + l_{D+1}}\right) = \left(\frac{\tilde{A}y)_i}{(Ay)_{D+1}}\right) \quad (5.1)$$

where $A$ — is an arbitrary $DD$ matrix, $B_i$ are the components of a $D$-dimensional column, $l_k$ — the components of $D + 1$ dimensional row, $\tilde{A}$ is the $(D + 1)(D + 1)$ matrix, and $y_s$ are the components of a $D + 1$ dimensional column ($y_1, y_2, ..., y_D, 1$).

Formula (5.1) may be considered as a realization of transformations of the $SL(D+1, R)$ group (in the case of real coordinates) from one hand, and as a projective transformation of the $D$-dimensional plane from the other.

It is not difficult to check, by direct calculations, that the Universal equation is invariant with respect to transformations (5.1) but in the order to prove that (5.1) is the maximal geometrical symmetry of this equation, the considerations and calculations of the previous sections should be taken into account.

6 Concluding remarks

The main result of this paper consists in the proof of the fact that the maximal group of geometrical symmetry of the Universal equation in $D$-dimensional space coincides with $SL(D + 1, R)$ — the group of projective transformations of $D$-dimensional space. The hidden symmetries of dynamical systems have as their corollary the additional number of conservation laws which in its turn allow to construct a more wide class of exact solutions for the systems of such a type. As it was understood recently [3], the solution of the Universal equation has a direct relation to the definite class of solutions of harmonical and bi–harmonical equations in four dimensions. The discovered maximal group of geometrical symmetry of the Universal equation may be used for these purposes.

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References


