Nonlinear Mathematical Physics 1996, V.3, N 1–2, 24–39.

Lie Symmetries, Infinite-Dimensional Lie Algebras and Similarity Reductions of Certain (2+1)-Dimensional Nonlinear Evolution Equations

M. LAKSHMANAN, M. SENTHIL VELAN

Center for Nonlinear Dynamics, Department of Physics,
Bharathidasan University Tiruchirapalli 620 024 India

Abstract

The Lie point symmetries associated with a number of $(2+1)$-dimensional generalizations of soliton equations are investigated. These include the Niznik – Novikov – Veselov equation and the breaking soliton equation, which are symmetric and asymmetric generalizations respectively of the KDV equation, the (2+1)-dimensional generalization of the nonlinear Schrödinger equation by Fokas as well as the (2+1)-dimensional generalized sine-Gordon equation of Konopelchenko and Rogers. We show that in all these cases the Lie symmetry algebra is infinite-dimensional; however, in the case of the breaking soliton equation they do not possess a centerless Virasoro-type subalgebra as in the case of other typical integrable (2+1)-dimensional evolution equations. We work out the similarity variables and special similarity reductions and investigate them.

1 Introduction

In the analysis of differential equations, the existence of a one-parameter Lie group of symmetries can dramatically aid in understanding the integrability properties and obtaining interesting solutions of them. Especially, this is true for nonlinear dynamical systems and soliton possessing nonlinear partial differential equations (pdes) [1–3]. It is now well accepted that in (1+1)-dimensions, soliton possessing pdes under similarity reductions get reduced to odes, whose solutions are free from movable critical points and thus obey the ARS (Ablowitz–Ramani–Segur) conjecture regarding the Painlevé property and integrable nonlinear dynamical systems [3]. This has been indeed verified for a large class of (1+1)-dimensional soliton systems [4].

In (2+1)-dimensions, integrable nonlinear pdes can show richer structures such as line solitons, line breathers, dromions and so on [3,5]. The Kadomtsev–Petviashvili and Davey–Stewartson equations are two of the earliest studied systems in higher dimensions which are direct extensions of Korteweg–de Vries (KDV) and nonlinear Schrödinger equations (NLS), respectively. These systems are solvable by the inverse scattering transform method, namely the d-bar method [3, 5]. From the symmetry point of view both the systems admit infinite-dimensional Lie symmetry vector fields [6, 7] of Kac–Moody–Virasoro type and admit interesting classes of similarity reductions. Other interesting (2+1)-dimensional evolution equations which were studied from a Lie symmetry point of view include fourth-order Infeld–Rowlands equations [8], three-wave interaction equation [9] and the generalized KDV equation [10] which also admit infinite-dimensional symmetry algebras.

There are many other important classes of (2+1)-dimensional nonlinear evolution equations which are direct extensions of (1+1)-dimensional soliton equations. These include the symmetric extensions of KdV, NLS and sine-Gordon equations, namely the Niznik – Novikov – Veselov (NNV) equation [11], (2+1)-dimensional generalization of the NLS equation of Fokas [12] and (2+1)-dimensional sine-Gordon (2DsG) equation of Konopelchenko and Rogers [13]. Besides there are other interesting equations such as the breaking soliton equation [5] and Strachan equation [14] which have interesting dynamical behaviour and fall within the IST formalism.

In this article, we aim to study the invariance and symmetry properties of the above type of (2+1)-dimensional evolution equations and show that in each of the cases there exists an infinite-dimensional Lie symmetry algebra. However, we point out that for the case of the breaking soliton equation, the Lie algebra does not possess a subalgebra of the Virasoro type. We obtain the various physically interesting similarity reductions and discuss the nature of solutions, wherever possible.

The plan of the paper is as follows. In sec.2 we discuss the Lie symmetries and similarity reductions associated with the NNV equation which is a symmetric generalization of the KdV equation in (2+1)-dimensions. We also find special solutions for certain cases. In Sec.3, we investigate the invariance properties associated with breaking soliton equation and present the possible similarity reductions associated with it. We also point out that it does not admit a Virasoro-type subalgebra which seems to be typical of integrable higher-dimensional nonlinear evolution equations. Finally in Secs.4 and 5 we present the Lie symmetries and similarity reductions associated with the (2+1)-dimensional NLS and 2DsG equations introduced by Fokas and Konopelchenko and Rogers, respectively, in the recent literature. In Sec.6 we present our conclusions, where we also mention briefly the nature of symmetries of the Strachan equation.

2 Lie symmetries and similarity reductions of the Nizhnik – Novikov – Veselov (NNV) equation

A symmetric generalization of the KdV equation in (2+1)–dimensions is the NNV equation [11]

$$u_t + u_{xxx} + u_{yyy} + u_x + u_y = 3(u\partial_y^{-1}u_x)_x + 3(u\partial_x^{-1}u_y)_y, \tag{1}$$

where ∂^{-1} denotes the integral over the subscripts:

$$\partial_y^{-1}u_x = \int\limits_{-\infty}^{y} u_x dy', \qquad \partial_x^{-1}u_y = \int\limits_{-\infty}^{x} u_y dx'.$$

Equation (1) admits the full Painlevé property, Hirota bilinearization and multidromion solutions [15]. Tamizhmani and Punithavathi [10] have studied the Lie symmetries of a nonsymmetrical version of eq.(1), by omitting the last term on the right–hand side and the linear terms.

To study the classical Lie symmetries associated with (1), we rewrite it as

$$u_t + u_{xxx} + u_{yyy} + u_x + u_y = 3(uv)_x + 3(uq)_y, \quad u_x = v_y, \quad u_y = a_x. \tag{2}$$

The invariance of eq.(2) under the infinitesimal point transformations

$$\begin{aligned}
x &\rightarrow X = x + \varepsilon\xi_1(t, x, y, u, v, q),\\
y &\rightarrow Y = y + \varepsilon\xi_2(t, x, y, u, v, q),\\
t &\rightarrow T = t + \varepsilon\xi_3(t, x, y, u, v, q),\\
u &\rightarrow U = u + \varepsilon\phi_1(t, x, y, u, v, q),\\
v &\rightarrow V = v + \varepsilon\phi_2(t, x, y, u, v, q),\\
q &\rightarrow Q = q + \varepsilon\phi_3(t, x, y, u, v, q), \quad \varepsilon << 1
\end{aligned}$$

leads to the expressions for the infinitesimals

$$\begin{aligned}
&\xi_1 = \frac{x}{3}\dot{f}(t) + g(t), \quad \xi_2 = \frac{y}{3}\dot{f}(t) + h(t), \quad \xi_3 = f(t), \quad \phi_1 = -\frac{2}{3}u\dot{f}(t),\\
&\phi_2 = \frac{2}{9}\dot{f}(t) - \frac{2}{3}v\dot{f}(t) - \frac{1}{9}x\ddot{f}(t) - \frac{1}{3}\dot{g}(t), \quad \phi_3 = \frac{2}{9}\dot{f}(t) - \frac{2}{3}g\dot{f}(t) - \frac{1}{9}y\ddot{f}(t) - \frac{1}{3}\dot{h}(t),
\end{aligned} \tag{3}$$

where $f(t), g(t)$ and $h(t)$ are arbitrary functions of t and a dot denotes differentiation with respect to time.

2.1 Lie algebra of symmetry vector fields

The presence of arbitrary functions f, g and h of t necessarily leads to an infinite-dimensional Lie algebra of symmetries. We can write a general element of this Lie algebra as

$$V = V_1(f) + V_2(g) + V_3(h),$$

where

$$\begin{aligned}
V_1(f) = {}& \frac{x}{3}\dot{f}(t)\frac{\partial}{\partial x} + \frac{y}{3}\dot{f}(t)\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t} - \frac{2}{3}u\dot{f}(t)\frac{\partial}{\partial u} +\\
& \left(\frac{2}{9}\dot{f}(t) - \frac{2}{3}v\dot{f}(t) - \frac{1}{9}x\ddot{f}(t)\right)\frac{\partial}{\partial v} + \left(\frac{2}{9}\dot{f}(t) - \frac{2}{3}q\dot{f}(t) - \frac{1}{9}y\ddot{f}(t)\right)\frac{\partial}{\partial q},\\
V_2(g) = {}& g(t)\frac{\partial}{\partial x} - \frac{1}{3}\dot{g}(t)\frac{\partial}{\partial v}, \quad V_3(h) = h(t)\frac{\partial}{\partial y} - \frac{1}{3}\dot{h}(t)\frac{\partial}{\partial q}.
\end{aligned}$$

The associated Lie algebra with these vector fields becomes

$$\begin{aligned}
&[V_1(f_1), V_1(f_2)] = V_1(f_1\dot{f}_2 - f_2\dot{f}_1), && [V_2(g_1), V_2(g_2)] = 0,\\
&[V_3(h_1), V_3(h_2)] = 0, && [V_1(f), V_2(g)] = V_2\left(f\dot{g} - \frac{1}{3}g\dot{f}\right),\\
&[V_1(f), V_3(h)] = V_3\left(f\dot{h} - \frac{1}{3}h\dot{f}\right), && [V_2(g), V_3(h)] = 0,
\end{aligned}$$

which is obviously an infinite–dimensional Lie algebra of symmetries. A Virasoro – Kac – Moody-type subalgebra is immediately obtained by restricting the arbitrary functions

F, g and h to Laurent polynomials so that we have the commutators

$$
\begin{aligned}
&[V_1(t^n), V_1(t^m)] = (m-n)V_1(t^{n+m-1}), && [V_1(t^n), V_2(t^m)] = (m-\frac{1}{3}n)V_2(t^{n+m-1}),\\
&[V_1(t^n), V_3(t^m)] = (m-\frac{1}{3}n)V_3(t^{n+m-1}), && [V_2(t^n), V_2(t^m)] = 0,\\
&[V_3(t^n), V_3(t^m)] = 0, && [V_2(t^n), V_3(t^m)] = 0.
\end{aligned}
$$

We have not pursued the further properties of these algebras here.

2.2 Similarity variables and similarity reductions

The similarity variables associated with the infinitesimal symmetries (3) can be obtained by solving the associated invariant surface condition or the related characteristic equation. The latter reads

$$
\begin{aligned}
&\frac{dx}{\frac{x}{3}\dot{f}(t)+g(t)} = \frac{dy}{\frac{y}{3}\dot{f}(t)+h(t)} = \frac{dt}{f(t)} = \frac{du}{-\frac{2}{3}u\dot{f}(t)} = \\
&\frac{dy}{\frac{2}{9}\dot{f}(t)-\frac{2}{3}v\dot{f}(t)-\frac{1}{9}x\ddot{f}(t)-\frac{1}{3}\dot{g}(t)} = \frac{dq}{\frac{2}{9}\dot{f}(t)-\frac{2}{3}q\dot{f}(t)-\frac{1}{9}y\ddot{f}(t)-\frac{1}{3}\dot{h}(t)}.
\end{aligned}
\tag{4}
$$

Integrating eq.(4) with the condition $f(t) \neq 0,\ g(t) \neq 0,\ h(t) \neq 0$, we get the following similarity variables:

$$
\begin{aligned}
&\tau_1 = \frac{x}{f^{1/3}(t)} - \int^t \frac{g(t')}{f^{4/3}(t')}dt', \quad \tau_2 = \frac{y}{f^{1/3}(t)} - \int^t \frac{h(t')}{f^{4/3}(t')}dt', \quad F = uf^{2/3}(t),\\
&G = vf^{2/3}(t) - \frac{1}{3}f^{2/3}(t) + \frac{1}{9}\dot{f}(t)\int^t \frac{g(t')}{f^{4/3}(t')}dt' + \frac{1}{9}\tau_1\dot{f}(t) + \frac{g(t)}{3f^{1/3}(t)},\\
&H = qf^{2/3}(t) - \frac{1}{3}f^{2/3}(t) + \frac{1}{9}\dot{f}(t)\int^t \frac{h(t')}{f^{4/3}(t')}dt' + \frac{1}{9}\tau_2\dot{f}(t) + \frac{h(t)}{3f^{1/3}(t)},
\end{aligned}
$$

where F, G and H are functions of τ_1 and τ_2. Under the above similarity transformations, eq.(2) gets reduced to a system of pde in two independent variables τ_1 and τ_2:

$$
F_{\tau_1\tau_1\tau_1} + F_{\tau_2\tau_2\tau_2} - 3(GF)_{\tau_1} - 3(FH)_{\tau_2} = 0, \quad F_{\tau_1} = G_{\tau_2}, \quad F_{\tau_2} = H_{\tau_1}. \tag{5}
$$

Since the original (2+1)-dimensional pde (2) satisfies the Painlevé property [15] for a general manifold, the (1+1)-dimensional similarity reduced pde (5) will also naturally satisfy the P-property and so is a candidate for a completely integrable system in (2+1)-dimensions.

2.3 Lie symmetries and similarity reductions of eqs.(5)

Now the reduced pde (5) in two independent variables can itself be further analyzed for its symmetry properties by looking at its own invariance property under the classical Lie algorithm again. In this case we obtain the following three-parameter Lie symmetries,

$$
\xi_1 = c_1\tau_1 + c_2,\ \xi_2 = c_1\tau_2 + c_3,\ \eta_1 = -2c_1F,\ \eta_2 = -2c_1G,\ \eta_3 = -2c_1H, \tag{6}
$$

where c_1, c_2 and c_3 are arbitrary constants. The corresponding Lie vector fields are

$$V_1 = \tau_1 \frac{\partial}{\partial \tau_1} + \tau_2 \frac{\partial}{\partial \tau_2} - 2F \frac{\partial}{\partial F} - 2G \frac{\partial}{\partial G} - 2H \frac{\partial}{\partial H}, \quad V_2 = \frac{\partial}{\partial \tau_1}, \quad V_3 = \frac{\partial}{\partial \tau_2},$$

leading to the 3–dimensional solvable Lie algebra

$$[V_1, V_2] = -V_2,\ [V_1, V_3] = -V_3,\ [V_2, V_3] = 0.$$

Solving the associated characteristic equation

$$\frac{d\tau_1}{c_1\tau_1 + c_2} = \frac{d\tau_2}{c_1\tau_2 + c_3} = \frac{dF}{-2c_1 F} = \frac{dG}{-2c_1 G} = \frac{dH}{-2c_1 H},$$

we obtain the similarity variables

$$\begin{aligned} & z = \frac{c_1\tau_1 + c_2}{c_1\tau_2 + c_3}, \\ & w_1(z) = (c_1\tau_2 + c_3)^2 F, \quad w_2(z) = (c_1\tau_2 + c_3)^2 G, \quad w_3(z) = (c_1\tau_2 + c_3)^2 H. \end{aligned} \tag{7}$$

The associated similarity reduced ode follows from eqs.(5) and (7) as

$$\begin{aligned} & c_1^2(1 - z^3) w_1''' - 12c_1^2 z^2 w_1'' - (36c_1^2 z + 3w_2 - 3zw_3) w_1' - \\ & \qquad 3w_1 w_2' + 3z w_1 w_3' + 12 w_1 w_3 - 24 c_1^2 w_1 = 0, \\ & w_1' + 2w_2 + z w_2' = 0, \quad w_3' + 2w_1 - z w_1' = 0 \quad \left(' = \frac{d}{dz}\right). \end{aligned} \tag{8}$$

While the exact solution of eq.(8) has not been found for the general three-parameter case, physically interesting solutions can be obtained for the special one-parameter choice $c_1 = 0,\ c_2 = c,\ c_3 = 1$ in eqs.(6). The associated similarity variables are

$$z = \tau_1 - c\tau_2,\ w_1 = F,\ w_2 = G,\ w_3 = H$$

and they lead to the "travelling wave" solutions. The reduced ode in this case becomes

$$(1 - c^3) w_1''' - 3w_1 w_2' - 3w_2 w_1' + 3c(w_1 w_3' w_1') = 0, \quad w_1' = -cw_2', w_3' = -cw_1'. \tag{9}$$

Integrating the two last eqs.(9), we get

$$w_1 = -cw_2 + I_1,\ w_3 = c^2 w_2 + I_2,$$

where I_1 and I_2 are constants. Substituting these expressions into the first of eqs.(9), one gets

$$w_2''' + K_1 w_2 w_2' + K_2 w_2' = k, \tag{10}$$

where K_1, K_2 and k are constants given in terms of c, I_1 and I_2. Integration of (10) leads to elliptic wave solutions.

2.4 Other special reductions

One can also obtain few other distinct similarity reductions for certain special choices of the arbitrary functions $f(t), g(t)$ and $h(t)$.

a) $f(t) = 0,\ g(t), h(t) \neq 0$
Similarity variables:

$$\tau_1 = x - \frac{g(t)}{h(t)}y,\ \tau_2 = t,\ F = u,\ G = v + \frac{\dot{g}(t)}{3g(t)}x,\ H = g + \frac{\dot{h}(t)}{3h(t)}x.$$

Reduced system:

$$F_{\tau_2} + F_{\tau_1\tau_1\tau_1} - \frac{g^3(\tau_2)}{h^3(\tau_2)}F_{\tau_1\tau_1\tau_1} + F_{\tau_1} - \frac{g(\tau_2)}{h(\tau_2)}F_{\tau_1} - 3GF_{\tau_1} + \frac{g'(\tau_2)}{g(\tau_2)}F -$$
$$3FG_{\tau_1} + 3\frac{g(\tau_2)}{h(\tau_2)}(HF)_{\tau_1} - \left(\frac{h'(\tau_2)}{h(\tau_2)} - \frac{g'(\tau_2)}{g(\tau_2)}\right)\tau_1 = 0,$$
$$F_{\tau_1} = -\frac{g(\tau_2)}{h(\tau_2)}G_{\tau_1}, \quad F_{\tau_1} = -\frac{h(\tau_2)}{g(\tau_2)}\left(H_{\tau_1} - \frac{h'(\tau_2)}{3h(\tau_2)}\right).$$

b) $f(t) = h(t) = 0$ and $g(t) \neq 0$
Similarity variables

$$\tau_1 = t,\ \tau_2 = y,\ \bar{\phi}_1 = u,\ \bar{\phi}_2 = v + \frac{g'(\tau_2)}{3g(\tau_2)}x,\ \bar{\Phi}_3 = q$$

lead to the solution

$$\bar{\phi}_1 = \bar{\phi}_1(\tau_1), \quad \bar{\phi}_2 = \bar{\phi}_2(\tau_1), \quad \bar{\phi}_3 = \frac{\tau_2}{3}\left(\frac{\bar{\phi}_{1\tau_1}}{\bar{\phi}_1} + \frac{g'(\tau_1)}{g(\tau_1)}\right) + \frac{\bar{\phi}_3(\tau_1)}{3}, \tag{11}$$

where $\bar{\phi}_1, \bar{\phi}_2$ and $\bar{\phi}_3$ are functions of τ_1. Rewriting (11) in terms of the original variables, one obtains the following particular solutions of eq.(2):

$$u = \bar{\phi}_1(t), \quad v = -\frac{\dot{g}(t)}{3g(t)}x + \bar{\phi}_2(t), \quad q = \frac{y}{3}\left(\frac{\dot{\bar{\phi}}_1}{\bar{\phi}_1} + \frac{\dot{g}}{g}\right) + \frac{\bar{\phi}_3(t)}{3},$$

where $g, \bar{\phi}_1, \bar{\phi}_2$ and $\bar{\phi}_3$ are arbitrary functions of t.
c) $f(t) = g(t) = 0,\ h(t) \neq 0$

As above, we obtain the special solution

$$u = \bar{\phi}_1(t), \quad v = \frac{x}{3}\left(\frac{\dot{\bar{\phi}}_1}{\bar{\phi}_1} + \frac{\dot{h}(t)}{h(t)}\right) + \frac{\bar{\phi}_2(t)}{3}, \quad q = -\frac{\dot{h}(t)}{3h(t)}y + \bar{\phi}_3(t),$$

where again $h, \bar{\phi}_1, \bar{\phi}_2$ and $\bar{\phi}_3$ are arbitrary functions of t.

3 Lie symmetries and similarity reductions of the breaking soliton equation

We now again wish to investigate the symmetries and similarity reductions associated with another important (2+1)-dimensional generalization of the KdV equation, namely, the breaking soliton equation of the form [5]

$$u_t + Au_{xxx} + Bu_{xxy} + 6Auu_x + 4Buu_y + 4Bu_x\partial_x^{-1}u_y = 0. \tag{12}$$

Eq.(12), in contrast to symmetric generalizations of eq.(1), is an asymmetric generalization of KdV in the (x, y) spatial variables. Note that for the choice $B = 0$ eq.(12) reduces to the KdV equation. Equation (12) possesses a Lax pair but the spectral parameter breaks to be a multivalued function and hence the solution may also get multivalued [5]. It also admits the Painlevé property for $A = 0$ and admits dromion-like solutions [16].

Making the substitution

$$u = \rho_x, \qquad \int_{-\infty}^{x} dx' u_y = \rho_y,$$

eq.(12) can be rewritten as

$$\rho_{xt} + A\rho_{xxxx} + B\rho_{xxxy} + 6A\rho_x\rho_{xx} + 4B\rho_x\rho_{xy} + 4B\rho_{xx}\rho_y = 0. \tag{13}$$

We will investigate the Lie symmetries of eq.(13) under the one-parameter (ε) group of transformations

$$\begin{array}{ll} x \to X = x + \varepsilon\xi_1(t, x, y, \rho), & y \to Y = y + \varepsilon\xi_2(t, x, y, \rho), \\ t \to T = t + \varepsilon\xi_3(t, x, y, \rho), & \rho \to \Delta = \rho + \varepsilon\phi(t, x, y, \rho), \end{array}$$

The infinitesimal transformations can be worked out to be

$$\begin{aligned} &\xi_1 = -c_1x + f(t), \quad \xi_2 = 4Bc_2t - c_1y - c_3, \quad \xi_3 = -3c_1t - c_4, \\ &\phi = -\frac{3A}{2B}c_2y + c_1\rho + c_2x + \frac{1}{4B}y\dot{f}(t) + g(t), \end{aligned} \tag{14}$$

where $f(t)$ and $g(t)$ are arbitrary functions, and c_1, c_2, c_3, c_4 are arbitrary parameters.

3.1 The Lie algebra

The Lie vector fields associated with the infinitesimal transformations can be written as

$$V = V_1(f) + V_2(g) + V_3 + V_4 + V_5 + V_6,$$

where f and g are arbitrary functions of t and

$$\begin{aligned} &V_1(f) = f(t)\frac{\partial}{\partial x} + \frac{1}{4B}y\dot{f}(t)\frac{\partial}{\partial \rho}, \quad V_2(g) = g(t)\frac{\partial}{\partial \rho}, \quad V_3 = -x\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - 3t\frac{\partial}{\partial t} + \rho\frac{\partial}{\partial \rho}, \\ &V_4 = 4Bt\frac{\partial}{\partial y} + \left(-\frac{3A}{2B}y + x\right)\frac{\partial}{\partial \rho}, \quad V_5 = -\frac{\partial}{\partial y}, \quad V_6 = -\frac{\partial}{\partial t}. \end{aligned}$$

The corresponding Lie algebra of the vector fields becomes

$$\begin{array}{lll} [V_1(f_1), V_1(f_2)] = 0, & [V_2(g_1), V_2(g_2)] = 0, & [V_1(f), V_2(g)] = 0, \\ [V_1, V_3] = V_1(-f + 3t\dot{f}), & [V_1, V_4] = V_2(f - t\dot{f}), & [V_1, V_5] = \dfrac{1}{4B}V_2(\dot{f}), \\ [V_1, V_6] = V_1(\dot{f}), & [V_2, V_3] = V_2(g + 3t\dot{g}), & [V_2, V_4] = 0, \\ [V_2, V_5] = 0, & [V_2, V_6] = V_2(\dot{g}), & [V_3, V_4] = -2V_4, \\ [V_3, V_5] = V_5, & [V_3, V_6] = 3V_6, & [V_4, V_5] = \dfrac{-3A}{2B}\dfrac{V_2}{g(t)}, \\ [V_4, V_6] = 4BV_6, & [V_5, V_6] = 0. & \end{array}$$

The above algebra is obviously infinite-dimensional, however, its further analysis is not performed here. However, one important point is that the above algebra does not contain a Virasoro algebra, which is typical of integrable (2+1)-dimensional systems such as the NNV equation discussed earlier and other integrable systems quoted in Introduction.

3.2 Similarity variables and reductions

Solving the characteristic equation associated with the infinitesimals (14), we obtain the following similarity variables

$$\tau_1 = \frac{x}{\Gamma^{1/3}} + \int_0^t \frac{f(t')dt'}{\Gamma^{4/3}}, \quad \tau_2 = \frac{y}{\Gamma^{1/3}} - \frac{1}{\Gamma^{1/3}}\left[\frac{4Bc_2t}{c_1} - \frac{4Bc_2}{2c_1^2}\Gamma - \frac{c_3}{c_1}\right],$$

$$\begin{aligned} F = {} & \rho\Gamma^{(1/3)} - \frac{6Ac_2^2}{c_1^2}t\Gamma^{(1/3)} + \frac{9Ac_2^2}{4c_1^3}\Gamma^{(4/3)} + \frac{3Ac_2c_3}{2Bc_1^2}\Gamma^{(1/3)} - -\frac{3Ac_2\tau_2}{4Bc_1}\Gamma^{(2/3)} - \\ & c_2\int^t \frac{1}{\Gamma^{1/3}}\left[\int^{t'} \frac{f(t'')}{\Gamma^{4/3}}dt''\right]dt' + \frac{c_2\tau_1}{2c_1}\Gamma^{(2/3)} + \frac{c_2}{c_1}\int^t \frac{t\dot{f}(t')dt'}{\Gamma^{2/3}} - \\ & \frac{c_2}{2c_1^2}\int^t \Gamma^{(1/3)}\dot{f}(t)dt - \frac{c_3}{4Bc_1}\int^t \frac{\dot{f}(t')dt'}{\Gamma^{2/3}} + \frac{\tau_2 f(t)}{4B\Gamma^{1/3}} + \\ & \frac{\tau_2}{4B}\int^t \frac{c_1 f(t')}{\Gamma^{4/3}}dt' + \int^t \frac{g(t')dt'}{\Gamma^{2/3}}, \end{aligned}$$

where $\Gamma = (3c_1t + c_4)$.

Using the above similarity variables, the (2+1)-dimensional breaking soliton equation (13) reduces to the 2-dimensional pde

$$\begin{aligned} & AF_{\tau_1\tau_1\tau_1\tau_1} + BF_{\tau_1\tau_1\tau_1\tau_2} + 6AF_{\tau_1}F_{\tau_1\tau_1} + 4BF_{\tau_1}F_{\tau_1\tau_2} + 4BF_{\tau_1\tau_1}F_{\tau_2} - \\ & \quad 2c_1F_{\tau_1} - c_1\tau_1F_{\tau_1\tau_1} - c_1\tau_2F_{\tau_1\tau_2} = 0. \end{aligned} \tag{15}$$

Eq.(15) can be further analyzed for its own invariance properties. Corresponding to a simple three-parameter Lie group of symmetries, one obtains the similarity variables

$$z = \tau_1 - \tau_2, \quad w = \tau_2 - F$$

so that (15) reduces to the ode

$$(A - B)w''' + (6A + 8B)w'w'' + 4Bw'' - 2c_1w' - c_1zw' = 0. \tag{16}$$

Eq.(16) can be shown to satisfy the Painlevé property.

3.3 Special similarity reductions

Besides the above general similarity reductions, one can find a number of special reductions corresponding to lesser parameter symmetries by choosing some of the arbitrary parameters c_1, c_2, c_3, c_4 and c_5 and arbitrary functions $f(t)$ and $g(t)$ to zero.

Important nontrivial cases are listed in Table I. One may notice that in cases 2,3,5–8, the similarity reduction yields essentially variable parameter KdV equations, while the case 4 leads to explicit solutions. The case 9 leads to the KdV equation.

Table I. Special similarity reductions of the breaking soliton equation

case	parameter restriction	Similarity variables	Reduction equation/ solution
1	$c_1 = 0$	$\tau_1 = x + \frac{1}{c_4}\int f(t')dt',\ \tau_2 = y + \frac{2Bc_2}{c_4}t^2 - \frac{c_3}{c_4}t$ $F = \rho + \frac{Ac_2}{c_4^2}t^3 - \frac{3Ac_2c_3}{4Bc_4^2}t^2 - \frac{3Ac_2}{2Bc_4}\tau_2 t + \frac{c_2}{c_4}\tau_1 t +$ $\frac{f(t)}{4Bc_4}\tau_2 + \frac{c_2}{c_4^2}\left[\int^t \left(\int^{t'} f(t'')dt''\right) dt'\right] + \int^t \frac{g(t')}{c_4}dt' -$ $\frac{Bc_2}{2c_4^2}\left[t^2 f(t) - 2\int^t f(t')dt'\right] + \frac{c_3}{4Bc_4^2}\left[tf(t) - \int^t f(t')dt'\right]$	$AF_{\tau_1\tau_1\tau_1\tau_1} + BF_{\tau_1\tau_1\tau_1\tau_2} + 6AF_{\tau_1}F_{\tau_1\tau_1} +$ $4BF_{\tau_1}F_{\tau_1\tau_2} + 4BF_{\tau_1\tau_1}F_{\tau_2} -$ $\frac{c_3}{c_4}F_{\tau_1\tau_2} - \frac{c_2}{c_4} = 0$
2	$c_1 = c_4 = 0$ $f(t) = 0$	$\tau_1 = x,\ \tau_2 = t,$ $F = \rho + \left[\frac{3A}{4B}c_2 y^2 - c_2\tau_1 y + g(\tau_2)y\right]\frac{1}{4Bc_2\tau_2 - c_3}$	$\chi = F_{\tau_1},$ $\chi_{\tau_2} + A\chi_{\tau_1\tau_1\tau_1} + 6A\chi\chi_{\tau_1} +$ $\frac{4B}{4Bc_2\tau_2 - c_3}\left[c_2\chi + (B + c_2\tau_1)\chi_{\tau_1}\right] = 0$
3	$c_1 = c_2 = c_4 = 0$	$\tau_1 = x + \frac{1}{c_3}f(t)y,\ \tau_2 = t,$ $F = \rho + \frac{f'(\tau_2)}{8Bc_3}y^2 + \frac{g(\tau_2)}{c_3}y$	$\chi = F_{\tau_1},$ $\chi_{\tau_2} + \left(A + \frac{B}{c_3}f(\tau_2)\right)\chi_{\tau_1\tau_1\tau_1} +$ $\left(6A + \frac{B}{c_3}f(\tau_2)\right)\chi\chi_{\tau_1} - \frac{4B}{c_3}g(\tau_2)\chi_{\tau_1} = 0$
4	$c_1 = c_2 =$ $c_3 = c_4 = 0$	$\tau_1 = y,\quad \tau_2 = t,$ $F = \rho - \frac{1}{4B}xy\frac{f'(\tau_2)}{f(\tau_2)} - \frac{g(\tau_2)}{f(\tau_2)}x$	$u = \frac{k_1 xy}{4B(k_1 t + k_2)} + \frac{k_3 x}{4B(k_1 t + k_2)} + m(y,t)$ k_1, k_2, k_3: arbitrary constants $m(y,t)$: arbitrary function
5	$c_1 = c_2 = c_4 = 0$ $f(t) = 0$	$\tau_1 = x,\ \tau_2 = t,\ F = \rho + \frac{1}{c_3}g(\tau_2)y$	$\chi = F_{\tau_1},$ $\chi_{\tau_2} + A\chi_{\tau_1\tau_1\tau_1} + 6\chi\chi_{\tau_1} - \frac{4}{3}B\beta(\tau_2)\chi_{\tau_1} = 0$

6	$c_1 = c_2 = c_4 = 0$ $g(t) = 0$	$\tau_1 = x + \frac{1}{3} f(t) y,\ \tau_2 = y,$ $F = \rho + \frac{1}{8Bc_3} f'(\tau_2) y^2$	$\chi = F_{\tau_1},$ $\chi_{\tau_2} + \left(6A + \frac{8}{3} B f(\tau_2)\right) \chi\chi_{\tau_1} +$ $\left(A + \frac{1}{3} B f(\tau_2)\right) \chi_{\tau_1\tau_1\tau_1} = 0$
7	$c_1 = c_3 = c_4 = 0$ $f(t) = 0$	$\tau_1 = x,\ \tau_2 = t,$ $F = \rho + \frac{3A}{16B\tau_2} y^2 + \frac{y}{4B\tau_2}\tau_1 + \frac{g(\tau_2)}{4Bc_2} y$	$\chi = F_{\tau_1},$ $\chi_{\tau_2} + 6A\chi\chi_{\tau_1} + A\chi_{\tau_1\tau_1\tau_1} + \frac{1}{\tau_2}\chi +$ $\left(\frac{\tau_1}{\tau_2} + \frac{g(\tau_2)}{c_2}\right) \chi_{\tau_1} = 0$
8	$c_1 = c_3 = c_4 = 0$ $g(t) = 0$	$\tau_1 = x - \frac{f(t)}{4Bc_2 t} y,\ \tau_2 = t,$ $F = \rho + \frac{3Ay^2}{16B^2\tau_2} + \frac{1}{4B\tau_2}\left[\frac{f(\tau_2)}{8Bc_2\tau_2} y^2 + \tau_1 y\right] \frac{f'(\tau_2)}{32B^2 c_2 \tau_2} y^2$	$\chi_{\tau_2} + \left(6A - \frac{f(\tau_2)}{c_2\tau_2}\right) \chi\chi_{\tau_1} + \left(A - \frac{f(\tau_2)}{4c_2\tau_2}\right) \times$ $\chi_{\tau_1\tau_1\tau_1} + \frac{1}{\tau_2}\chi_{\tau_1} + \frac{\tau_1\chi_\tau}{\tau_2} f(\tau_2) \frac{\chi\chi_{\tau\tau}}{c_2\tau_2} = 0$
9	$c_1 = c_2 = c_4 = 0$ $f(t) = g(t) = 0$	$\tau_1 = x,\ \tau_2 = t,\ F = \rho$	$\chi = F_{\tau_1},$ $\chi_{\tau_2} + \chi\chi_{\tau_1} + \chi_{\tau_1\tau_1\tau_1} = 0$ (KdV equation)
10	$c_1 = c_2 = c_3 = 0$ $c_4 = f(t)$	$\tau_1 = y,\ \tau_2 = t,\ F = \rho - \frac{\tau_1 g'(\tau_2)}{4Bg(\tau_2)} x$	$u = \frac{y}{4B} \int \frac{k_1}{k_1 t + k_2} + F(y,t),$ F is an arbitrary function

4 Lie symmetries and similarity reductions of the (2+1)-dimensional generalized nonlinear Schrödinger equation

The recently introduced (2+1)-dimensional generalized nonlinear Schrödinger equation of Fokas [12]

$$
\begin{aligned}
&iq_t - (\alpha-\beta)q_{xx} + (\alpha+\beta)q_{yy} - 2\lambda q\Big[(\alpha+\beta)v - (\alpha-\beta)u\Big] = 0,\\
&v_x = \mid q\mid_y^2, \qquad v_y = \mid q \mid_x^2,
\end{aligned} \tag{17}
$$

admits Lie symmetries in the following form when the complex scalar field is written as

$$q = a + ib,$$

where $a = a(x,y,t),\ b = b(x,y,t)$ are real functions. The associated Lie symmetry vector fields turn out to be

$$V = V_1(f) + V_2(g) + V_3(h) + V_4(l) + V_5(m),$$

where

$$
\begin{aligned}
V_1(f) = {}& \frac{1}{2}x\dot f(t)\frac{\partial}{\partial x} + \frac{1}{2}y\dot f(t)\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t} - \left(\frac{1}{2}a\dot f(t) - \frac{1}{BA}bx^2\ddot f(t) + \frac{1}{8B}by^2\ddot f(t)\right)\frac{\partial}{\partial a} - \\
& \left(\frac{1}{2}b\dot f(t) - \frac{1}{8A}ax^2\ddot f(t) - \frac{1}{8B}ay^2\ddot f(t)\right)\frac{\partial}{\partial b} - (v\dot f(t) + \frac{1}{16B^2C}y^2 \dddot f(t))\frac{\partial}{\partial v} - \\
& \left(u\dot f(t) + \frac{1}{16A^2C}x^2 \dddot f(t)\right)\frac{\partial}{\partial u},\\
V_2(g) = {}& g(t)\frac{\partial}{\partial x} + \frac{1}{2A}bx\dot g(t)\frac{\partial}{\partial a} - \frac{1}{2A}ax\dot g(t)\frac{\partial}{\partial b} - \frac{1}{4A^2C}x\ddot g(t)\frac{\partial}{\partial u},\\
V_3(h) = {}& h(t)\frac{\partial}{\partial y} - \frac{1}{2B}by\dot h(t)\frac{\partial}{\partial a} + \frac{1}{2B}ay\dot h(t)\frac{\partial}{\partial b} - \frac{1}{4B^2C}y\ddot h(t)\frac{\partial}{\partial v},\\
V_4(l) = {}& -bl(t)\frac{\partial}{\partial a} + al(t)\frac{\partial}{\partial b} + \frac{1}{2AC}\dot l(t)\frac{\partial}{\partial u}, \quad V_5(m) = m\frac{\partial}{\partial v} + \frac{B}{A}m\frac{\partial}{\partial u},
\end{aligned}
$$

where $A = (\alpha-\beta),\ B = (\alpha+\beta)$ and $C = \lambda$. The nonzero commutation relations for the Lie vector fields are

$$
\begin{aligned}
&[V_1(f_1),\ V_1(f_2)] = V_1(f_1\dot f_2 - f_2\dot f_1), && [V_2(g_1),\ V_2(g_2)] = -\frac{1}{2A}V_4(g_1\dot g_2 - g_2\dot g_1),\\
&[V_3(h_1),\ V_3(h_2)] = -\frac{1}{2B}V_4(h_1\dot h_2 - h_2\dot h_1), && [V_1(f),\ V_2(g)] = V_2\left(f\dot g - \frac{g\dot f}{2}\right),\\
&[V_1(f),\ V_3(h)] = V_3\left(f\dot h - \frac{h\dot g}{2}\right), && [V_1(f),\ V_4(h)] = V_4(f\dot l),\\
&[V_1(f),\ V_5(m)] = V(m\dot f + f\dot m).
\end{aligned} \tag{18}
$$

One sees that Virasoro–Kac–Moody-type Lie algebras occur for all the three integrable cases of the generalized (2+1)-dimensional NLS, the only difference being that coefficient A in eqs.(18) takes different values for different cases. These forms are similar to that of the Davey–Stewartson equations derived by Champagne and Winternitz [7]. Note that the above symmetries get modified for the limiting cases $A = 0$ $(\alpha = \beta)$ and $B = 0$ $(\alpha = -\beta)$. They will be reported separately.

The similarity reductions can be performed as for the previous systems and the results are similar to that of Champagne and Winternitz [7] but for minor details. The general similarity variables in this case turn out to be

$$\begin{aligned}
&\tau_1 = \frac{x}{f^{1/2}(t)} - \int^t \frac{q(t')}{f^{3/2}(t')}dt', \quad \tau_2 = \frac{y}{f^{1/2}(t)} - \int^t \frac{h(t')}{f^{3/2}(t')}dt', \\
&a = \frac{F_2 \sin U}{f^{1/2}(t)}, \quad b = \frac{F_2 \cos U}{f^{1/2}(t)}, \\
&v = \frac{1}{f(t)} \int^t \left(-\frac{\ddot{h}(t')y}{4B^2C} - \frac{\dddot{f}(t')y^2}{16B^2C} + l \right) dt' + \frac{F_3}{f(t)}, \\
&u = \frac{1}{f(t)} \int^t \left(-\frac{\ddot{g}(t')x}{4A^2C} - \frac{\dddot{f}(t')x^2}{16B^2C} + \frac{g(t')l(t')}{A} + \frac{\dot{m}(t')}{2AC} \right) dt' + \frac{F_4}{f(t)},
\end{aligned}$$

where

$$U = \int^t \frac{1}{f(t')} \left(-m(t') + \frac{\dot{g}(t')}{2A}x + \frac{\ddot{f}(t')}{8A}x^2 - \frac{\dot{h}(t')}{2B}y - \frac{\ddot{f}(t')}{8B}y^2 \right) dt'.$$

Under this similarity transformation the reduced equation takes the form

$$\begin{aligned}
&AF_2F_{1\tau_1}^2 - BF_2F_{1\tau_2}^2 - AF_{2\tau_1\tau_1} + BF_{2\tau_2\tau_2} - 2BCF_2F_3 + 2ACF_2F_4 = 0, \\
&2AF_{2\tau_1}F_{1\tau_1} + AF_2F_{1\tau_1\tau_1} - 2BF_{2\tau_1}F_{1\tau_2} - BF_2F_{1\tau_2\tau_2} = 0, \\
&F_{3\tau_1} - 2F_2F_{2\tau_2} = 0, \quad F_{4\tau_2} - 2F_2F_{2\tau_1} = 0.
\end{aligned} \tag{19}$$

Now performing the invariance analysis for the eq.(19), we find that the system is invariant under the following five-parameter symmetry group:

$$\begin{aligned}
&\xi_1 = -(1/2)c_2\tau_1 + c_4, \quad \xi_2 = -(1/2)c_2\tau_2 + c_5, \\
&\phi_1 = c_1, \quad \phi_2 = (1/2)c_2F_2, \quad \phi_3 = c_2F_3 + c_3, \quad \phi_4 = c_2F_4 + (B/A)c_3,
\end{aligned}$$

where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants.

4.1 Similarity variables

The similarity variables associated with the above infinitesimal symmetries are

$$\begin{aligned}
&z = \frac{\tau_1}{c_2\tau_1 - 2c_5} - \frac{2c_4}{c_2(c_2\tau_2 - 2c_5)}, \\
&w_1 = F_1 + \frac{2c_1}{c_2} \log{(c_2\tau_2 - 2c_5)}, \quad w_2 = (c_2\tau_2 - 2c_5)F_2, \\
&w_3 = \left(F_3 + \frac{c_3}{c_2}\right) c_2(c_2\tau_2 - 2c_5), \quad w_4 = \left(F_4 + \frac{Bc_3}{Ac_2}\right) c_2(c_2\tau_2 - 2c_5).
\end{aligned}$$

Table II. Special similarity reductions of (2+1)–dimensional sine–Gordon equations

case	parameter restriction	Similarity variables	Reduced system
1	$f(x)=0$	$\tau_2=x,\ \tau_2=\int^y \frac{dy'}{g(y')}-\int^t \frac{dt'}{h(t')},$ $F=\rho h(t)-\int^t l(t')dt',\ G=\theta-\int^t \frac{m(t')}{h(t')}dt'$	$G_{\tau_1\tau_2\tau_2}-\frac{1}{2}F_{\tau_1}G_{\tau_2}-\frac{1}{2}F_{\tau_2}G_{\tau_1}=0,$ $F_{\tau_1\tau_2}+\frac{1}{2}\Big[G_{\tau_1\tau_2}G_{\tau_2}+G_{\tau_1}G_{\tau_2\tau_2}\Big]=0$
2	$g(y)=0$	$\tau_1=\int^x \frac{dx'}{f(x')}-\int^t \frac{dt'}{h(t')},\ \tau_2=y,$ $F=\rho h(t)-\int^t l(t')dt',\ G=\theta-\int^t \frac{m(t')}{h(t')}dt'$	$G_{\tau_1\tau_2\tau_2}+\frac{1}{2}F_{\tau_1}G_{\tau_2}+\frac{1}{2}F_{\tau_2}G_{\tau_1}=0,$ $F_{\tau_1\tau_2}+\frac{1}{2}\Big[G_{\tau_2}G_{\tau_1\tau_1}+G_{\tau_1}G_{\tau_2\tau_2}\Big]=0$
3	$h(t)=0$	$\tau_1=\int^x \frac{dx'}{f(x')}-\int^y \frac{dy'}{g(y')},\ \tau_2=t,$ $F=\rho-l(t)\int^y \frac{dy'}{g(y')},\ \ G=\theta-m(t)\int^y \frac{dy'}{g(y')}$	$G_{\tau_1\tau_1\tau_2}+F_{\tau_1}G_{\tau_1}-\frac{1}{2}m(\tau_2)F_{\tau_1}-\frac{1}{2}l(\tau_2)G_{\tau_1}=0,$ $G_{\tau_1}G_{\tau_1\tau_2}-F_{\tau_1\tau_1}-\frac{1}{2}m(\tau_2)G_{\tau_1\tau_2}-\frac{1}{2}m'(\tau_2)G_{\tau_1}=0,$
4	$f(x)=0, g(y)=0$	$\tau_1=x,\ \tau_2=y,\ F=\rho h(t)-\int^t l(t')dt',$ $G=\theta-\int^t \frac{m(t')}{h(t')}dt'$	$F_{\tau_1\tau_2}=0,\ F_{\tau_1}G_{\tau_1}+F_{\tau_2}G_{\tau_1}=0$
5	$f(x)=0, h(t)=0$	$\tau_1=x,\ \tau_2=t,\ F=\rho-l(t)\int^y \frac{dy'}{g(y')},$ $G=\theta-m(t)\int^y \frac{dy'}{g(y')}$	$m(\tau_2)F_{\tau_1}+l(\tau_2)G_{\tau_1}=0,\ m_{\tau_2}G_{\tau_1\tau_2}+m'(\tau_2)G_{\tau_1}=0$
6	$g(y)=0, h(t)=0$	$\tau_1=y,\ \tau_2=t,\ F=\rho-l(t)\int^x \frac{dx'}{f(x')},$ $G=\theta-m(t)\int^x \frac{dx'}{f(x')}$	$l(\tau_2)G_{\tau_1}+m(\tau_2)F_{\tau_1}=0,\ m_{\tau_2}G_{\tau_1\tau_2}+m'(\tau_2)G_{\tau_1}=0$

The reduced odes take the form

$$
\begin{aligned}
&2Aw_{1z}w_{2z} + Aw_2w_{1zz} - 6Bc_1c_2w_2 - 4Bc_1c_2zw_{2z} - 2Bc_2^2zw_2w_{1z} - \\
&\quad 2Bc_2^2z^2w_{1z}w_{2z} + Bc_2^2z^2w_{1zz}w_2 = 0, \\
&Aw_2w_{1z}^2 - 4Bc_1^2w_2 - Bc_2^2z^2w_2w_{1z}^2 - 4Bc_1c_2w_2w_{1z} - Aw_{2zz} + +2Bc_2^2w_2 + 4Bc_2^2w_{2z} + \\
&\quad Bc_2^2z^2w_{2zz} - BCw_2w_3 + \frac{2ACw_2w_4}{c_2} = 0, \\
&2w_2w_{2z} + 2w_4 + zw_{4z} = 0, \quad w_{3z} + 2c_2^2w_2^2 + 2c_2^2zw_2w_{2z} = 0.
\end{aligned}
$$

Since the original pde (17) passes the P-test and the reduced ode will also admit the P–property.

5 Lie symmetries of the (2+1)-dimensional sine-Gordon equation

The (2+1)-dimensional integrable sine–Gordon equation introduced by Konopelchenko and Rogers [13] in appropriate variables has the form

$$
\theta_{xyt} + \frac{1}{2}\theta_y\rho_x + \frac{1}{2}\theta_x\rho_y = 0, \quad \rho_{xy} - \frac{1}{2}(\theta_x\theta_y)_t = 0.
$$

We list below the main results alone. Infinitesimal symmetries are

$$
\xi_1 = f(x),\ \xi_2 = g(y),\ \xi_3 = h(t),\ \phi_1 = m(t),\ \phi_2 = -\rho\dot{h}(t) + l(t),
$$

where f, g, h, m and l are arbitrary functions of x, y and t, respectively. Here $\xi_1, \xi_2, \xi_3, \phi_1$ and ϕ_2 are the infinitesimals associated with the variables x, y, t, θ and ρ, respectively. Vector fields are as follows:

$$
V = V_1(f) + V_2(g) + V_3(h) + V_4(l) + V_5(m),
$$

where

$$
V_1 = f(x)\frac{\partial}{\partial x},\ V_2 = g(y)\frac{\partial}{\partial y},\ V_3 = h(t)\frac{\partial}{\partial t} - \rho\dot{h}(t)\frac{\partial}{\partial \rho},\ V_4 = l(t)\frac{\partial}{\partial \rho},\ V_5 = m(t)\frac{\partial}{\partial \theta}.
$$

The nonzero commutation relations among the vector fields are

$$
\begin{aligned}
&[V_1(f_1), V_1(f_2)] = V_1(f_1f_2' - f_2f_1'), \quad [V_2(g_1), V_2(g_2)] = V_2(g_1g_2' - g_2g_1'), \\
&[V_3(h_1), V_3(h_2)] = V_3(h_1h_2' - h_2\dot{h}_1), \quad [V_3(h), V_4(l)] = V_4(l\dot{h} + h\dot{l}), \\
&[V_3(h), V_5(m)] = V_5(h\dot{m}).
\end{aligned}
$$

All other commutators vanish.

General similarity variables

$$
\begin{aligned}
&\tau_1 = \int^x \frac{dx'}{f(x')} - \int^t \frac{dt'}{h(t')}, \qquad \tau_2 = \int^y \frac{dy'}{g(y')} - \int^t \frac{dt'}{h(t')}, \\
&F = \rho h(t) - \int^t \frac{l(t')}{h(t')}dt', \qquad G = \theta - \int^t \frac{m(t')}{h(t')}dt'.
\end{aligned}
$$

Reduced (1+1)-dimensional equation

$$\begin{aligned} &G_{\tau_1\tau_1\tau_2} + G_{\tau_1\tau_2\tau_2} - \frac{F_{\tau_1}G_{\tau_2}}{2} - \frac{F_{\tau_2}G_{\tau_1}}{2} = 0, \\ &F_{\tau_1\tau_2} + \frac{G_{\tau_1\tau_2}}{2}(G_{\tau_1} + G_{\tau_2}) + \frac{G_{\tau_2}G_{\tau_1\tau_1}}{2} + \frac{G_{\tau_1}G_{\tau_2\tau_2}}{2} = 0. \end{aligned} \tag{20}$$

Eq.(20) has a five-dimensional symmetry Lie algebra, leading to the reduced coupled odes:

$$c_1^2(\tau^3 - 1)w_{2\tau\tau\tau} + 2c_1^2(\tau^2 - 1)w_{2\tau\tau} - 6c_1^2\tau w_{2\tau} - 2\tau w_{1\tau}w_{2\tau} - w_1 w_{2\tau} + w_{1\tau} + 2c_1 = 0,$$
$$c_1^2(\tau^3 - 1)w_{2\tau\tau} + 2c_1^2(\tau^2 - 1)w_{2\tau\tau} - 6c_1^2\tau w_{2\tau} + 2\tau w_{1\tau}w_{2\tau} - w_1 w_{2\tau} + w_{1\tau} + 2c_1 = 0.$$

Special reductions: They are given in Table II.

6 Conclusions

In this paper we have carried out invariance analysis for certain (2+1)-dimensional nonlinear evolution equations. Particularly, we have considered the NNV equation, breaking soliton equation, generalized NLS equation, and 2DsG equation. While all these systems admit an infinite-dimensional Lie algebra of symmetry vector fields, the breaking soliton equation is not of the Kac–Moody–Virasoro-type while the others are. Our recent investigation shows that another system which is integrable but yet has a non Virasoro-type infinite-dimensional Lie algebra is the Strachan equation [14]. The fuller implications of the existence of both types of infinite-dimensional Lie algebras of symmetry vector fields in (2+1)-dimensions and their connection to integrability deserve much further study. Among these, the integrable breaking soliton equation does not admit a Virasoro-type subalgebra, whereas other above integrable systems admit. Our present investigation also shows that this type of algebras also does not exist in the higher-dimensional NLS introduced by Strachan. Currently we are investigating the Lie algebraic structure and similarity reductions associated with them. The results will be published elsewhere.

Acknowledgement:

The work of M.S is supported by a Senior Research Fellowship by the Council of Scientific and Industrial Research. The work of M.L forms part of a National Board for Higher Mathematics (Department of Atomic Energy) and also a Department of Science and Technology, Government of India have sponsored the research project.

References

[1] Olver P.J., Applications of Lie Groups to Differential Equations, Springer, New York, 1986.

[2] Bluman G.W. and Kumei S., Symmetries and Differential Equations, Springer, New York, 1989.

[3] Ablowitz M. J. and Clarkson P.A., Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, Cambridge, 1990.

[4] Lakshmanan M. and Kaliappan P., Lie transformations, nonlinear evolution equations and Painlevé forms, *J. Math. Phys.,* 1983, V.24, 795.

[5] Konopelchenko B.G., Solitons in Multidimensions, World Scientific, Singapore, 1993.

[6] David D., Kamran N., Levi D. and Winternitz P., Symmetry reduction for the Kadomstev–Petviashvili equation using a loop algebra, *J. Math. Phys.,* 1986, V.27, 1225.

[7] Champagne B. and Winternitz P., On the infinite dimensional symmetry group of the Davey–Stewartson equations, *J. Math. Phys.,* 1988, V.9, 1.

[8] Faucher M. and Winternitz P., Symmetry analysis of the Infeld–Rowlands equation, *Phys. Rev. E,* 1993, V.48, 3066.

[9] Martina L. and Winternitz P., Analysis and applications of symmetry group of the multidimensional three wave interaction problem, *Ann. Phys.,* (N.Y.), 1989, V.196, 231.

[10] Tamizhmani K.M. and Punithavathi P., The infinite-dimensional Lie algebraic structure and the symmetry reductions of a nonlinear higher–dimensional equation, *J. Phys. Soc. Japan,* 1990, V.59, 843.

[11] Novikov S.P. and Veselov A.P., A multidimensional Schrödinger operator: Inverse scattering transform and evolutional equations, *Physica D,* 1986, V.18, 267.

[12] Fokas A.S., On the simplest integrable equation in 2+1, *Inv. Prob.,* 1994, V.10, L19.

[13] Konopelchenko B. G. and Rogers C., On generalized Loewner systems: Novel integrable equations in (2+1) dimensions, *J. Math. Phys.,* 1993, V.34, 214.

[14] Strachan I.A.B., Wave solutions of a (2+1) dimensional generalization of the nonlinear Schrödinger equations, *Inv. Prob.,* 1992, V.8, L21.

[15] Radha R. and Lakshmanan M., Singularity analysis and localized coherent structures in (2+1) dimensional generalized Korteweg–deVries equations, *J. Math. Phys.,* 1994, V.35, 4746.

[16] Radha R. and Lakshmanan M., Dromion–like structures in the (2+1) dimensional breaking soliton equations, *Phys. Lett.,* 1995, V.197, 7.